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Readjustment of the Paper [J. Kaur and S. S. Bhatia, Integrability and L^1-Convergence of Double Cosine Trigonometric Series, Anal. Theory Appl., 27(1) (2011), pp. 32–39.]

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Abstract. In this paper, we show that new modified double cosine trigonometric sums introduced in [1] are inappropriate, the class of double sequences J_d introduced there is unusable for such sums and consequently the results obtained in it are completely incorrect. We here introduce appropriate modified double cosine trigonometric sums making the class J_d usable considering a particular double cosine trigonometric series.

Key Words: L^1-convergence, double null sequence, cosine trigonometric series, modified sums.

AMS Subject Classifications: 42A20, 42B05

1 Introduction and auxiliary statements

For a function $f(x,y)$ with two independent variables x and y we write $f\in L^1(T^2)$ if

$$\|f\|=\iint_{T^2}|f(x,y)|dxdy<+\infty,$$

where $T^2:=[0,\pi]\times[0,\pi]$.

Let

$$f(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k a_{j,k}\cos jx\cos ky \tag{1.1}$$

be a double cosine series on the positive quadrant $T^2:=[0,\pi]\times[0,\pi]$ of the two dimensional torus, where $\lambda_0=1/2$ and $\lambda_i=1$ for $i=1,2,\cdots$, and $\{a_{j,k}\}$ is a double sequence of real numbers.

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Let us denote by

$$S_{m,n}(x,y):=\sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k a_{j,k}\cos jx\cos ky, \quad m,n\geq 0,$$

the partial sums of the series (1.1) and

$$f(x,y)=\lim_{m+n\to\infty}S_{m,n}(x,y).$$

In 2011, J. Kaur and S. S. Bhatia [1] introduced some new modified double cosine trigonometric sums as follows

$$g_{m,n}(x,y)=\frac{a_{0,0}}{2}+\sum_{j=1}^{m}\sum_{k=1}^{n}\Big(\sum_{i=j}^{m}\sum_{\ell=k}^{n}\Delta_{11}\left(a_{i,\ell}\cos ix\cos\ell y\right)\Big).$$

Also, they defined the following class of numerical sequences.

Definition 1.1. A double null sequence $\{a_{j,k}\}$ of positive numbers is said to belong to the class J_d if there exists a double sequence $\{A_{j,k}\}$ such that

$$A_{j,k}\downarrow 0, \quad j+k\to\infty, \tag{1.2a}$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}jkA_{j,k}<\infty, \tag{1.2b}$$

and

$$\left|\Delta_{p,q}\Big(\frac{a_{j,k}}{jk}\Big)\right|\leq\frac{A_{j,k}}{jk}, \quad 1\leq p+q\leq 2, \tag{1.3}$$

for any nonnegative integers p,q and $j,k\in\{1,2,3,\cdots\}$.

Moreover, they have presented the following results:

Theorem 1.1 (see [1]). *If a double sequence $\{a_{j,k}\}$ belongs to the class J_d, then $\|g_{m,n}-f\|\to 0$ as $j+k\to\infty$.*

Corollary 1.1 (see [1]). Under condition of Theorem 1.1, the sum-function f of the series (1.1) is an integrable function and (1.1) is the Fourier series of f.

Corollary 1.2 (see [1]). If a double sequence $\{a_{j,k}\}$ belongs to the class J_d, then $\|S_{m,n}-f\|\to 0$ as $j+k\to\infty$.

Unfortunately, the sums $g_{m,n}(x)$ are not appropriate so that Theorem 1.1, Corollary 1.1 and Corollary 1.2 will be true. Indeed, after some elementary calculations the authors have written (see relation (3.1) on page 35, in [1])

$$\begin{aligned} g_{m,n}(x,y) =& \frac{a_{0,0}}{2} + \sum_{j=1}^{m}\sum_{k=1}^{n}\Big(\sum_{i=j}^{m}\sum_{\ell=k}^{n}\Delta_{11}(a_{i,\ell}\cos ix\cos \ell y)\Big) \\ =& S_{m,n}(x,y) - \sum_{j=1}^{m}\sum_{k=1}^{n}[a_{j,n+1}\cos jx\cos(n+1)y + a_{m+1,k}\cos(m+1)x\cos y] \\ &+ mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y, \end{aligned}$$

which is not true.

