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On Fatou Type Convergence of Convolution Type Double Singular Integral Operators

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Dedicated to my academic father Professor Paul L. Butzer

Abstract. In this paper, some approximation formulae for a class of convolution type double singular integral operators depending on three parameters of the type

$$(T_\lambda f)(x,y)=\int_a^b\int_a^b f(t,s)K_\lambda(t-x,s-y)dsdt,\quad x,y\in(a,b),\quad \lambda\in\Lambda\subset[0,\infty),\qquad (0.1)$$

are given. Here f belongs to the function space $L_1(\langle a,b\rangle^2)$, where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$. In this paper three theorems are proved, one for existence of the operator $(T_\lambda f)(x,y)$ and the others for its Fatou-type pointwise convergence to $f(x_0,y_0)$, as (x,y,λ) tends to (x_0,y_0,λ_0). In contrast to previous works, the kernel functions $K_\lambda(u,v)$ don't have to be 2π-periodic, positive, even and radial. Our results improve and extend some of the previous results of [1,6,8,10,11,13] in three dimensional frame and especially the very recent paper [15].

Key Words: Fatou-type convergence, convolution type double singular integral operators, μ-generalized Lebesgue point.

AMS Subject Classifications: 41A35, 44A35, 42A85

1 Introduction

It is well-known that convolution type singular integral operators of various types create an important subject of mathematical investigations and are often applicable in physics. In particular, linear singular integrals of the convolution type occur in approximation theory, as a solutions of partial differential equations and in the theory of Fourier series. Let us mention, for example, the Fejér, Abel-Poisson, Poisson and Gauss-Weierstrass singular integrals (see [5]). All of these convolution type linear singular integrals which we mentioned above are depend on two parameters.

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It was Taberski [6] who took the first step to obtain some approximation results for linear singular integral operators depending on two parameters, which contain the single parameter sequence of convolution type singular integral operators widely developed in [5].

Taberski has shown that the pointwise convergence at which the points are Lebesgue points of integrable functions in $L_1(\langle -\pi,\pi\rangle)$, by the family of convolution type singular integral operators of the form

$$U(f;x,\lambda)=\int_{-\pi}^{\pi} f(t)K(t-x,\lambda)dt, x\in(-\pi,\pi), \tag{1.1}$$

where $K(t,\lambda)$ is a kernel which satisfies suitable assumptions. His results are valid on some subsets of the plane, i.e., Fatou type convergence was discussed. Since that time a new theory has been developed in order to give a unitary approach to the study of convergence and of the order of approximation for a general family of integral operators. In various papers, based mainly on Taberski's idea, some extensions and generalizations of the pointwise convergence theorems for the operators (1.1) and for some kind of its nonlinear counterpart have been given (see for example [1–4, 8–10] and [13]).

In papers [11–14] and [15] pointwise convergence of integrable functions in $L_1([-a,a]\times[-b,b])$ and $L_1(\langle -\pi,\pi\rangle^2)$ by a three parameter family of convolution type singular integral operators of the form

$$U(f;x,\lambda)=\int_{-a}^{a}\int_{-b}^{b} f(t,s)K_\lambda(t-x,s-y)dsdt, \quad (x,y)\in([-a,a],[-b,b]), \tag{1.2}$$

where $\lambda\in\Lambda\subset[0,\infty)$, have been investigated.

In the present paper, we extend and generalized the very recent results of Yilmaz [15], in which the author considers the operators (1.2) with $f\in L_1(\langle -\pi,\pi\rangle^2)$ and with a special kind of kernel, namely with a radial kernel, and obtained the pointwise convergence of (1.2) to $f(x_0,y_0)$ at a continuity point (x_0,y_0) of f.

In particular, we obtain precise approximation formulae for the Fatou-type pointwise convergence of $(T_\lambda f)(x,y)$ to $f(x_0,y_0)$ in $L_1(\langle a,b\rangle^2)$, by a family of convolution type singular integral operators of the form

$$(T_\lambda f)(x,y)=\int_a^b\int_a^b f(t,s)K_\lambda(t-x,s-y)dsdt, \quad x,y\in(a,b), \quad \lambda\in\Lambda\subset[0,\infty).$$

We note that, in this paper three theorems are proved, one for existence of the operator $(T_\lambda f)(x,y)$ and the others for its Fatou-type pointwise convergence to $f(x_0,y_0)$, as (x,y,λ) tends to (x_0,y_0,λ_0). In contrast to previous works, the kernel functions $K_\lambda(u,v)$ don't have to be 2π-periodic, positive, even and radial. Our results improve and extend some of the previous results of [1, 6, 8, 10, 11, 13] in three dimensional frame and especially the very recent paper [15]. Moreover, taking $\lambda\in\mathbb{N}$, $x=x_0$, and $K\geq 0$ this two-parameters family is reducedto the single parameter sequence of convolution type singular integral operators widely developed in [5].

