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Variable Exponent Herz Type Hardy Spaces and Their Applications

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Abstract. In this paper, the authors introduce certain Herz type Hardy spaces with variable exponents and establish the characterizations of these spaces in terms of atomic and molecular decompositions. Using these decompositions, the authors obtain the boundedness of some singular integral operators on the Herz type Hardy spaces with variable exponents.

Key Words: Hardy space, Herz space, variable exponent, maximal function, atom, molecule.

AMS Subject Classifications: 42B35, 42B25, 42B20

1 Introduction

In 1991, Kováčik and Rákosník introduced basic properties of variable Lebesgue and Sobolev spaces in [23]. After that many spaces with variable exponents appeared, for example: Bessel potential spaces with variable exponent, Besov and Triebel-Lizorkin spaces with variable exponents, Morrey spaces with variable exponents, Hardy spaces with variable exponent and so on, see [2,3,9,11,15,17,21,22,31,41] and reference therein. Indeed, the atomic, molecular and wavelet decompositions of variable exponent Besov and Triebel-Lizorkin spaces were given in [3,9,21,22,42]. The duality and reflexivity of spaces $B^s_{p(\cdot),q}$ and $F^s_{p(\cdot),q}$ were discussed in [33]. The atomic and molecular decompositions of Hardy spaces with variable exponent and their applications for the boundedness of singular integral operators were obtained in [31,34]. Variable exponent spaces have many applications, for instance in differential equations, see the article [16].

In parallel to the above, there are other type function spaces, Herz type spaces, have attracted many authors' interests for last three decades. Actually, many properties of classic Lebesgue spaces have been generalized to Herz type spaces. We outline some recent results we have concerned. In 2010, Izuki proved the boundedness of sublinear operators

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on Herz space with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}$ and $K^{\alpha,q}_{p(\cdot)}$ in [19]. In 2012, Almeida and Drihem obtained boundedness results for a wide class of classical operators on Herz spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $K^{\alpha(\cdot),q}_{p(\cdot)}$ in [1]. Shi and the second author in [35] considered Herz type Besov and Triebel-Lizorkin spaces with one variable exponent. Then the authors in [10] established the boundedness of vector-valued Hardy-Littlewood maximal operator in spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $K^{\alpha(\cdot),q}_{p(\cdot)}$ and gave characterizations of Herz type Besov and Triebel-Lizorkin spaces with variable exponents by maximal functions. In [40], Hongbin Wang and Zongguang Liu introduced a certain Herz type Hardy spaces with variable exponent which is a generalization of classical Herz type Hardy spaces, for the latter, see the monograph [30] by Shanzhen Lu, Dachun Yang and Guoen Hu. For more information about the generalizations, see [18, 24–26, 29, 38, 43, 44].

Inspired by the previous articles, we would like to declare that the goal of this paper is to introduce new Herz type Hardy spaces with variable exponents and give their applications. The structure of the paper is as follows. In the rest of the section we shall recall some definitions and notations. In Section 2, we shall define the Herz type Hardy spaces with variable exponents $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}$, and give their atomic characterizations. In Section 3, we shall present the molecular characterizations of $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}$.

Throughout this paper, $|S|$ denotes the Lebesgue measure and χ_S the characteristic function for a measurable set $S\subset\mathbb{R}^n$. For a multi-index $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$, we denote $|\beta|=\beta_1+\beta_2+\cdots+\beta_n$. We also use the notation $a\lesssim b$ if there exist a constant $C>0$ such that $a\le Cb$. If $a\lesssim b$ and $b\lesssim a$ we will write $a\approx b$. Finally we claim that C is always a positive constant but it may change from line to line.

Definition 1.1. Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a measurable function. Denote

$$L^{p(\cdot)}(\mathbb{R}^n):=\{f \text{ is measurable on } \mathbb{R}^n:\rho_{p(\cdot)}(f/\lambda)<\infty \text{ for some constant } \lambda>0\},$$

where $\rho_{p(\cdot)}(f):=\int_{\mathbb{R}^n}|f(x)|^{p(x)}dx$, and

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}:=\inf\{\lambda>0:\rho_{p(\cdot)}(f/\lambda)\le 1\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Let $L^1_{\rm loc}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $f\in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x):=\sup_{r>0}r^{-n}\int_{B(x,r)}|f(y)|dy,\ \forall x\in\mathbb{R}^n,$$

where $B(x,r):=\{y\in\mathbb{R}^n:|x-y|<r\}$. We also use the following notation: $p_-:=\operatorname{essinf}\{p(x):x\in\mathbb{R}^n\}$ and $p_+:=\operatorname{esssup}\{p(x):x\in\mathbb{R}^n\}$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_->1$

and $p_+ < \infty$. $\mathcal{B}(\mathbb{R}^n)$ is the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. It is well known that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies the following global log-Hölder continuous then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

Definition 1.2. Let $\alpha(\cdot)$ be a real function on $\mathbb{R}^n$.

(i) $\alpha(\cdot)$ is called log-Hölder continuous on $\mathbb{R}^n$ if there exists $C>0$ such that

$$|\alpha(x)-\alpha(y)| \le \frac{C}{\log(e+1/|x-y|)}, \quad \forall x,y \in \mathbb{R}^n, \quad |x-y| < \frac{1}{2}.$$

(ii) $\alpha(\cdot)$ is called log-Hölder continuous at origin if there exists $C>0$ such that

$$|\alpha(x)-\alpha(0)| \le \frac{C}{\log(e+1/|x|)}, \quad \forall x \in \mathbb{R}^n.$$

(iii) $\alpha(\cdot)$ is called satisfying the log-Hölder decay condition if there exist $\alpha_\infty \in \mathbb{R}$ and a constant $C>0$ such that

$$|\alpha(x)-\alpha_\infty| \le \frac{C}{\log(e+|x|)}, \quad \forall x \in \mathbb{R}^n.$$

We denote by $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ the class of all variable exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which are log-Hölder continuous at the origin and at the infinity respectively.

