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A Note on Padé Approximants Pairs as Limits of Algebraic Polynomials Pairs of Weighted Best Approximation in Orlicz Spaces

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Abstract. In this short note, we show the behavior in Orlicz spaces of best approximations by algebraic polynomials pairs on union of neighborhoods, when the measure of them tends to zero.

Key Words: Best approximation pair, Padé approximant pair, Orlicz spaces.

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1 Introduction

Let $\emptyset \neq X \subset \mathbb{R}$ be an open and bounded set. We denote by $\mathcal{M}=\mathcal{M}(X)$ the equivalence class of all real Lebesgue measurable functions on X. Let Φ be the set of convex functions $\phi:\mathbb{R}_+\to\mathbb{R}_+$, with $\phi(x)>0$ for $x>0$ and $\phi(0)=0$.

For each $\phi\in\Phi$, define

$$L^{\phi}=L^{\phi}(X)=\Big\{f\in\mathcal{M}:\ \int_X \phi(\alpha|f(x)|)dx<\infty \ \text{ for some } \ \alpha>0\Big\}.$$

The space L^{ϕ} is called an Orlicz space determined by ϕ. This space is endowed with the Luxemburg norm defined by

$$\|f\|_{\phi}=\inf\Big\{\lambda>0:\int_X \phi\Big(\frac{|f(x)|}{\lambda}\Big)dx\leq 1\Big\},$$

and so it becomes a Banach space. Sometimes we write $\|\cdot\|_{L^{\phi}(W)}$ instead of $\|f\chi_W\|_{\phi}$, where χ_W denotes the characteristic function on $W\subset X$.

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We assume that $\phi\in\Phi$ satisfies the Δ_2-condition, that is, there exists a constant $\gamma>0$ such that $\phi(2x)\le\gamma\phi(x)$ for all $x\ge 0$. In this case,

$$\int_X \phi\Big(\frac{|f(x)|}{\|f\|_\phi}\Big)dx=1.$$

A detailed treatment about these subjects may be found in [5].

Given $x_1<\cdots<x_k$ in X, $k\ge 1$, for $\delta>0$ small enough we define a net of pairwise disjoint sets $V_j=V_j(\delta):=x_j+\varepsilon_j(\delta)A_j\subset X$, $1\le j\le k$, where $\varepsilon_j=\varepsilon_j(\delta)\searrow 0$ as $\delta\to 0$, and each interval A_j, independent of δ, has Lebesgue measure 1.

Let $a\in\mathbb{R}$, $n,m\in\mathbb{N}\cup\{0\}$ and let Π^n be the class of algebraic polynomials with real coefficients of degree at most n. For $r\in\{0,1\}$, let $\Pi^m(a,r)=\{Q\in\Pi^m:Q(a)=r\}$ and we consider the sets

$$\mathcal{S}^n_m(a):=\Pi^n\times\Pi^m(a,1)\quad\text{and}\quad \Pi^n_m(a):=\Big\{\frac{P}{Q}:(P,Q)\in\Pi^n\times\Pi^m(a,0),\ Q\neq 0\Big\}.$$

Given a function $f\in L^\phi$, we say that $(P_\delta,Q_\delta)\in\mathcal{S}^n_m(a)$ is a *best* $\|\cdot\|_\phi$*-approximant pair* of f from $\mathcal{S}^n_m(a)$ respect to $\|\cdot\|_{L^\phi(V)}$ if

$$\|fQ_\delta-P_\delta\|_{L^\phi(V)}\le\|fQ-P\|_{L^\phi(V)},\quad (P,Q)\in\mathcal{S}^n_m(a), \tag{1.1}$$

where $V=\bigcup_{j=1}^k V_j$. It is easy to see that (P_δ,Q_δ) exists. In fact, let $Q_*(x)=1$, $x\in\mathbb{R}$. Then $\Pi^m(a,1)=Q_*+\Pi^m(a,0)$ and we see that existence of a minimizing pair for (1.1) is equivalent to the existence of a minimum of

$$\|f-R\|_{L^\phi(V)},\quad R\in\mathcal{R}^n_m(f,a), \tag{1.2}$$

where

$$\mathcal{R}^n_m(f,a):=f\Pi^m(a,0)+\Pi^n$$

is a finite dimensional subspace of L^ϕ. Clearly, (1.2) is minimized by some $R_0=fQ_0+P_0\in\mathcal{R}^n_m(f,a)$, so that (P_0,Q_*-Q_0) is a best $\|\cdot\|_\phi$-approximant pair of f from $\mathcal{S}^n_m(a)$ respect to $\|\cdot\|_{L^\phi(V)}$.