Firstly, we shall give conditions which provide the existence of the operators.

We take a family of kernel $\mathbb{K}=(K_\lambda)_{\lambda\in\Lambda}$ of functions $K_\lambda:\mathbb{R}^2\to\mathbb{R}$ satisfying the following conditions;

(a) As function of t and s, K_λ, (t,s) is defined on $(-\infty,\infty)$ and Lebesgue integrable for each fixed $\lambda\in\Lambda$ (Λ is a given set of numbers with an accumulation point λ_0).

(b) $\lim_{\lambda\to\lambda_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_\lambda(t,s)dsdt=1$.

(c) $\lim_{\lambda\to\lambda_0}\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(t,s)|dsdt=0$,for every $\delta>0$.

(d) $\lim_{\lambda\to\lambda_0}\big[\sup_{|t|\geq\delta}|K_\lambda(t,0)|\big]=\lim_{\lambda\to\lambda_0}\big[\sup_{|s|\geq\delta}|K_\lambda(0,s)|\big]=0$, for every $\delta>0$.

(e) There exist $\delta_0,\delta_1>0$ such that $|K_\lambda(t,\cdot)|$ is non-decreasing on $(-\delta_0,0]$ and non-increasing on $[0,\delta_0)$ as a function of t and $|K_\lambda(\cdot,x)|$ is non-decreasing on $(-\delta_1,0]$ and non-increasing on $[0,\delta_1)$ as a function of s, for each $\lambda\in\Lambda$.

2 Existence of the operator

We denote $\widetilde{f}\in L_1(\mathbb{R}^2)$ as

$$\widetilde{f}(t,s):=\begin{cases} f(t,s), & (t,s)\in\langle a,b\rangle^2,\\ 0, & (t,s)\notin\langle a,b\rangle.^2\end{cases} \tag{2.1}$$

The proofs of the theorems are based on the following existence theorem.

Theorem 2.1. *Let $1\leq p<\infty$. We assume that the kernel function $K_\lambda(t,s)$ of the operators* (0.1) *satisfies (a). If $f\in L_p(\langle a,b\rangle^2)$, then $T_\lambda f\in L_p(\langle a,b\rangle^2)$ and*

$$\begin{aligned}&\|T_\lambda f\|_{L_p(\langle a,b\rangle^2)}\leq\|K_\lambda\|_{L_1(\mathbb{R}^2)}\|f\|_{L_p(\langle a,b\rangle^2)}, \quad 1<p<\infty,\\ &\|T_\lambda f\|_{L_1(\langle a,b\rangle^2)}\leq\|K_\lambda\|_{L_1(\mathbb{R}^2)}\|f\|_{L_1(\langle a,b\rangle^2)}, \quad p=1,\end{aligned}$$

for every $\lambda\in\Lambda$. This implies that $(T_\lambda f)(x,y)$ defines a continuous transformation over $L_p(\langle a,b\rangle^2)$, where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$.

Proof. Let $p=1$. From the assumptions on K_λ and applying the Hölder-Minkowsky Inequality, we have

$$\begin{aligned}|T_\lambda f|_{L_1(\langle a,b\rangle^2)}&=\int_a^b\int_a^b\Big|\int_a^b\int_a^b f(t,s)K_\lambda(t-x,s-y)dsdt\Big|dxdy\\ &\leq\int_a^b\int_a^b|f(t,s)|\Big(\int_a^b\int_a^b|K_\lambda(t-x,s-y)|dxdy\Big)dsdt\\ &\leq\|K_\lambda\|_{L_1(\mathbb{R}^2)}\|f\|_{L_1(\langle a,b\rangle^2)}.\end{aligned}$$

So let $p>1$.

$$\Big(\int_a^b\int_a^b\Big|(T_\lambda f)(x,y)\Big|^p dxdy\Big)^{1/p}=\Big(\int_a^b\int_a^b\Big|\int_a^b\int_a^b f(t,s)K_\lambda(t-x,s-y)dsdt\Big|^p dxdy\Big)^{1/p}.$$