We call $p'(\cdot)$ the conjugate exponent to $p(\cdot)$, that is $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ and form [6], we know that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ is equivalent to $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For more details of the set $\mathcal{B}(\mathbb{R}^n)$, see [4–8, 32].

From now on, in order to writing simplicity and reading easily, we omit the domain of space. For example, we denote $L^{p(\cdot)}(\mathbb{R}^n)$ by $L^{p(\cdot)}$ and $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ by $\|\cdot\|_{L^{p(\cdot)}}$ differently. We shall use the following results.

Lemma 1.1 (see [23]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $r_p = 1+1/p^- - 1/p^+$.

Lemma 1.2 (see [19, Remark 2.8]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exist $0<\delta_1,\delta_2<1$ depending only on $p(\cdot)$ and n such that for balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1} \quad \text{and} \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2}.$$

Lemma 1.3 (see [19, Lemma 2.9]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant $C>0$ such that for balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}}\|\chi_B\|_{L^{p'(\cdot)}} \le C.$$

To give the definition of Herz spaces with variable exponents, let us introduce the following notations. Let $k\in\mathbb{Z}$, $B_k:=\{x\in\mathbb{R}^n:|x|\le 2^k\}$, $R_k:=B_k\backslash B_{k-1}$, $\chi_k:=\chi_{R_k}$. The symbol $\mathbb{N}_0$ denotes the set of all non-negative integers. For $m\in\mathbb{N}_0$, we denote $\tilde{\chi}_m:=\chi_{R_m}$ if $m\ge 1$ and $\tilde{\chi}_0:=\chi_{B_0}$.

Definition 1.3. Let $0<q\le\infty, p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and non-homogeneous Herz space $K^{\alpha(\cdot),q}_{p(\cdot)}$ are defined respectively by

$$\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}:=\Big\{f\in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n\backslash 0):\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}<\infty\Big\}$$

and

$$K^{\alpha(\cdot),q}_{p(\cdot)}:=\Big\{f\in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n):\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}<\infty\Big\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}:=\Big\{\sum_{k=-\infty}^{\infty}\|2^{\alpha(\cdot)k}f\chi_k\|^q_{L^{p(\cdot)}}\Big\}^{\frac{1}{q}}\quad\text{and}\quad\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}:=\Big\{\sum_{m=0}^{\infty}\|2^{\alpha(\cdot)m}f\tilde{\chi}_m\|^q_{L^{p(\cdot)}}\Big\}^{\frac{1}{q}}.$$

Here, there is the usual modification when $q=\infty$.

Proposition 1.1 (see [1, Proposition 3.8])**.** Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $q\in(0,\infty]$. If $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, then

$$K^{\alpha(\cdot),q}_{p(\cdot)}=K^{\alpha_\infty,q}_{p(\cdot)}.$$

If $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}^{\log}_0(\mathbb{R}^n)\cap\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, then

$$\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\approx\Big(\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\|f\chi_k\|^q_{L^{p(\cdot)}}\Big)^{1/q}+\Big(\sum_{k=0}^{\infty}2^{kq\alpha_\infty}\|f\chi_k\|^q_{L^{p(\cdot)}}\Big)^{1/q}.$$

From Theorem 2 in [10], if we replace $\{f_j\}_{j\in\mathbb{N}}$ by f, then we have the following lemma.

Lemma 1.4. *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $0<q<\infty$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}^{\log}_0(\mathbb{R}^n)\cap\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$, such that $-n\delta_1<\alpha(0)\le\alpha_\infty<n\delta_2$, where $0<\delta_1,\delta_2<1$ are constants from Lemma 1.2. Suppose that T is a sublinear and bounded operator on $L^{p(\cdot)}$ satisfying size condition*

$$|Tf(x)|\le C\int_{\mathbb{R}^n}|x-y|^{-n}|f(y)|dy$$

for all $f\in L^1_{\rm loc}(\mathbb{R}^n)$ with compact support and a.e. $x\notin{\rm supp}\, f$, and there exists a positive constant C_1 such that for all locally integrable functions on $\mathbb{R}^n$,

$$\|Tf\|_{L^{p(\cdot)}}\le C\|f\|_{L^{p(\cdot)}}.$$

Then there exists a positive constant C_2 such that

$$\|Tf\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\le C_2\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\quad\textit{and}\quad\|Tf\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}\le C_2\|f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}$$

for all locally integrable functions f on $\mathbb{R}^n\backslash\{0\}$ and $\mathbb{R}^n$, respectively.