Firstly, the term $a_{m+1,k}\cos(m+1)x\cos y$ inside brackets must be $a_{m+1,k}\cos(m+1)x\cos ky$, and secondly it seems that authors predicted that

$$S_{m,n}(x,y) = \frac{a_{0,0}}{2} + \sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\cos jx\cos ky$$

is true, but of course it is not since the correct expression for $S_{m,n}(x,y)$ is

$$S_{m,n}(x,y) = \frac{a_{0,0}}{4} + \frac{1}{2}\sum_{j=1}^{m}a_{j,0}\cos jx + \frac{1}{2}\sum_{k=1}^{n}a_{0,k}\cos kx + \sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\cos jx\cos ky.$$

After some detailed investigations we found out that if we want to make the class J_d usable we have to consider double cosine series of the form

$$f_1(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{j,k}\cos jx\cos ky, \tag{1.4}$$

with its partial sums

$$S^{\cos}_{m,n}(x,y) := \sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\cos jx\cos ky, \quad m,n\geq 1,$$

and

$$f_1(x,y) = \lim_{m+n\to\infty} S^{\cos}_{m,n}(x,y).$$

Then, we introduce the appropriate double modified cosine sums

$$X_{m,n}(x,y) = \sum_{j=1}^{m}\sum_{k=1}^{n}\Big(\sum_{i=j}^{m}\sum_{\ell=k}^{n}\Delta_{1,1}(a_{i,\ell}\cos ix\cos \ell y)\Big).$$

Based on what we have said above, it is clear that the object of this paper is to obtain the correct versions of Theorem 1.1, Corollary 1.1 and Corollary 1.2, considering new introduced sums $X_{m,n}(x,y)$. To do this, we need the following auxiliary statements.

Lemma 1.1 (see [2])**.** *Let* $n\geq 1$, r_1 *be a nonnegative integer and* $x\in[\varepsilon,\pi]$. *Then*

$$|\widetilde{D}_n^{(r_1)}(x)|\leq \frac{C_\varepsilon n^{r_1}}{x},$$

where C_ε *is a positive constant depending only on* ε, $0<\varepsilon<\pi$ *and* $\widetilde{D}_n^{(r_1)}(x)$ *is the* r_1*-th derivative of the conjugate Dirichlet kernel.*

Lemma 1.2 (see [2])**.** *The following estimation*

$$\|\widetilde{D}_n^{(r_1)}\|=\mathcal{O}(n^{r_1}\log n),\quad n\to\infty,\quad r_1\in\{0,1,2,\cdots\},$$

holds true.

2 Main results

Theorem 2.1. *If a double sequence* $\{a_{j,k}\}$ *belongs to the class* J_d, *then* $\|f_1-X_{m,n}\|\to 0$ *as* $j+k\to\infty$.

Proof. Firstly, after some simple calculations we have

$$\begin{aligned}
X_{m,n}(x,y)=&\sum_{j=1}^{m}\sum_{k=1}^{n}\Big(\sum_{i=j}^{m}\sum_{\ell=k}^{n}\Delta_{1,1}(a_{i,\ell}\cos ix\cos\ell y)\Big)\\
=&\sum_{j=1}^{m}\sum_{k=1}^{n}\Big\{\sum_{i=j}^{m}\Big[\sum_{\ell=k}^{n}\Delta_{0,1}(\Delta_{1,0}(a_{i,\ell}\cos ix\cos\ell y))\Big]\Big\}\\
=&\sum_{j=1}^{m}\sum_{k=1}^{n}\Big\{\sum_{i=j}^{m}(\Delta_{1,0}(a_{i,k}\cos ix\cos ky)-\Delta_{1,0}(a_{i,n+1}\cos ix\cos(n+1)y))\Big\}\\
=&\sum_{j=1}^{m}\sum_{k=1}^{n}\{a_{j,k}\cos jx\cos ky-a_{m+1,k}\cos(m+1)x\cos ky\\
&-a_{j,n+1}\cos jx\cos(n+1)y+a_{m+1,n+1}\cos(m+1)x\cos(n+1)y\}\\
=&S^{\cos}_{m,n}(x,y)-\sum_{j=1}^{m}\sum_{k=1}^{n}\{a_{m+1,k}\cos(m+1)x\cos ky+a_{j,n+1}\cos jx\cos(n+1)y\}\\
&+\sum_{j=1}^{m}\sum_{k=1}^{n}a_{m+1,n+1}\cos(m+1)x\cos(n+1)y\\
=&\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}\cos jx\cos ky-m\cos(m+1)x\sum_{k=1}^{n}a_{m+1,k}\cos ky\\
&-n\cos(n+1)y\sum_{j=1}^{m}a_{j,n+1}\cos jx+mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y. \qquad (2.1)
\end{aligned}$$

Moreover, the equality (2.1) can be rewritten as follows

$$X_{m,n}(x,y)=\sum_{j=1}^{m}\sum_{k=1}^{n}\Big(\frac{a_{j,k}}{jk}\Big)(\sin jx)'(\sin ky)'-m\cos(m+1)x\sum_{k=1}^{n}a_{m+1,k}\cos ky$$
$$-n\cos(n+1)y\sum_{j=1}^{m}a_{j,n+1}\cos jx+mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y. \tag{2.2}$$