In view of (2.1), we can rewrite

$$\begin{aligned}\|T_\lambda f\|_{L_p(\langle a,b\rangle^2)} &= \Big(\int_a^b\int_a^b \Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{f}(t,s)K_\lambda(t-x,s-y)dsdt\Big|^p dxdy\Big)^{1/p} \\ &= \Big(\int_a^b\int_a^b \Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{f}(t+x,s+y)K_\lambda(t,s)dsdt\Big|^p dxdy\Big)^{1/p}.\end{aligned}$$

Applying the Hölder-Minkowsky Inequality, one has

$$\begin{aligned}\|T_\lambda f\|_{L_p(\langle a,b\rangle^2)} &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K_\lambda(t,s)|\Big(\int_a^b\int_a^b \Big|\widetilde{f}(t+x,s+y)\Big|dxdy\Big)^{1/p}dsdt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K_\lambda(t,s)|\Big(\int_{a+s}^{b+s}\int_{a+t}^{b+t}|\widetilde{f}(u,v)|dudv\Big)^{1/p}dsdt \\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K_\lambda(t,s)|\Big(\int_a^b\int_a^b |f(u,v)|dudv\Big)^{1/p}dsdt \\ &\le \|K_\lambda\|_{L_1(\mathbb{R}^2)}\|f\|_{L_p(\langle a,b\rangle^2)}.\end{aligned}$$

This completes the proof of the existence theorem. □

3 Main results

In the present section, we assume that $\langle a,b\rangle$ is any arbitrary finite interval in $\mathbb{R}$, such as $[a,b]$, $[a,b)$, $(a,b]$ or (a,b).

Theorem 3.1. *Suppose that the kernel function $K_\lambda(t,s)$ of (0.1) satisfies the conditions (a)-(e). Let (x_0,y_0) be a μ-generalized-Lebesgue point of function $f(x,y)\in L_1(\langle a,b\rangle^2)$, i.e., for some (x_0,y_0) satisfy the conditions*

$$\lim_{h\to 0^+}\frac{1}{\mu(h)}\int_{x_0}^{x_0+h}|f(t,s)-f(x_0,s)|dt=0, \tag{3.1}$$

and

$$\lim_{h\to 0^+}\frac{1}{\mu(h)}\int_{y_0}^{y_0+h}|f(t,s)-f(t,y_0)|ds=0, \tag{3.2}$$

holds uniformly with respect to almost all $s\in\langle a,b\rangle$ and $t\in\langle a,b\rangle$, respectively. Here $\mu(t)$ is a function defined on $[0,b-a]$, increasing, absolutely continuous and $\mu(0)=0$. If (x,y,λ) tends to (x_0,y_0,λ_0) on any planar set Z on which the functions

$$\int_{y_0-\delta}^{y_0+\delta}|K_\lambda(t-x,s-y)|\mu'(|s-y_0|)ds+2|K_\lambda(t-x,0)|\mu(|y-y_0|), \tag{3.3}$$

and

$$\int_{x_0-\delta}^{x_0+\delta}|K_\lambda(t-x,s-y)|\mu'(|t-x_0|)dt+2|K_\lambda(0,s-y)|\mu(|x-x_0|),\quad 0<\delta<\min\{\delta_0,\delta_1\}, \tag{3.4}$$

are bounded, then

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} |(T_\lambda f)(x,y)-f(x_0,y_0)|=0. \tag{3.5}$$

Proof. Suppose that

$$x_0+\delta<b, \quad x_0-\delta>a, \quad \text{and} \quad 0<x_0-x<\frac{\delta}{2}, \tag{3.6}$$

and

$$y_0+\delta<b, \quad y_0-\delta>a \quad \text{and} \quad 0<y_0-y<\frac{\delta}{2}, \tag{3.7}$$

for any $0<\delta<\min\{\delta_0,\delta_1\}$.