2 The atomic characterizations and their applications

In this section, we will introduce Herz type Hardy spaces with variable exponents $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}$. To do this, we need recall some notations. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) := \sup_{\phi\in\mathcal{A}_N} |\phi^*_\nabla(f)(x)|, \quad \forall x\in\mathbb{R}^n,$$

where

$$\mathcal{A}_N := \Big\{\phi\in\mathcal{S}(\mathbb{R}^n):\ \sup_{|\alpha|,|\beta|\le N,\ x\in\mathbb{R}^n} |x^\alpha D^\beta\phi(x)|\le 1\Big\},$$

and $N>n+1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) := \sup_{|y-x|<t} |\phi_t * f(y)|, \quad \forall x\in\mathbb{R}^n, \quad \text{with } \phi_t(x)=t^{-n}\phi(x/t).$$

The grand maximal operator G_N was firstly introduced by C. Fefferman and E. Stein in [12] to study Hardy spaces. For the systematical treatment of Hardy spaces, one can see [14,36,39]. E. Nakai and Y. Sawano generalized them to variable exponent case in [31].

Definition 2.1. Let $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, $0<q\le\infty$, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $N>n+1$. The homogeneous Herz type Hardy space with variable exponents $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and non-homogeneous Herz type Hardy space with variable exponents $HK^{\alpha(\cdot),q}_{p(\cdot)}$ are defined respectively by

$$H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)} := \Big\{f\in\mathcal{S}'(\mathbb{R}^n):\ G_N f\in\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}\Big\},$$

and

$$HK^{\alpha(\cdot),q}_{p(\cdot)} := \Big\{f\in\mathcal{S}'(\mathbb{R}^n):\ G_N f\in K^{\alpha(\cdot),q}_{p(\cdot)}\Big\},$$

where

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}} := \|G_N f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}} \quad \text{and} \quad \|f\|_{HK^{\alpha(\cdot),q}_{p(\cdot)}} := \|G_N f\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}.$$

Remark 2.1. If $\alpha(\cdot)=\alpha$, these spaces were considered by Wang and Liu in [40]. If $p(\cdot)$ and $\alpha(\cdot)$ are constant, these are the classical Herz type Hardy spaces, see [30].

Let $\psi(r)=1$ for $r\in[0,1]$, $\psi(r)=r^{-N}$ for $r\in(1,+\infty)$. Then there exists $C>0$ such that $\phi(x)\le C\psi(|x|)$ for all $\phi\in\mathcal{A}_N$. Therefore by [36, Proposition in Page 57], there exists $C>0$ such that $G_N f(x)\le CMf(x)$ for all $x\in\mathbb{R}^n$. This means that $G_N f$ satisfies the size condition in Lemma 1.4. By Lemma 1.4, if $-n\delta_1<\alpha(0)\le\alpha_\infty<n\delta_2$ and $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, then

$$H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}\cap L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n\backslash\{0\}) = \dot{K}^{\alpha(\cdot),q}_{p(\cdot)} \quad \text{and} \quad HK^{\alpha(\cdot),q}_{p(\cdot)}\cap L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n) = K^{\alpha(\cdot),q}_{p(\cdot)}.$$

Thus we are interested in the case $n\delta_2 \le \alpha(0),\alpha_\infty < \infty$. In this case, we will establish characterizations of the spaces $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}$ in terms of central atomic decomposition. For $t \in \mathbb{R}$ we denote by [t] the largest integer less than or equal to t.

Definition 2.2. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ and non-negative integer $s \ge [\alpha_r - n\delta_2]$, where $n\delta_2 \le \alpha_r < \infty$ and δ_2 is defined in Lemma 1.2. Here $\alpha_r = \alpha(0)$, if $r<1$, $\alpha_r = \alpha_\infty$, if $r \ge 1$.

(i) A function a on $\mathbb{R}^n$ is said to be a central $(\alpha(\cdot),p(\cdot))$-atom, if it satisfies:

(1) supp $a \subset B(0,r)$;

(2) $\|a\|_{L^{p(\cdot)}} \le |B(0,r)|^{-\alpha_r/n}$;

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \le s$.

(ii) A function a on $\mathbb{R}^n$ is said to be a central $(\alpha_r,p(\cdot))$-atom of restricted type, if it satisfies

(1) supp $a \subset B(0,r)$, $r \ge 1$;

(2) $\|a\|_{L^{p(\cdot)}} \le |B(0,r)|^{-\alpha_\infty/n}$;

(3) $\int_{\mathbb{R}^n} a(x)x^\beta dx = 0$, $|\beta| \le s$.

Remark 2.2. If $p(\cdot) = p$, $\alpha(\cdot) = \alpha$ are constant, then taking $\delta_2 = 1 - 1/p$ we can get the classical case.

Theorem 2.1. *Let $0<q<\infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ and $n\delta_2 \le \alpha(0),\alpha_\infty < \infty$, where δ_2 is defined in Lemma 1.2.*

(i) $f \in H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ if and only if $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot),p(\cdot))$-atom with support contained in B_k and

$$\sum_{k=-\infty}^{\infty} |\lambda_k|^q < \infty.$$

Moreover,

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}} \approx \inf\Big(\sum_{k=-\infty}^{\infty} |\lambda_k|^q\Big)^{1/q},$$

where the infimum is taken over all above decompositions of f.

(ii) $f \in HK^{\alpha(\cdot),q}_{p(\cdot)}$ if and only if $f = \sum_{k=0}^{\infty} \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot),p(\cdot))$-atom of restricted type with support contained in B_k and

$$\sum_{k=0}^{\infty} |\lambda_k|^q < \infty.$$

Moreover

$$\|f\|_{HK^{\alpha(\cdot),q}_{p(\cdot)}} \approx \inf\Big(\sum_{k=0}^{\infty} |\lambda_k|^q\Big)^{1/q},$$

where the infimum is taken over all above decompositions of f.