We observe that if $f\notin\Pi^n_m(a)$, then $\mathcal{R}^m_n(f,a)$ has dimension $n+m+1$ and $\mathcal{R}^n_m(f,a)=f\Pi^m(a,0)\oplus\Pi^n$

If the net (P_δ,Q_δ) has a limit in $\mathcal{S}^n_m(a)$ as $\delta\to 0$, this limit is called a best local $\|\cdot\|_\phi$-approximation of type (n,m) of f from $\mathcal{S}^n_m(a)$ on $\{x_1,\cdots,x_k\}$.

We denote by $PC^t(X)$ the class of functions with $t-1$ continuous derivatives and bounded, piecewise continuous t^{th} derivative on X. Let $f\in PC^t(X)$, $(P,Q)\in\Pi^n\times\Pi^m$, and let $\langle i_j\rangle$ be an ordered N-tuple of nonnegative integers with $i_j\le t$ and $\sum_{j=1}^N i_j=n+m+1$. If

$$(fQ-P)^{(i)}(y_j)=0,\quad j=1,2,\cdots,N,\quad i=0,1,\cdots,i_j-1, \tag{1.3}$$

then (P,Q) is said to be a $\langle i_j\rangle$-Padé approximant pair of f in $\{y_1,\cdots,y_N\}$. If $Q\neq 0$ and

$$\Big(f-\frac{P}{Q}\Big)^{(i)}(y_j)=0,\quad j=1,2,\cdots,N,\quad i=0,1,\cdots,i_j-1,$$

then the rational function P/Q is called a $\langle i_j\rangle$-Padé approximant of f in $\{y_1,\cdots,y_N\}$. Clearly, the problem (1.3) always has a nontrivial solution for (P,Q), since it is a homogeneous system of $n+m+1$ equations in $n+m+2$ unknowns.

From now on, we make an assumption on the k-tuple $\langle\varepsilon_j\rangle:=(\varepsilon_1,\cdots,\varepsilon_k)$ which allows us to compare the following expressions, as functions of δ,

$$v_j(\alpha):=\|\mathcal{X}_{V_j}\|_\phi\varepsilon_j^\alpha=\frac{\varepsilon_j^\alpha}{\phi^{-1}\big(\frac{1}{\varepsilon_j}\big)},\quad \text{where } \alpha \text{ is a nonnegative integer.}$$

More precisely, for each pair of nonnegative integers α and β, and any pair j, l, $1\le j,l\le k$, we suppose

$$v_l(\alpha)=\mathcal{O}(v_j(\beta))\quad\text{or}\quad v_j(\beta)=o(v_l(\alpha))\quad\text{as } \delta\to 0. \tag{1.4}$$

Let $\langle i_j\rangle$ be an ordered k-tuple of nonnegative integers. We say that $v_l(i_l)$ is maximal if $v_j(i_j)=O(v_l(i_l))$ for all j, $1\le j\le k$. We denote it by $v_l(i_l)=\max\{v_j(i_j)\}$. A k-tuple $\langle i_j\rangle$ of nonnegative integers is said to be $\|\cdot\|_\phi$-balanced if for each $i_r>0$,

$$\frac{1}{v_l(i_r-1)}\max\{v_j(i_j)\}=o(1).$$

An integer $N\ge 0$ is called $\|\cdot\|_\phi$-balanced if there exists a $\|\cdot\|_\phi$-balanced k-tuple $\langle i_j\rangle$, with $N=\sum_{j=1}^k i_j$. It easy to see that a such k-tuple is unique.

As it was seen in [2], for each $N\ge 0$ there exists the smallest $\|\cdot\|_\phi$-balanced integer greater than or equal to N, and the greatest $\|\cdot\|_\phi$-balanced integer smaller than or equal to N, which we denote by $\overline{N}$ and $\underline{N}$, respectively. We write $\sum_{j=1}^k\bar{i}_j=\overline{N}$ and $\sum_{j=1}^k\underline{i}_j=\underline{N}$, where $\langle\bar{i}_j\rangle$ and $\langle\underline{i}_j\rangle$ are $\|\cdot\|_\phi$-balanced k-tuples. If N is not a $\|\cdot\|_\phi$-balanced integer, then $\underline{N}<N<\overline{N}$ and there are no $\|\cdot\|_\phi$-balanced integers between $\underline{N}$ and $\overline{N}$.