We define the following sets

$$F_{n,k}:=\langle c_n,c_{n+1}\rangle\times\langle d_k,d_{k+1}\rangle, \quad 0\le n,k\le 5, \tag{3.8}$$

where

$$\begin{array}{lllll} c_0=-\infty, & c_1=a, & c_2=x_0-\delta, & c_3=x_0, & c_4=x_0+\delta, \\ c_5=b, & c_6=\infty, & d_0=-\infty, & d_1=a, & d_2=y_0-\delta, \\ d_3=y_0, & d_4=y_0+\delta, & d_5=b, & d_6=\infty. & \end{array}$$

According to condition (b) and (2.1), we shall write

$$\begin{aligned} |(T_\lambda f)(x,y)-f(x_0,y_0)| &= \Big| \int_a^b \int_a^b f(t,s)K_\lambda(t-x,s-y)dsdt - f(x_0,y_0)\Big| \\ &= \Big| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{f}(t,s)K_\lambda(t-x,s-y)dsdt - f(x_0,y_0)\Big| \\ &\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\widetilde{f}(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt \\ &\quad + |f(x_0,y_0)|\Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\lambda(t-x,s-y)dsdt-1\Big| \\ &=: A_\lambda(x,y)+B_\lambda(x,y). \end{aligned}$$

In view of the sets (3.8), we can re-write $A_\lambda(x,y)$ as;

$$A_\lambda(x,y)=\sum_{n,k=0}^{5} I_{n,k}(x,y,\lambda),$$

where

$$I_{n,k}(x,y,\lambda)=\int_{F_{n,k}} |\widetilde{f}(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt. \tag{3.9}$$

To prove the theorem, it is sufficient to show that $I_{n,k}(x,y,\lambda)\to 0$ as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$ on Z, whenever $0\le n,k\le 5$.

Let us consider $I_{0,k}(x,y,\lambda)$ for $k=0,1,\cdots,5$.

$$\begin{aligned}
I_{0,k}(x,y,\lambda)&=\int_{F_{0,k}}|\widetilde{f}(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&=\int_{-\infty}^{a}\int_{d_k}^{d_{k+1}}|0-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&\le|f(x_0,y_0)|\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv.
\end{aligned}$$

For $k=0,1,\cdots,5$.

$$\begin{aligned}
I_{5,k}(x,y,\lambda)&=\int_{F_{5,k}}|\widetilde{f}(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&=\int_{b}^{+\infty}\int_{d_k}^{d_{k+1}}|0-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&\le|f(x_0,y_0)|\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv.
\end{aligned}$$

Similarly for $n=1,2,\cdots,4$, one has

$$\begin{aligned}
I_{n,0}(x,y,\lambda)&=\int_{F_{n,0}}|\widetilde{f}(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&=\int_{c_n}^{c_{n+1}}\int_{-\infty}^{a}|0-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&\le|f(x_0,y_0)|\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv.
\end{aligned}$$

For $n=1,2,3,4$,

$$\begin{aligned}
I_{n,5}(x,y,\lambda)&=\int_{F_{n,5}}|\widetilde{f}(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&=\int_{c_n}^{c_{n+1}}\int_{b}^{+\infty}|0-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\
&\le|f(x_0,y_0)|\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv,\\
I_{1,1}(x,y,\lambda)&\le2\|f\|_{L_1(\langle a,b\rangle^2)}\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv,\\
I_{4,4}(x,y,\lambda)&\le2\|f\|_{L_1(\langle a,b\rangle^2)}\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv,
\end{aligned}$$

which in view of $d)$ tend to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Now

$$\begin{aligned}
I_{1,2}(x,y,\lambda) &= \int_{F_{1,2}} |f(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt \\
&= \int_a^{x_0-\delta}\int_{y_0-\delta}^{y_0} |f(t,s)-f(t,y_o)+f(t,y_0)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt \\
&\le \int_a^{x_0-\delta}\int_{y_0-\delta}^{y_0} |f(t,s)-f(t,y_o)||K_\lambda(t-x,s-y)|dsdt \\
&\quad + \int_a^{x_0-\delta}\int_{y_0-\delta}^{y_0} |f(t,y_0)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt \\
&= \int_a^{x_0-\delta}\int_{y_0-\delta}^{y_0} |f(t,s)-f(t,y_o)||K_\lambda(t-x,s-y)|dsdt \\
&\quad + \int_a^{x_0-\delta} |f(t,y_0)-f(x_0,y_0)|\Big[\int_{y_0-\delta}^{y_0}|K_\lambda(t-x,s-y)|ds\Big]dt. \qquad (3.10)
\end{aligned}$$

In view of (3.7) and the condition (e), one has

$$\begin{aligned}
&\int_a^{x_0-\delta} |f(t,y_0)-f(x_0,y_0)|\Big[\int_{y_0-\delta}^{y_0}|K_\lambda(t-x,s-y)|ds\Big]dt \\
\le&\delta\int_a^{x_0-\delta} |f(t,y_0)-f(x_0,y_0)||K_\lambda(t-x,0)|dt \\
\le&\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\int_a^{x_0-\delta} |f(t,y_0)-f(x_0,y_0)|dt \\
\le&2\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)},
\end{aligned}$$

which in view of d) tends to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Thus, it is sufficient to show that the first term on the right hand side of (3.10) tends to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$ on Z.