Proof. We only prove (i). (ii) can be proved in the similar way. To prove the necessity, choose $\phi\in C_0^\infty(\mathbb{R}^n)$ such that $\phi\ge 0$, $\int_{\mathbb{R}^n}\phi(x)dx=1$, supp $\phi\subset\{x:|x|\le 1\}$. For $j\in\mathbb{N}_0$, let $\phi_{(j)}(x):=2^{jn}\phi(2^jx)$, $\forall x\in\mathbb{R}^n$. For each $f\in\mathcal{S}'(\mathbb{R}^n)$, set $f^{(j)}(x)=f*\phi_{(j)}(x)$, $\forall x\in\mathbb{R}^n$. It is obvious that $f^{(j)}\in C^\infty(\mathbb{R}^n)$ and $\lim_{j\to\infty}f^{(j)}=f$ in $\mathcal{S}'(\mathbb{R}^n)$. Let ψ be a radial smooth function such that supp $\psi\subset\{x:1/2-\varepsilon\le|x|\le 1+\varepsilon\}$ with $0<\varepsilon<1/4$, $\psi(x)=1$ for $1/2\le|x|\le 1$. Let $\psi_k(x):=\psi(2^{-k}x)$ for $k\in\mathbb{Z}$ and

$$\tilde{A}_{k,\varepsilon}:=\{x:2^{k-1}-2^k\varepsilon\le|x|\le 2^k+2^k\varepsilon\}.$$

Observe that supp $\psi_k\subset\tilde{A}_{k,\varepsilon}$ and supp $\psi_k(x)=1$ for $x\in A_k:=\{x:2^{k-1}\le|x|\le 2^k\}$. Obviously, $1\le\sum_{k=-\infty}^{\infty}\psi_k(x)\le 2$, $|x|>0$. Let

$$\Phi_k(x):=\begin{cases}\psi_k(x)\Big/\sum\limits_{l=-\infty}^{\infty}\psi_l(x), & x\neq 0,\\ 0, & x=0.\end{cases}$$

Then

$$\sum_{k=-\infty}^{\infty}\Phi_k(x)=1$$

for $x\neq 0$. For some $m\in\mathbb{N}$, we denote by $\mathcal{P}_m$ the class of all the real polynomials with the degree less than m. Let $P_k^{(j)}(x):=P_{\tilde{A}_{k,\varepsilon}}(f^{(j)}\Phi_k)(x)\chi_{\tilde{A}_{k,\varepsilon}}\in\mathcal{P}_m(\mathbb{R}^n)$ be the unique polynomial satisfying

$$\int_{\tilde{A}_{k,\varepsilon}}\big(f^{(j)}(x)\Phi_k(x)-P_k^{(j)}(x)\big)x^\beta dx=0,\quad |\beta|\le m=\max\{[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}.$$

Here and below we denote $\alpha_k:=\alpha_{2^k}$ and it is well defined in Definition 2.2. Write

$$f^{(j)}(x)=\sum_{k=-\infty}^{\infty}\big(f^{(j)}(x)\Phi_k(x)-P_k^{(j)}(x)\big)+\sum_{k=-\infty}^{\infty}P_k^{(j)}(x):=I_{(j)}+II_{(j)}.$$

For the term $I_{(j)}$, let

$$g_k^{(j)}(x):=f^{(j)}(x)\Phi_k(x)-P_k^{(j)}(x)\quad\text{and}\quad a_k^{(j)}(x):=g_k^{(j)}(x)/\lambda_k,$$

where

$$\lambda_k:=b|B_{k+1}|^{\alpha_{k+1}/n}\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}$$

and b is a constant which will be chosen later. Note that

$$\text{supp }a_k^{(j)}\subset B_{k+1},\quad I_{(j)}=\sum_{k=-\infty}^{\infty}\lambda_k a_k^{(j)}(x).$$

Now we estimate $\|g_k^{(j)}\|_{L^{p(\cdot)}}$. To do this, let $\{\phi_\gamma^k:|\gamma|\le m\}$ be the orthogonal polynomials restricted to $\tilde{A}_{k,\varepsilon}$ with respect to the weight $1/|\tilde{A}_{k,\varepsilon}|$, which are obtained from $\{x^\beta:|\beta|\le m\}$ by Gram-Schmidt method, that is

$$\langle\phi_\nu^k,\phi_\mu^k\rangle=\frac{1}{|\tilde{A}_{k,\varepsilon}|}\int_{\tilde{A}_{k,\varepsilon}}\phi_\nu^k(x)\phi_\mu^k(x)dx=\delta_{\nu\mu}.$$

It is easy to see that

$$P_k^{(j)}(x)=\sum_{|\gamma|\le m}\langle f^{(j)}\Phi_k,\phi_\gamma^k\rangle\phi_\gamma^k(x)\quad\text{for }\ x\in\tilde{A}_{k,\varepsilon}.$$

On the other hand, from

$$\frac{1}{|\tilde{A}_{k,\varepsilon}|}\int_{\tilde{A}_{k,\varepsilon}}\phi_\nu^k(x)\phi_\mu^k(x)dx=\delta_{\nu\mu},$$

we infer that

$$\frac{1}{|\tilde{A}_{1,\varepsilon}|}\int_{\tilde{A}_{1,\varepsilon}}\phi_\nu^k(2^{k-1}y)\phi_\mu^k(2^{k-1}y)dy=\delta_{\nu\mu}.$$