Applying double summation by parts to (2.2) we obtain

$$\begin{aligned}X_{m,n}(x,y)=&\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\Delta_{11}\Big(\frac{a_{j,k}}{jk}\Big)\widetilde{D}'_j(x)\widetilde{D}'_k(y)+\sum_{j=1}^{m-1}\Delta_{10}\Big(\frac{a_{j,n}}{jn}\Big)\widetilde{D}'_j(x)\widetilde{D}'_n(y)\\&+\sum_{k=1}^{n-1}\Delta_{01}\Big(\frac{a_{m,k}}{mk}\Big)\widetilde{D}'_m(x)\widetilde{D}'_k(y)+\frac{a_{m,n}}{mn}\widetilde{D}'_m(x)\widetilde{D}'_n(y)\\&-m\cos(m+1)x\sum_{k=1}^{n}a_{m+1,k}\cos ky-n\cos(n+1)y\sum_{j=1}^{m}a_{j,n+1}\cos jx\\&+mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y=\sum_{s=1}^{7}R_s(x,y).\end{aligned}$$

Based on Lemma 1.1 we have $|\widetilde{D}'_m(u)|\le C_\varepsilon m/u$, $0<u\le\pi$, therefore using (1.3) and (1.2b) we clearly have

$$\begin{aligned}|R_1(x,y)|\le&\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\Big|\Delta_{11}\Big(\frac{a_{j,k}}{jk}\Big)\Big||\widetilde{D}'_j(x)||\widetilde{D}'_k(y)|\\\le&\frac{C_\varepsilon}{xy}\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\Big(\frac{A_{j,k}}{jk}\Big)jk\le\frac{C_\varepsilon}{xy}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}A_{j,k}<+\infty,\end{aligned}$$

for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Also, we have

$$\begin{aligned}|R_2(x,y)|\le&\sum_{j=1}^{m-1}\Big|\Delta_{10}\Big(\frac{a_{j,n}}{jn}\Big)\Big||\widetilde{D}'_j(x)||\widetilde{D}'_n(y)|\le\frac{C_\varepsilon}{xy}\sum_{j=1}^{m-1}\Big|\sum_{k=n}^{\infty}\Delta_{11}\Big(\frac{a_{j,k}}{jk}\Big)\Big|jn\\\le&\frac{C_\varepsilon}{xy}\sum_{j=1}^{m-1}\sum_{k=n}^{\infty}\Big|\Delta_{11}\Big(\frac{a_{j,k}}{jk}\Big)\Big|jn\le\frac{C_\varepsilon}{xy}\sum_{j=1}^{m-1}\sum_{k=n}^{\infty}\frac{A_{j,k}}{jk}jn\\\le&\frac{C_\varepsilon}{xy}\sum_{j=1}^{m-1}\sum_{k=n}^{\infty}\frac{A_{j,k}}{n}n=\frac{C_\varepsilon}{xy}\sum_{j=1}^{m-1}\sum_{k=n}^{\infty}A_{j,k}\to0\quad\text{as}\quad n\to\infty,\end{aligned}$$

uniformly in m and for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Similarly,

$$|R_3(x,y)| \le \frac{C_\varepsilon}{xy}\sum_{j=m}^{\infty}\sum_{k=1}^{n-1} A_{j,k} \to 0 \quad \text{as} \quad m\to\infty,$$

uniformly in n and for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Then based on Lemma 1.1 and on the fact that $\{a_{jk}\}$ is a double null sequence we have

$$|R_4(x,y)| = \frac{a_{m,n}}{mn}|\widetilde{D}'_m(x)||\widetilde{D}'_n(y)| \le \frac{C_\varepsilon}{xy}\frac{a_{m,n}}{mn}mn = \frac{C_\varepsilon}{xy}a_{m,n} \to 0 \quad \text{as} \;\; m+n\to\infty$$

for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Next, since $\{a_{jk}\}\in J_d$ we have

$$\begin{aligned}
|R_5(x,y)| \le & m\sum_{k=1}^{n}|a_{m+1,k}| = m(m+1)\sum_{k=1}^{n}\Big|\frac{a_{m+1,k}}{(m+1)k}\Big|k \\
= & m(m+1)\sum_{k=1}^{n}\Big|\sum_{j=m+1}^{\infty}\Delta_{10}\Big(\frac{a_{j,k}}{jk}\Big)\Big|k \\
\le & m(m+1)\sum_{k=1}^{n}\sum_{j=m+1}^{\infty}\Big|\Delta_{10}\Big(\frac{a_{j,k}}{jk}\Big)\Big|k \\
\le & m(m+1)\sum_{j=m+1}^{\infty}\sum_{k=1}^{n}\frac{A_{j,k}}{jk}k \\
\le & m(m+1)\sum_{j=m+1}^{\infty}\sum_{k=1}^{n}\frac{A_{j,k}}{m+1} \\
\le & \frac{1}{2}\sum_{j=m+1}^{\infty}\sum_{k=1}^{n} jA_{j,k} \to 0 \quad \text{as} \quad m\to\infty,
\end{aligned} \tag{2.3}$$