Denote

$$F(t,s):=\int_s^{y_0}|f(t,m)-f(t,y_o)|dm,$$

then, by virtue of (3.1), to each $\varepsilon>0$ there exists a $\delta>0$ such that

$$F(t,s)\le\varepsilon\mu(y_0-s) \qquad (3.11)$$

for all $0<y_0-s\le\delta$. We now fix this δ and estimate

$$\int_a^{x_0-\delta}\int_{y_0-\delta}^{y_0}|f(t,s)-f(t,y_0)||K_\lambda(t-x,s-y)|dsdt.$$

Following the method given in [7], one has

$$\begin{aligned}&\int_a^{x_0-\delta}\int_{y_0-\delta}^{y_0}|f(t,s)-f(t,y_0)||K_\lambda(t-x,s-y)|dsdt\\ \leq&\varepsilon\int_a^{x_0-\delta}\Big(\int_{y_0-\delta}^{y_0}|K_\lambda(t-x,s-y)|\mu'(y_0-s)ds+2|K_\lambda(t-x,0)|\mu(|y-y_0|)\Big)dt.\end{aligned}$$

In view of (3.3) and being the arbitrariness of ε, $I_{1,2}(x,y,\lambda)$ tends to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

By using the same method one can easily prove that the integrals $I_{2,1}(x,y,\lambda)$, $I_{2,4}(x,y,\lambda)$, $I_{4,2}(x,y,\lambda)$ tend to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Similarly, we also find the following inequalities:

$$\begin{aligned}I_{1,3}(x,y,\lambda)\leq&\int_a^{x_0-\delta}\varepsilon\Big(\int_{y_0}^{y_0+\delta}\|K_\lambda(t-x,s-y)|\mu'(s-y_0)ds+2|K_\lambda(t-x,0)|\mu(|y-y_0|)\Big)dt\\ &+2\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)},\\ I_{3,1}(x,y,\lambda)\leq&\int_a^{y_0-\delta}\varepsilon\Big(\int_{x_0}^{x_0+\delta}|K_\lambda(t-x,s-y)|\mu'(t-x_0)dt+2|K_\lambda(0,s-y)|\mu(|x-x_0|)\Big)ds\\ &+2\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)},\\ I_{4,2}(x,y,\lambda)\leq&\int_{x_0+\delta}^{b}\varepsilon\Big(\int_{y_0-\delta}^{y_0}|K_\lambda(t-x,s-y)|\mu'(y_0-s)ds+2|K_\lambda(t-x,0)|\mu(|y-y_0|)\Big)dt\\ &+2\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)},\\ I_{2,4}(x,y,\lambda)\leq&\int_{y_0+\delta}^{b}\varepsilon\Big(\int_{x_0-\delta}^{x_0}|K_\lambda(t-x,s-y)|\mu'(x_0-t)dt+2|K_\lambda(0,s-y)|\mu(|x-x_0|)\Big)ds\\ &+2\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)},\\ I_{4,3}(x,y,\lambda)\leq&\int_{x_0+\delta}^{b}\varepsilon\Big(\int_{y_0}^{y_0+\delta}|K_\lambda(t-x,s-y)|\mu'(s-y_0)ds+2|K_\lambda(t-x,0)|\mu(|y-y_0|)\Big)dt\\ &+2\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)},\\ I_{3,4}(x,y,\lambda)\leq&\int_{y_0+\delta}^{b}\varepsilon\Big(\int_{x_0}^{x_0+\delta}|K_\lambda(t-x,s-y)|\mu'(t-x_0)dt+2|K_\lambda(0,s-y)|\mu(|x-x_0|)\Big)ds\\ &+2\delta\Big|K_\lambda\Big(\frac{\delta}{2},0\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)}.\end{aligned}$$