Thus we can get $\phi_\nu^k(2^{k-1}y)=\phi_\nu^1(y)$. a.e. That is, $\phi_\nu^k(x)=\phi_\nu^1(2^{1-k}x)$ a.e. for $x\in\tilde{A}_{k,\varepsilon}$. Therefore $|\phi_\nu^k(x)|\le C$, and for $x\in\tilde{A}_{k,\varepsilon}$, by the generalized Hölder inequality we have

$$|P_k^{(j)}(x)|\lesssim\frac{1}{|\tilde{A}_{k,\varepsilon}|}\int_{\tilde{A}_{k,\varepsilon}}|f^{(j)}(x)\Phi_k(x)|dx\lesssim\frac{1}{|\tilde{A}_{k,\varepsilon}|}\|f^{(j)}\Phi_k\|_{L^{p(\cdot)}}\|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{p'(\cdot)}}.$$

By Lemma 1.3 we have

$$\begin{aligned}\|g_k^{(j)}\|_{L^{p(\cdot)}}&\le\|f^{(j)}\Phi_k\|_{L^{p(\cdot)}}+\|P_k^{(j)}\|_{L^{p(\cdot)}}\\&\lesssim\|f^{(j)}\Phi_k\|_{L^{p(\cdot)}}+\frac{1}{|\tilde{A}_{k,\varepsilon}|}\|f^{(j)}\Phi_k\|_{L^{p(\cdot)}}\|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{p'(\cdot)}}\|\chi_{\tilde{A}_{k,\varepsilon}}\|_{L^{p(\cdot)}}\\&\lesssim\|f^{(j)}\Phi_k\|_{L^{p(\cdot)}}+\|f^{(j)}\Phi_k\|_{L^{p(\cdot)}}\lesssim\|(f*\phi_{(j)})\Phi_k\|_{L^{p(\cdot)}}\\&\le C\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}.\end{aligned}$$

Choose $b=C$, then $\|a_k^{(j)}\|_{L^{p(\cdot)}}\le|B_{k+1}|^{-\alpha_{k+1}/n}$ and each $a_k^{(j)}$ ia a central $(\alpha(\cdot),p(\cdot))$-atom with support contained in B_{k+1}. Furthermore, by Proposition 1.1 we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty}|\lambda_k|^q&\lesssim\sum_{k=-\infty}^{\infty}|B_{k+1}|^{q\alpha_{k+1}/n}\Big(\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}\Big)^q\\&\lesssim\sum_{k+1\ge0}|B_{k+1}|^{q\alpha_\infty/n}\Big(\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}\Big)^q\end{aligned}$$

$$+\sum_{k+1<0}|B_{k+1}|^{q\alpha(0)/n}\Big(\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}\Big)^q$$
$$\lesssim\|G_Nf\|^q_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}.$$

It remains to estimate $II_{(j)}$. Let $\{\psi^k_d:|\gamma|\le m\}$ be the dual basis of $\{x^\beta:|\beta|\le m\}$ with respect to the weight $1/|\tilde{A}_{k,\varepsilon}|$ on $\tilde{A}_{k,\varepsilon}$, that is

$$\langle\psi^k_\gamma,x^\beta\rangle=\frac{1}{|\tilde{A}_{k,\varepsilon}|}\int_{\tilde{A}_{k,\varepsilon}}x^\beta\psi^k_\gamma(x)dx=\delta_{\beta\gamma}.$$

Similar to the method of [27], let

$$h^{(j)}_{k,\gamma}(x):=\sum_{l=-\infty}^{k}\Big(\frac{\psi^k_\gamma(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|}-\frac{\psi^{k+1}_\gamma(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|}\Big)\int_{\mathbb{R}^n}f^{(j)}(y)\Phi_l(y)y^\gamma dy.$$

We can write

$$\begin{aligned}II_{(j)}&=\sum_{k=-\infty}^{\infty}\sum_{|\gamma|\le m}\langle f^{(j)}\Phi_k,x^\gamma\rangle\psi^k_\gamma(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)=\sum_{|\gamma|\le m}\sum_{k=-\infty}^{\infty}\Big(\int_{\mathbb{R}^n}f^{(j)}\Phi_kx^\gamma dx\Big)\frac{\psi^k_\gamma(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|}\\&=\sum_{|\gamma|\le m}\sum_{k=-\infty}^{\infty}\Big(\sum_{l=-\infty}^{k}\int_{\mathbb{R}^n}f^{(j)}(x)\Phi_l(x)x^\gamma dx\Big)\times\Big(\frac{\psi^k_\gamma(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|}-\frac{\psi^{k+1}_\gamma(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|}\Big)\\&=\sum_{|\gamma|\le m}\sum_{k=-\infty}^{\infty}\alpha_{k,\gamma}h^{(j)}_{k,\gamma}(x)/\alpha_{k,\gamma}=:\sum_{|\gamma|\le m}\sum_{k=-\infty}^{\infty}\alpha_{k,\gamma}a^{(j)}_{k,\gamma}(x),\end{aligned}$$

where

$$\alpha_{k,\gamma}:=\tilde{b}\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}|B_{k+2}|^{\alpha_{k+2}/n}$$

and $\tilde{b}$ is a constant which will be chosen later. Note that

$$\int_{\mathbb{R}^n}\sum_{l=-\infty}^{k}|\Phi_l(x)x^\gamma|dx=\sum_{l=-\infty}^{k}\int_{\tilde{A}_{k,\varepsilon}}|\Phi_l(x)x^\gamma|dx\lesssim2^{k(n+|\gamma|)}.$$