uniformly in n and for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Similarly, we have verified that

$$|R_6(x,y)| \le \frac{1}{2}\sum_{j=1}^{m}\sum_{k=n+1}^{\infty} kA_{j,k} \to 0 \quad \text{as} \;\; n\to\infty, \tag{2.4}$$

uniformly in m and for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Doing almost the same reasoning we have proved that

$$|R_7(x,y)| \le \frac{1}{4}\sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty} jkA_{j,k} \to 0 \quad \text{as} \;\; m+n\to\infty \tag{2.5}$$

for all x and y such that $0<x\le\pi$, $0<y\le\pi$.

Subsequently,

$$\lim_{m+n\to\infty} X_{m,n}(x,y) = \lim_{m+n\to\infty} S^{\cos}_{m,n}(x,y) = f_1(x,y)$$

exists and $f_1(x,y)\in L^1(T^2)$.

Applying Lemma 1.2, doing the same reasoning as in [1] and using some estimates which we already have obtained (during the proof) we get

$$\|X_{m,n}-f_1\|\to 0 \quad \text{as } \ j+k\to\infty.$$

The proof of the theorem is completed. □

Corollary 2.1. Under condition of Theorem 2.1, the sum-function f_1 of the series (1.4) is an integrable function and (1.4) is the Fourier series of f_1.

Proof. We already have proved in the proof of Theorem 2.1 that $f_1(x,y)\in L^1(T^2)$. Also, it is a well-known fact that L^1-convergence implies that in weak convergence. Moreover, for $r,\ell\geq 1$ fixed and

$$\begin{aligned} X_{m,n}(x,y) =& \sum_{j=1}^{m}\sum_{k=1}^{n} a_{j,k}\cos jx\cos ky - m\cos(m+1)x\sum_{k=1}^{n} a_{m+1,k}\cos ky \\ & - n\cos(n+1)y\sum_{j=1}^{m} a_{j,n+1}\cos jx + mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f_1(x,y)\cos rx\cos\ell y dxdy \\ =& \lim_{m+n\to\infty}\frac{4}{\pi^2}\int_0^\pi\int_0^\pi X_{m,n}(x,y)\cos rx\cos\ell y dxdy \\ =& \lim_{m+n\to\infty}\Big\{a_{r,\ell} - m\sum_{k=1}^{n} a_{m+1,k} - n\sum_{j=1}^{m} a_{j,n+1} + mna_{m+1,n+1}\Big\} = a_{r,\ell}, \end{aligned}$$

(because of (2.3), (2.4) and (2.5)) which means that (1.4) is the Fourier series of f_1. □

Corollary 2.2. If a double sequence $\{a_{j,k}\}$ belongs to the class J_d, then $\|S^{\cos}_{m,n}-f_1\|\to 0$ as $j+k\to\infty$.

Proof. According to (2.1) it is clear that

$$\begin{aligned} \|S^{\cos}_{m,n}-f_1\| =& \|S^{\cos}_{m,n}-X_{m,n}+X_{m,n}-f_1\| \leq \|f_1-X_{m,n}\| + \|X_{m,n}-S^{\cos}_{m,n}\| \\ \leq& \|f_1-X_{m,n}\| + \int_0^\pi\int_0^\pi \Big| m\cos(m+1)x\sum_{k=1}^{n} a_{m+1,k}\cos ky\Big| dxdy \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\pi\int_0^\pi\Big|n\cos(n+1)y\sum_{j=1}^m a_{j,n+1}\cos jx\Big|dxdy\\
&+\int_0^\pi\int_0^\pi|mna_{m+1,n+1}\cos(m+1)x\cos(n+1)y|dxdy\\
\le&\|f_1-X_{m,n}\|+\int_0^\pi\int_0^\pi\Big|m\sum_{k=1}^n a_{m+1,k}\Big|dxdy\\
&+\int_0^\pi\int_0^\pi\Big|n\sum_{j=1}^m a_{j,n+1}\Big|dxdy+\int_0^\pi\int_0^\pi|mna_{m+1,n+1}|dxdy.
\end{aligned}
$$

Note that the first term tends to zero based on Theorem 2.1 as well as the second, third and fourth terms according to (2.3), (2.4) and (2.5) respectively. The proof is finished. □

References

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[2] Sh. Y. Sheng, The extensions of the theorems of Č. V. Stanojević and V. B. Stanojević, Proc. Amer. Math. Soc., 110(4) (1990), 895–904.