Consequently, one has

$$I_{1,2}(x,y,\lambda)+I_{1,3}(x,y,\lambda)+I_{4,2}(x,y,\lambda)+I_{4,3}(x,y,\lambda)\to 0$$

as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Now, if $t<x_0-\delta$ then
$$t-x<x_0-x-\delta<-\frac{\delta}{2}<0,$$
while if $s<y_0-\delta$ then
$$s-y<y_0-y-\delta<-\frac{\delta}{2}<0,$$
and if $t>x_0+\delta$ then
$$t-x>x_0+\delta-x>\delta>0,$$
while if $s>y_0+\delta$ then
$$s-y>y_0-y+\delta>\delta>0.$$
So, one has
$$\begin{aligned}I_{4,1}(x,y,\lambda)=&\int_{x_0+\delta}^{b}\int_{a}^{y_0-\delta}|f(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\ \le&2\Big|K_\lambda\Big(\delta,-\frac{\delta}{2}\Big)\Big|\|f\|_{L_1(\langle a,b\rangle^2)}\le 2|K_\lambda(\delta,0)|\|f\|_{L_1(\langle a,b\rangle^2)},\end{aligned}$$
and
$$\begin{aligned}I_{1,4}(x,y,\lambda)=&\int_{a}^{x_0-\delta}\int_{y_0+\delta}^{b}|f(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\ \le&2\Big|K_\lambda\Big(-\frac{\delta}{2},\delta\Big)\Big|\|f\|_{L_1}(\langle a,b\rangle^2)\le 2|K_\lambda(0,\delta)|\|f\|_{L_1(\langle a,b\rangle^2)}.\end{aligned}$$
Finally, let us consider the integrals $I_{2,2}(x,y,\lambda)$ and $I_{3,3}(x,y,\lambda)$. According to (2.1) we write
$$\begin{aligned}I_{2,2}(x,y,\lambda)=&\int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0}|f(t,s)-f(t,y_0)+f(t,y_0)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\ \le&\int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0}|f(t,s)-f(t,y_0)||K_\lambda(t-x,s-y)|dsdt\\ &+\int_{y_0-\delta}^{y_0}\int_{x_0-\delta}^{x_0}|f(t,y_0)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dtds.\end{aligned}$$
The two terms of the righthand-side of the last inequality estimate as the integral $I_{1,2}(x,y,\lambda)$. Consequently, $I_{2,2}(x,y,\lambda)$ and $I_{3,3}(x,y,\lambda)$ tend to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Combining the above estimations, we get
$$A_\lambda(x,y)\to 0$$
as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

These inequalities are shown to be also valid for
$$-\frac{\delta}{2}<x_0-x<0\quad\text{and}\quad -\frac{\delta}{2}<y_0-y<0.$$

Therefore our theorem now follows, in view of conditions (b), (c) and (d) and (3.3).

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}(T_\lambda f)(x,y)=f(x_0,y_0)$$

and this proves (3.5). □

If we consider the case $\langle a,b\rangle=\mathbb{R}$, we have the following theorem.

Theorem 3.2. *Suppose that the kernel function $K_\lambda(t,s)$ of* (0.1) *satisfies the conditions (a)-(e). If (x,λ) tends to (x_0,λ_0) on any planar set Z on which the functions* (3.3) *and* (3.4) *are bounded. Then at each point (x_0,y_0) for which* (3.1) *and* (3.2) *holds, we have for $f(x,y)\in L_1(\mathbb{R}^2)$*

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}|(T_\lambda f)(x,y)-f(x_0,y_0)|=0.$$

Proof. Suppose that (3.6) and (3.7) hold true for any $0<\delta<\min\{\delta_0,\delta_1\}$.

We define the following sets

$$F_{n,k}:=\langle c_n,c_{n+1}\rangle\times\langle d_k,d_{k+1}\rangle,\quad 1\le n,k\le 4, \tag{3.12}$$

where

$$\begin{array}{lllll} c_1=-\infty, & c_2=x_0-\delta, & c_3=x_0, & c_4=x_0+\delta, & c_5=\infty,\\ d_1=-\infty, & d_2=y_0-\delta, & d_3=y_0, & d_4=y_0+\delta, & d_5=\infty. \end{array}$$

According to condition (b), 2.1 and in view of the sets (3.12), we have

$$|(T_\lambda f)(x,y)-f(x_0,y_0)|\le A_\lambda(x,y)+|f(x_0,y_0)|\Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_\lambda(t-x,s-y)dsdt-1\Big|,$$

where

$$A_\lambda(x,y)=\sum_{n,k=1}^{4}I_{n,k}(x,y,\lambda),$$

and $I_{n,k}(x,y,\lambda)$ is as given in (3.9).