By a computation, we have

$$\Big|\int_{\mathbb{R}^n}f^{(j)}(y)\sum_{l=-\infty}^{k}\Phi_l(y)y^\gamma dy\Big|\lesssim2^{k(n+|\gamma|)}G_Nf(x),\quad x\in B_{k+2}.$$

This together with the inequality that

$$\Big|\frac{\psi^k_\gamma(x)\chi_{\tilde{A}_{k,\varepsilon}}(x)}{|\tilde{A}_{k,\varepsilon}|}-\frac{\psi^{k+1}_\gamma(x)\chi_{\tilde{A}_{k+1,\varepsilon}}(x)}{|\tilde{A}_{k+1,\varepsilon}|}\Big|\lesssim2^{-k(n+|\gamma|)}\sum_{l=k-1}^{k+1}\chi_l(x),$$

show that

$$\|h_{k,\gamma}^{(j)}\|_{L^{p(\cdot)}}\le C\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}.$$

Take $\tilde{b}=C$. It is readily to verify that each $a_{k,\gamma}^{(j)}$ is a central $(\alpha(\cdot),p(\cdot))$-atom with support contained in

$$\tilde{A}_{k,\varepsilon}\cup\tilde{A}_{k+1,\varepsilon}\subset B_{k+2},$$

and

$$\alpha_{k,\gamma}=C\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}|B_{k+2}|^{\alpha_{k+2}/n},$$

where C is a constant independent of j, f, k and γ. Moreover, by Proposition 1.1 we have

$$\begin{aligned}\sum_{k,\gamma}|\alpha_k|^q&\lesssim\sum_{k=-\infty}^{\infty}|B_{k+2}|^{\alpha_{k+2}q/n}\Big(\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}\Big)^q\\&=\sum_{k+2\ge 0}|B_{k+2}|^{\alpha_\infty q/n}\Big(\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}\Big)^q\\&\quad+\sum_{k+2<0}|B_{k+2}|^{\alpha(0)q/n}\Big(\sum_{l=k-1}^{k+1}\|(G_Nf)\chi_l\|_{L^{p(\cdot)}}\Big)^q\\&\lesssim\|G_Nf\|^q_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}<\infty,\end{aligned}$$

where C is independent of j and f.

Thus, we obtain that

$$f^{(j)}(x)=\sum_{d=-\infty}^{\infty}\lambda_d a_d^{(j)}(x),$$

where each $a_d^{(j)}$ is a central $(\alpha(\cdot),p(\cdot))$-atom with support contained in $\tilde{A}_{d,\varepsilon}\cup\tilde{A}_{d+1,\varepsilon}\subset B_{d+2},\lambda_d$ is independent of j and

$$\Big(\sum_{d=-\infty}^{\infty}|\lambda_d|^q\Big)^{1/q}\lesssim\|G_Nf\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}<\infty,$$

where C is independent of j and f.

Since

$$\sup_{j\in\mathbb{N}_0}\|a_0^{(j)}\|_{L^{p(\cdot)}}\le|B_2|^{-\alpha_2/n},$$

by the Banach-Alaoglu theorem we can obtain a subsequence $\{a_0^{(j_{n_0})}\}$ of $\{a_0^{(j)}\}$ converging in the weak $*$ topology of $L^{p(\cdot)}$ to some $a_0\in L^{p(\cdot)}$. It is easy to verify that a_0 is a central $(\alpha(\cdot),p(\cdot))$-atom supported on B_2. Next, since

$$\sup_{j_{n_0}\in\mathbb{N}_0}\|a_1^{(j_{n_0})}\|_{L^{p(\cdot)}}\le|B_3|^{-\alpha_3/n},$$

another application of the Banach-Alaoglu theorem yields a subsequence $\{a_1^{(j_{n_1})}\}$ of $\{a_1^{(j_{n_0})}\}$ which converges weak $*$ in $L^{p(\cdot)}$ to a central $(\alpha(\cdot),p(\cdot))$-atom a_1 with support in B_3. Furthermore,

$$\sup_{j_{n_1}\in\mathbb{N}_0}\|a_{-1}^{(j_{n_1})}\|_{L^{p(\cdot)}}\le|B_1|^{-\alpha_1/n}.$$

Similarly, there exists a subsequence $\{a_{-1}^{(j_{n_{-1}})}\}$ of $\{a_{-1}^{(j_{n_1})}\}$ which converges weak $*$ in $L^{p(\cdot)}$ to some $a_{-1}\in L^{p(\cdot)}$, and a_{-1} is a central $(\alpha(\cdot),p(\cdot))$-atom supported on B_1. Repeating the above procedure for each $d\in\mathbb{Z}$, we can find a subsequence $\{a_d^{(j_{n_d})}\}$ of $\{a_d^{(j)}\}$ converging weak $*$ in $L^{p(\cdot)}$ to some $a_d\in L^{p(\cdot)}$ which is a central $(\alpha(\cdot),p(\cdot))$-atom supported on B_{d+2}. By usual diagonal method we obtain a subsequence $\{j_\nu\}$ of $\mathbb{N}_0$ such that for each $d\in\mathbb{Z}$, $\lim_{\nu\to\infty}a_d^{(j_\nu)}=a_d$ in the weak $*$ topology of $L^{p(\cdot)}$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$.