Generally, if $\langle i_j\rangle$ is a $\|\cdot\|_\phi$-balanced k-tuple, let K be the set of indexes j with the property that $v_j(i_j)=\max\{v_l(i_l)\}$. As a consequence of the algorithm established in [2] for computing the $\|\cdot\|_\phi$-balanced integers, we deduce that the smallest $\|\cdot\|_\phi$-balanced integer greater than $\sum_{j=1}^k i_j$ is $\sum_{j=1}^j i'_j$, where $\langle i'_j\rangle$ is a $\|\cdot\|_\phi$-balanced k-tuple, $i'_j=i_j+1$ for $j\in K$ and $i'_j=i_j$ for $j\notin K$. Therefore the cardinal of K is $\sum_{j=1}^k i'_j-\sum_{j=1}^k i_j$.

In this work we study the behavior in L^ϕ of best $\|\cdot\|_\phi$-approximant pairs of f from $\mathcal{S}_m^n(a)$ respect to $\|\cdot\|_{L^\phi(V)}$ when $\delta\to 0$, i.e., we prove the existence and characterization of the best local $\|\cdot\|_\phi$-approximations of type (n,m) at f from $\mathcal{S}_m^n$. This result has been proved in (see [1, Theorem 3]) for L^p spaces, $1\le p\le\infty$, and $m=0$. In Orlicz spaces the existence of best local approximation of type $(n,0)$ can be seen in [2] in the case where $n+1$ is a $\|\cdot\|_\phi$-balanced integer. The case where $n+1$ is not a $\|\cdot\|_\phi$-balanced integer was investigated in [4].

2 Main results

Let $(S,T)\in\Pi^n\times\Pi^m$ be such that $T\neq 0$. We recall that S/T is normal if it is irreducible and either $\deg S=n$ or $\deg T=m$. The null rational function 0 is normal if and only if $\deg T=0$.

Lemma 2.1. *Let $(S,T)\in\Pi^n\times\Pi^m$ be such that $\frac{S}{T}$ is normal. If $T(a)\neq 0$, then*

$$S\Pi^m(a,0)+T\Pi^n=\Pi^{n+m}.$$

Proof. Clearly $S\Pi^m(a,0)+T\Pi^n\subset\Pi^{n+m}$. As $\frac{S}{T}$ is normal, it is well known that $S\Pi^m+T\Pi^n=\Pi^{n+m}$ (see for instance [3]). Let $F\in\Pi^{n+m}$ and let $(P_0,Q_0)\in\Pi^n\times\Pi^m$ be such that $F=SQ_0+TP_0$. By hypothesis, we can define $Q=Q_0-Q_0(a)/T(a)T\in\Pi^m$ and $P=P_0+Q_0(a)/T(a)S\in\Pi^n$. It is easy to see that $F=SQ+TP$ and $Q\in\Pi^m(a,0)$. So, the proof is complete. □

Theorem 2.1. *Let $f\in PC^t(X)\setminus\Pi^n_m(a)$ and let $\langle i_j\rangle$ be an ordered N-tuple of nonnegative integers with $i_j\leq t$ and $\sum_{j=1}^N i_j=n+m+1$. Assume that there is $\langle i_j\rangle$-Padé approximant pair of f in $\{y_1,\cdots,y_N\}$, say $(S,T)\in\Pi^n\times\Pi^m$, such that $\frac{S}{T}$ is normal. If $T(a)\neq 0$, then given an arbitrary set of real numbers $\{b_{i,j}\}$, there exists a unique $R\in\mathcal{R}^n_m(f,a)$ such that $R^{(i)}(y_j)=b_{i,j}$, $j=1,2,\cdots,N$, $i=0,1,\cdots,i_j-1$.*