By Theorem 3.1, it is sufficient only to consider the integrals $I_{1,k}(x,y,\lambda)+I_{4,k}(x,y,\lambda)$ and $I_{n,1}(x,y,\lambda)+I_{n,4}(x,y,\lambda)$. Indeed, the proofs of the other integrals are similar to that of in Theorem 3.1. Hence, it is sufficient only to show that

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}[I_{1,k}(x,y,\lambda)+I_{4,k}(x,y,\lambda)]=0$$

and

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)}[I_{n,1}(x,y,\lambda)+I_{n,4}(x,y,\lambda)]=0.$$

According to (3.6), when $t\notin[x_0-\delta,x_0+\delta]$, we have $t<x_0-\delta$ or $t>x_0+\delta$. If $t<x_0-\delta$ then

$$t-x<x_0-x-\delta<-\frac{\delta}{2}<0,$$

while if $t>x_0+\delta$

$$t-x>x_0-x+\delta>\delta>\frac{\delta}{2}>0.$$

Clearly for $k=1,\cdots,4$, one has

$$\begin{aligned}I_{1,k}(x,y,\lambda)+I_{4,k}(x,y,\lambda)=&\int_{F_{1,k}}|f(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\&+\int_{F_{4,k}}|f(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\&\le 8|f(x_0,y_0)|\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv\end{aligned}$$

and for $n=1,\cdots,4$,

$$\begin{aligned}I_{n,1}(x,y,\lambda)+I_{n,4}(x,y,\lambda)=&\int_{F_{n,1}}|f(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\&+\int_{F_{n,4}}|f(t,s)-f(x_0,y_0)||K_\lambda(t-x,s-y)|dsdt\\&\le 8|f(x_0,y_0)|\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(u,v)|dudv\end{aligned}$$

for any $\delta>0$. Hence according to conditions (c) and (d) and in view of Theorem 3.1, $A_\lambda(x,y)$ tends to zero as $(x,y,\lambda)\to(x_0,y_0,\lambda_0)$.

Thus the proof of the theorem is now complete. □

Example 3.1. We consider the function

$$K_\lambda(t,s)=\begin{cases}\dfrac{5\lambda^2}{4}, & t,s\in\Big[0,\dfrac{1}{\lambda}\Big],\\ -\dfrac{\lambda^2}{4}, & t,s\in\Big[-\dfrac{1}{\lambda},0\Big),\\ 0, & t,s\notin\Big[-\dfrac{1}{\lambda},\dfrac{1}{\lambda}\Big],\end{cases}\tag{3.13}$$

where $\Lambda=[1,\infty)$ is a set of indices with natural topology and $\lambda_0=\infty$ is an accumulation point of Λ in this topology.

From (3.13), one has

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_\lambda(t,s)dsdt=-\frac{\lambda^2}{4}\int_{[-\frac{1}{\lambda},0)}\int_{[-\frac{1}{\lambda},0)}dsdt+\frac{3\lambda^2}{4}\int_{[0,\frac{1}{\lambda}]}\int_{[0,\frac{1}{\lambda}]}dsdt=1<\infty.$$

Furthermore, it is easy to see that

$$\lim_{\lambda\to\infty}\iint_{\mathbb{R}^2/\langle-\delta,\delta\rangle^2}|K_\lambda(t,s)|dsdt=0,$$

and

$$\lim_{\lambda\to\infty}\big[\sup_{|t|\geq\delta}|K_\lambda(t,0)|\big]=\lim_{\lambda\to\infty}\big[\sup_{|s|\geq\delta}|K_\lambda(0,s)|\big]=0$$

for every $\delta>0$.

In addition, by (3.13), there exist $\delta_0,\delta_1>0$ such that $|K_\lambda(t,\cdot)|$ is non-decreasing on $(-\delta_0,0]$ and non-increasing on $[0,\delta_0)$ as a function of t and $|K_\lambda(\cdot,s)|$is non-decreasing on $(-\delta_1,0]$ and non-increasing on $[0,\delta_1)$ as a function of s, for each $\lambda\in\Lambda$.

Further, $|K_\lambda(t,\cdot)|$ and $|K_\lambda(\cdot,s)|$ are non-decreasing on $(-\infty,0]$ and non-increasing on $[0,\infty)$ as a function of t and s, for each $\lambda\in\Lambda$, respectively.

This implies that the kernel function K_λ satisfies the conditions (a)-(e).