Now we only need to prove that $f=\sum_{d=-\infty}^{\infty}\lambda_d a_d$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$. For each $\varphi\in\mathcal{S}(\mathbb{R}^n)$, note that

$$\operatorname{supp} a_d^{(j_\nu)}\subset(\tilde{A}_{d,\varepsilon}\cup\tilde{A}_{d+1,\varepsilon})\subset(A_{d-1}\cup A_d\cup A_{d+1}\cup A_{d+2}).$$

We have

$$\langle f,\varphi\rangle=\lim_{\nu\to\infty}\sum_{d=-\infty}^{\infty}\lambda_d\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx.$$

See [27] for the details.

Recall that $m=\max\{[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}$. If $d\le 0$, then by Lemmas 1.1 and 1.2 we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx\Big|&=\Big|\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\Big(\varphi(x)-\sum_{|\beta|\le m}\frac{D^\beta\varphi(0)}{\beta!}x^\beta\Big)dx\Big|\\
&\lesssim\int_{\mathbb{R}^n}|a_d^{(j_\nu)}(x)|\cdot|x|^{m+1}dx\lesssim 2^{d(m+1)}\int_{\mathbb{R}^n}|a_d^{(j_\nu)}(x)|dx\\
&\lesssim 2^{d(m+1-\alpha_{d+2})}\|\chi_{B_{d+2}}\|_{L^{p'(\cdot)}}\lesssim 2^{d(m+1-\alpha_{d+2})}\Big(\frac{|B_{d+2}|}{|B_2|}\Big)^{\delta_2}\|\chi_{B_2}\|_{L^{p'(\cdot)}}\\
&\lesssim 2^{d(m+1-\alpha_{d+2}+n\delta_2)}\frac{|B_2|}{|B_0|}\|\chi_{B_0}\|_{L^{p'(\cdot)}}\\
&\lesssim 2^{d(m+1-\alpha_{d+2}+n\delta_2)}\inf\Big\{\lambda>0:\int_{B_0}\lambda^{-p'(x)}dx\le 1\Big\}\\
&\lesssim 2^{d(m+1-\alpha_{d+2}+n\delta_2)}\inf\Big\{1\ge\lambda>0:\int_{B_0}\lambda^{-(p')^+}dx\le 1\Big\}\\
&\approx 2^{d(m+1-\alpha_{d+2}+n\delta_2)}|B_0|^{1/(p')^+}\approx 2^{d(m+1-\alpha_{d+2}+n\delta_2)}.
\end{aligned}$$

If $d>0$, let $k_0\in\mathbb{N}_0$ such that $\min\{k_0+\alpha(0)-n,k_0+\alpha_\infty-n\}>0$, then by Lemmas 1.2 and

1.1 again we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)dx\Big| &\lesssim \int_{\mathbb{R}^n} |a_d^{(j_\nu)}(x)||x|^{-k_0}dx \lesssim 2^{-d(k_0+\alpha_{d+2})}\|\chi_{B_{d+2}}\|_{L^{p'(\cdot)}}\\ &\lesssim 2^{-d(k_0+\alpha_{d+2})}\frac{|B_{d+2}|}{|B_0|}\|\chi_{B_0}\|_{L^{p'(\cdot)}}\\ &\lesssim 2^{-d(k_0+\alpha_{d+2}-n)}\inf\Big\{\lambda>0:\int_{B_0}\lambda^{-p'(x)}dx\le 1\Big\}\\ &\lesssim 2^{-d(k_0+\alpha_{d+2}-n)}\inf\Big\{1\ge\lambda>0:\int_{B_0}\lambda^{-(p')^+}dx\le 1\Big\}\\ &\approx 2^{-d(k_0+\alpha_{d+2}-n)}.\end{aligned}$$

Let

$$\mu_d=\begin{cases} |\lambda_d|2^{d(m+1-\alpha_{d+2}+n\delta_2)}, & d\le 0,\\ |\lambda_d|2^{-d(k_0+\alpha_{d+2}-n)}, & d>0.\end{cases}$$

Then

$$\sum_{d=-\infty}^{\infty}|\mu_d|\lesssim\Big(\sum_{d=-\infty}^{\infty}|\lambda_d|^q\Big)^{1/q}\lesssim\|G_Nf\|_{\dot K^{\alpha(\cdot),q}_{p(\cdot)}}<\infty$$

and

$$|\lambda_d|\Big|\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx\Big|\lesssim|\mu_d|,$$

which implies that

$$\langle f,\varphi\rangle=\sum_{d=-\infty}^{\infty}\lim_{\nu\to\infty}\lambda_d\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)dx=\sum_{d=-\infty}^{\infty}\lambda_d\int_{\mathbb{R}^n}a_d(x)\varphi(x)dx.$$

This establishes the identity we wanted.

To prove the sufficiency, we consider the two cases $0<q\le 1$ and $1<q<\infty$.