Proof. Let $1\leq j\leq N$, $1\leq i\leq i_j-1$. As

$$(fT-S)^{(i)}(y_j)=0 \tag{2.1}$$

and S/T is irreducible, then $T(y_j)\neq 0$. Hence $(f-S/T)^{(i)}(y_j)=0$. Let $M_j\in\mathbb{R}^{i_j\times i_j}$ be the lower triangular matrix define by

$$(M_j)_{\alpha\beta}=\begin{pmatrix}\alpha-1\\ \beta-1\end{pmatrix}\Big(\frac{1}{T}\Big)^{(\alpha-\beta)}(y_j)\quad\text{for }\ \alpha\geq\beta, \tag{2.2}$$

and let $M\in\mathbb{R}^{(n+m+1)\times(n+m+1)}$ be the block diagonal matrix given by

$$M=\begin{pmatrix}M_1 & & 0\\ & \ddots & \\ 0 & & M_N\end{pmatrix}. \tag{2.3}$$

Let $\{e_s\}_{s=1}^{n+m+1}$ be the canonical basis of $\mathbb{R}^{1\times(n+m+1)}$. As $\sum_{j=1}^N i_j=n+m+1$, for each $0\leq s\leq n+m+1$ there are unique $1\leq u_s\leq N$ and $0\leq v_s\leq i_j-1$ such that $s=\sum_{j=0}^{u_s-1}i_j+v_s$, where $i_0=0$. Since M is nonsingular there exists a unique $y_{u_s,v_s}=(a^{u_s,v_s}_{0,1},\cdots,a^{u_s,v_s}_{i_1-1,1},\cdots,a^{u_s,v_s}_{0,N},\cdots,a^{u_s,v_s}_{i_N-1,N})^t\in\mathbb{R}^{1\times(n+m+1)}$ such that

$$My_{u_s,v_s}=e_s. \tag{2.4}$$

Set $H_{u_s,v_s}\in\Pi^{n+m}$ satisfying $H^{(i)}_{u_s,v_s}(y_j)=a^{u_s,v_s}_{i,j}$, $j=1,2,\cdots,N$, $i=0,1,\cdots,i_j-1$. From Lemma 2.1, there is $(P_{u_s,v_s},Q_{u_s,v_s})\in\Pi^n\times\Pi^m(a,0)$ such that $SQ_{u_s,v_s}-TP_{u_s,v_s}=H_{u_s,v_s}$. According to (2.2)-(2.4), we have

$$\begin{aligned}(fQ_{u_s,v_s}-P_{u_s,v_s})^{(i)}(y_j)&=\Big(\frac{S}{T}Q_{u_s,v_s}-P_{u_s,v_s}\Big)^{(i)}(y_j)\\&=\sum_{r=0}^{i}\binom{i}{r}\Big(\frac{1}{T}\Big)^{(i-r)}(y_j)(SQ_{u_s,v_s}-TP_{u_s,v_s})^{(r)}(y_j)\\&=\sum_{r=0}^{i}\binom{i}{r}\Big(\frac{1}{T}\Big)^{(i-r)}(y_j)a^{u_s,v_s}_{r,j}=(My_{u_s,v_s})_{\sum_{d=0}^{j-1}i_d+i}=\delta_{(u_s,v_s)(j,i)},\end{aligned}$$

where δ is the Kronecker's delta function. Now, taking

$$P=\sum_{u=1}^{N}\sum_{v=0}^{i_u-1}b_{u,v}P_{u,v}$$

and

$$Q=\sum_{u=1}^{N}\sum_{v=0}^{i_u-1}b_{u,v}Q_{u,v},$$

we obtain $R=fQ-P\in\mathcal{R}^n_m(f,a)$ satisfying $R^{(i)}(y_j)=b_{i,j}$. Finally, suppose that there exist $R_l=fQ_l-P_l\in\mathcal{R}^n_m(f,a)$, $l=1,2$, such that $R^{(i)}_l(y_j)=b_{i,j}$, $j=1,2,\cdots,N$, $i=0,1,\cdots,i_j-1$. Thus $(f(Q_1-Q_2)-(P_1-P_2))^{(i)}(y_j)=0$. Since $f\in PC^t(X)\setminus\Pi^n_m(a)$, then

$$|||(P,Q)|||:=\max\big\{\big|(fQ-P)^{(i)}(y_j)\big|:1\le j\le N,\ 0\le i\le i_j-1\big\} \tag{2.5}$$

is a norm on $\Pi^n\times\Pi^m(a,0)$. Therefore, $Q_1=Q_2$ and $P_1=P_2$. This finishes the proof. □

Next, we present the first important result for the case where $n+m+1$ is a $\|\cdot\|_\phi$- balanced integer.