We assume that $\mu(t)=t$. If we use (3.13) in Theorem 3.1 and Theorem 3.2, we find

$$\begin{aligned}&\int_{y_0-\delta}^{y_0+\delta}|K_\lambda(t-x,s-y)|ds+2|K_\lambda(t-x,0)|\mu(|y-y_0|)\\=&\int_{y_0-\delta}^{y_0+\delta}|K_\lambda(t-x,s-y)|ds+\frac{5\lambda^2}{2}\mu(|y-y_0|)\end{aligned}$$

and

$$\begin{aligned}&\int_{x_0-\delta}^{x_0+\delta}|K_\lambda(t-x,s-y)|\mu'(|t-x_0|)dt+2|K_\lambda(0,s-y)|\mu(|x-x_0|)\\=&\int_{x_0-\delta}^{x_0+\delta}|K_\lambda(t-x,s-y)|dt+\frac{5\lambda^2}{2}\mu(|x-x_0)|).\end{aligned}$$

The first parts of the right hand side of these equalities are finite. Furthermore

$$\lim_{(x,,y,\lambda)\to(x_0,y_0,\infty)}\lambda^2|x-x_0|=M<\infty,$$

if and only if the rates of convergence of $\lambda^2\to\infty$, $x\to x_0$ and $y\to y_0$ are equivalent.

4 Graphical examples

Example 4.1. We note that in the graphs below, the graph in red belongs to the $\lambda=1$, the graph in green belongs to for $\lambda=2$ and finally the graph in blue belongs to for $\lambda=3$ for the kernel function (3.13), respectively.

Example 4.2. The following graph belongs to the integral of the kernel function.

Example 4.3. The last graph belong to the approximation. The graph in red belongs to the original function $f(x,y)$, the graph in green belongs to for $(T_2f)(x,y)$ and finally the graph in blue belongs to for $(T_5f)(x,y)$.

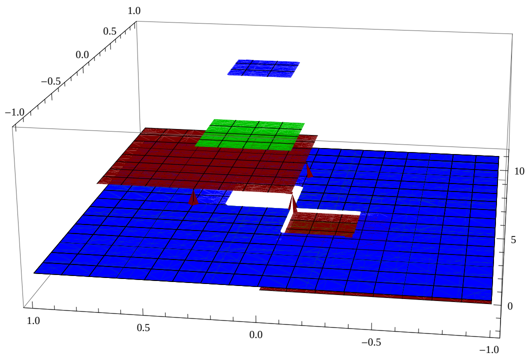

Figure 1: $K_1(t,s)$, $K_2(t,s)$, $K_3(t,s)$, $t,s\in[-1,1]$.

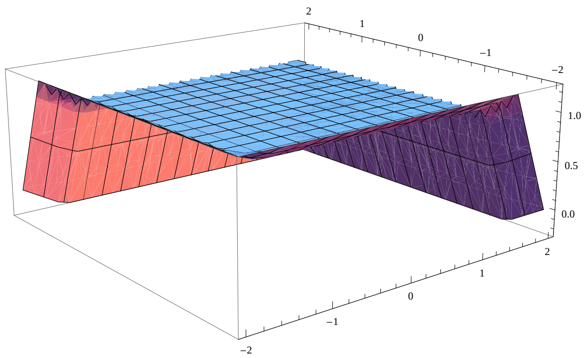

Figure 2: $\int_{-2}^{2}\int_{-2}^{2}K_2(t-x,s-y)dsdt$, $x,y\in[-2,2]$.

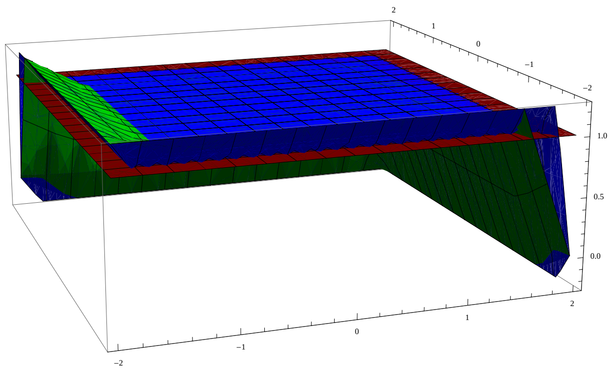

Figure 3: $f(x,y)=1$, $(T_2f)(x,y)$, $(T_5f)(x,y)$, $x,y\in[-2,2]$.

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