Theorem 2.2. *Let $a\in X$, $f\in PC^t(X)$ and let $\langle i_j\rangle$ be an ordered k-tuple $\|\cdot\|_\phi$-balanced with $i_j\le t$ and $\sum_{j=1}^k i_j=n+m+1$. Suppose that $(S,T)\in\Pi^n\times\Pi^m$ is a $\langle i_j\rangle$-Padé approximant pair of f in $\{x_1,\cdots,x_k\}$ with S/T normal. If $T(a)\neq0$, then the best local $\|\cdot\|_\phi$- approximation of type (n,m) at f from $\mathcal{S}^n_m(a)$ on $\{x_1,\cdots,x_k\}$ is $(S/T(a),T/T(a))$.*

Proof. Clearly $f\in PC^t(X)\setminus\Pi^n_m(a)$. Let $\{(P_\delta,Q_\delta)\}_{\delta>0}$ be a net of best $\|\cdot\|_\phi$-approximant pairs of f from $\mathcal{S}^n_m(a)$ respect to $\|\cdot\|_{L^\phi(V)}$. As seen in Introduction $f(Q_*-Q_\delta)+P_\delta\in\mathcal{R}^n_m(f,a)$ minimizes (1.2) from $\mathcal{R}^n_m(f,a)$. So, Theorem 2.1 and Theorem 4.3 in [2] imply that the net $\{f(Q_*-Q_\delta)+P_\delta\}_{\delta>0}$ converges to $fQ+P$ in $\mathcal{R}^n_m(f,a)$, as $\delta\to0$, defined by the $n+m+1$ interpolation conditions

$$(f(Q_*-Q)-P)^{(i)}(x_j)=0,\quad j=1,2,\cdots,k,\quad i=0,1,\cdots,i_j-1. \tag{2.6}$$

Since

$$\Big(P-\frac{S}{T(a)},Q_*-Q-\frac{T}{T(a)}\Big)\in\Pi^n\times\Pi^m(a,0)$$

and

$$\Big(f\frac{T}{T(a)}-\frac{S}{T(a)}\Big)^{(i)}(x_j)=0,\quad j=1,2,\cdots,k,\quad i=0,1,\cdots,i_j-1,$$

from (2.5) and (2.6) we have

$$\Big|\Big|\Big|\Big(P-\frac{S}{T(a)},Q_*-Q-\frac{T}{T(a)}\Big)\Big|\Big|\Big|=0.$$

We conclude that $P=S/T(a)$ and $Q_*-Q=T/T(a)$ and finally that the net $\{(P_\delta,Q_\delta)\}_{\delta>0}$ converges to $(\frac{S}{T(a)},\frac{T}{T(a)})$ as $\delta\to 0$. □

Corollary 2.1. Assume the same hypotheses of Theorem 2.2. Then S/T is the $\langle i_j\rangle$-Padé approximant of f in $\{x_1,\cdots,x_k\}$. In addition, if $\{(P_\delta,Q_\delta)\}_{\delta>0}$ is a net of best $\|\cdot\|_\phi$-approximant pairs of f from $\mathcal{S}^n_m(a)$ respect to $\|\cdot\|_{L^\phi(V)}$, then P_δ/Q_δ converge to S/T, uniformly on some neighborhood of $\{x_1,\cdots,x_k,a\}$ as $\delta\to 0$.

Next, we give a results about best local $\|\cdot\|_\phi$-approximation of type (n,m) when $n+m+1$ is not a balanced integers and ϕ satisfies a certain asymptotic condition. Henceforth, we assume that there exists

$$\lim_{\alpha\to\infty}\frac{\phi(\alpha x)}{\phi(\alpha)}$$

for all $x\geq 0$, and therefore this limit is x^p for some $p\geq 1$.

Theorem 2.3. *Let $N=n+m+1$ be a non $\|\cdot\|_\phi$-balanced integer with*

$$\sum_{j=1}^{k}\underline{i}_j+d=N,\quad 0<d<\overline{N}-\underline{N}.$$

For each $j\in K$ suppose

$$\lim_{\delta\to 0}\frac{v_j(\underline{i}_j)}{E}=e_j>0,$$

where $E=\max\{v_j(i_j)\}$. Let $a\in X\setminus\{x_j\}_{j=1}^k$ and let $f\in PC^t(X)$ be such that $\underline{i}_j\leq t$. Assume that there is $(S,T)\in\Pi^n\times\Pi^m$ a $(\underline{i}_1,\cdots,\underline{i}_k,d)$-Padé approximant pair of f in $\{x_1,\cdots,x_k,a\}$ such that S/T is normal. Then, for $\delta\to 0$, the limit of any convergent subsequence of $\{(P_\delta,Q_\delta)\}_{\delta>0}$, a net of best $\|\cdot\|_\phi$-approximant pairs of f from $\mathcal{S}^n_m(a)$, is a solution of the following minimization problem in $\mathbb{R}^{\overline{N}-\underline{N}}$:

$$\left\{\begin{array}{l}\displaystyle\min_{(P,Q)\in\mathcal{S}^n_m(a)}\Big\|\Big\langle e_j\frac{J_{A_j}(\underline{i}_j,p)}{\underline{i}_j!}(fQ-P)^{(\underline{i}_j)}(x_j)\Big\rangle_{j\in K}\Big\|_{l_p},\\ \text{with the constraints } (fQ-P)^{(i)}(x_j)=0,\ j=1,2,\cdots,k \text{ and } i=0,1,\cdots,\underline{i}_j-1,\end{array}\right.\tag{2.7}$$

where, for $j \in K$, $J_{A_j}(\underline{i}_j,p)$ is the minimum L_p norm over A_j of an $\underline{i}_j^{\text{th}}$ degree polynomial with unit leading coefficient. In particular, if (2.7) *has a unique solution (P,Q), then the net $\{(P_\delta,Q_\delta)\}_{\delta>0}$ converges to (P,Q) and therefore this is a best local $\|\cdot\|_\phi$-approximation of type (n,m) at f from $\mathcal{S}_m^n(a)$ on $\{x_1,\cdots,x_n\}$.*

Proof. As in the proof of Theorem 2.2, we have $f \in PC^t(X)\setminus\Pi_m^n(a)$ and $f(Q_*-Q_\delta)+P_\delta \in \mathcal{R}_m^n(f,a)$ minimizes (1.2) from $\mathcal{R}_m^n(f,a)$, By hypothesis, there exists a $(\underline{i}_1,\cdots,\underline{i}_k,d)$-Padé approximant pair of f in $\{x_1,\cdots,x_k,a\}$, (S,T), such that S/T is normal, so $T(a)\neq 0$. From Theorem 2.1 and Theorem 3.1 in [4], we obtain that the limit of any convergent subsequence of $\{f(Q_*-Q_\delta)+P_\delta\}_{\delta>0}$, is a solution of the following minimization problem in $\mathbb{R}^{\overline{N}-\underline{N}}$:

$$\begin{cases} \min\limits_{R\in\mathcal{R}_m^n(f,a)} \left\| \left\langle e_j \frac{J_{A_j}(\underline{i}_j,p)}{\underline{i}_j!}(fQ_*-R)^{(\underline{i}_j)}(x_j)\right\rangle_{j\in K}\right\|_{l_p}, \\ \text{with the constraints } (fQ_*-R)^{(i)}(x_j)=0,\ j=1,2,\cdots,k \text{ and } i=0,1,\cdots,\underline{i}_j-1. \end{cases}$$

Hence, the limit of any convergent subsequence of $\{(P_\delta,Q_\delta)\}_{\delta>0}$ satisfies (2.7). In particular, if (2.7) has a unique solution, say (P,Q), then $\{(P_\delta,Q_\delta)\}_{\delta>0}$ converges to (P,Q), as $\delta\to 0$. ☐

Corollary 2.2. Assume the same hypotheses of Theorem 2.3 and suppose that the problem (2.7) has a unique solution, say (P,Q). If $\{(P_\delta,Q_\delta)\}_{\delta>0}$ is a net of best $\|\cdot\|_\phi$- approximant pairs of f from $\mathcal{S}_m^n(a)$ respect to $\|\cdot\|_{L^\phi(V)}$, then $\frac{P_\delta}{Q_\delta}$ converges to P/Q, uniformly on some neighborhood of a, as $\delta\to 0$.

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References

[1] C. Chui, H. Diamond and R. Raphael, On best data approximation, Approx. Theory Appl., 1 (1984), 37–56.

[2] H. Cuenya, S. Favier, F. Levis and C. Ridolfi, Weighted best local $\|\cdot\|$-approximation in Orlicz spaces, Jaen J. Approx., 2(1) (2010), 113–127.

[3] E. W. Cheney and A. A. Glodstein. Mean square approximation by generalized rational functions, Math. Zeitschr., 95 (1967), 232–241.

[4] H. Cuenya, F. Levis, M. Marano and C. Ridolfi, Best local approximation in Orlicz spaces, Numer. Funct. Anal. Optim., 32(11) (2011), 1127–1145.

[5] M. A. Krasnosel'skii and Ya. B. Rutickii, Convex Function and Orlicz Spaces, Noordhoff, Groningen, 1961.