If $0<q\le 1$, it suffices to show that for each central $(\alpha(\cdot),p(\cdot))$-atom a,

$$\|G_Na\|_{\dot K^{\alpha(\cdot),q}_{p(\cdot)}}<C,$$

with the constant $C>0$ independent of a. For a fixed central $(\alpha(\cdot),p(\cdot))$-atom a, with supp $a\subset B(0,2^{-4})$. By Proposition 1.1, we can write

$$\begin{aligned}\|G_Na\|^q_{\dot K^{\alpha(\cdot),q}_{p(\cdot)}} &=\sum_{k\le -1}2^{\alpha(0)kq}\|(G_Na)\chi_k\|^q_{L^{p(\cdot)}}+\sum_{k\ge 0}2^{\alpha_\infty kq}\|(G_Na)\chi_k\|^q_{L^{p(\cdot)}}\\ &=\sum_{k\le -1}|B_k|^{\alpha(0)q/n}\|(G_Na)\chi_k\|^q_{L^{p(\cdot)}}+\sum_{k\ge 0}2^{\alpha_\infty q/n}\|(G_Na)\chi_k\|^q_{L^{p(\cdot)}}\\ &:=I+II.\end{aligned}$$

By the $L^{p(\cdot)}-$boundedness of the grand maximal operator G_N we have

$$I\le \|(G_N a)\|^q_{L^{p(\cdot)}}\sum_{k\le -1}|B_k|^{\alpha(0)q/n}\lesssim \|a\|^q_{L^{p(\cdot)}}\sum_{k\le -1}|B_k|^{\alpha(0)q/n}\le C.$$

To estimate II, we need a pointwise estimate for $G_N a(x)$ on A_k. Let $\phi\in\mathcal{A}_N$, $m\in\mathbb{N}$ such that $\alpha_k-n\delta_2<m+1$. Denote by P_m the m-th order Taylor series expansion. If $|x-y|<t$, then from the vanishing moment condition of a we have

$$\begin{aligned}|a*\phi_t(y)|&=t^{-n}\Big|\int_{\mathbb{R}^n}a(z)\Big(\phi\Big(\frac{y-z}{t}\Big)-P_m\Big(\frac{y}{t}\Big)\Big)dz\Big|\\&\lesssim t^{-n}\int_{\mathbb{R}^n}|a(z)|\Big|\frac{z}{t}\Big|^{m+1}(1+|y-\theta z/t|)^{-(n+m+1)}dz\\&\lesssim\int_{\mathbb{R}^n}|a(z)||z|^{m+1}(t+|y-\theta z|)^{-(n+m+1)}dz,\end{aligned}$$

where $0<\theta<1$. Since $x\in A_k$ for $k\ge 0$, we have $|x|\ge 2^{-2}$. From $|x-y|<t$ and $|z|<2^{-3}$, we have

$$t+|y-\theta z|\ge |x-y|+|y-\theta z|\ge |x|-|z|\ge |x|/2.$$

Thus,

$$\begin{aligned}|a*\phi_t(y)|&\lesssim\int_{\mathbb{R}^n}|a(z)||z|^{m+1}(|x-y|+|y-\theta z|)^{-(n+m+1)}dz\\&\lesssim 2^{-4(m+1)}|x|^{-(n+m+1)}\int_{\mathbb{R}^n}|a(z)|dz\\&\lesssim 2^{-4(m+1)}|x|^{-(n+m+1)}|B_{-4}|^{-\alpha(0)/n}\|\chi_{B_{-4}}\|_{L^{p'(\cdot)}}.\end{aligned}$$

Therefore, we have

$$G_N a(x)\lesssim 2^{-4(m+1)-k(m+n+1)}|B_{-4}|^{-\alpha(0)/n}\|\chi_{B_{-4}}\|_{L^{p'(\cdot)}},\quad x\in A_k,\quad \text{and}\quad k\ge 0.$$

So by Lemmas 1.2 and 1.3, we have

$$\begin{aligned}II&\lesssim\sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\cdot 2^{q[-4(m+1)-k(n+m+1)]}|B_{-4}|^{-\alpha(0)q/n}\|\chi_{B_{-4}}\|^q_{L^{p'(\cdot)}}\|\chi_{B_k}\|^q_{L^{p(\cdot)}}\\&\lesssim\sum_{k\ge 0}2^{\alpha_\infty kq}\cdot 2^{q[-4(m+1)-k(n+m+1)]}\cdot 2^{4\alpha(0)q}\|\chi_{B_{-4}}\|^q_{L^{p'(\cdot)}}(|B_k|\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}})^q\\&\lesssim\sum_{k\ge 0}2^{q(k+4)[\alpha_\infty-(m+1)]}\Big(\frac{|B_{-4}|}{|B_k|}\Big)^{q\delta_2}\\&\lesssim\sum_{k\ge 0}2^{q(k+4)[\alpha_\infty-(m+1)-n\delta_2]}\\&<\infty.\end{aligned}$$

This proves the desired estimate for the case $0<q\le 1$.

If $1<q<\infty$, write

$$\begin{aligned}
\|G_N f\|^q_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}} &= \sum_{k\le -1} 2^{\alpha(0)kq}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}} + \sum_{k\ge 0} 2^{\alpha_\infty kq}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}} \\
&= \sum_{k\le -1} |B_k|^{\alpha(0)q/n}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}} + \sum_{k\ge 0} 2^{\alpha_\infty q/n}\|(G_N f)\chi_k\|^q_{L^{p(\cdot)}} \\
&\lesssim \sum_{k\le -1} |B_k|^{\alpha(0)q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l|\|a_l\|_{L^{p(\cdot)}}\Big)^q + \sum_{k\le -1} |B_k|^{\alpha(0)q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|\|(G_N a_l)\chi_k\|_{L^{p(\cdot)}}\Big)^q \\
&\quad + \sum_{k\ge 0} |B_k|^{\alpha_\infty q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l|\|a_l\|_{L^{p(\cdot)}}\Big)^q + \sum_{k\ge 0} |B_k|^{\alpha_\infty q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|\|(G_N a_l)\chi_k\|_{L^{p(\cdot)}}\Big)^q \\
&:= III+IV+V+VI.
\end{aligned}$$

Next we shall estimate III, IV, V, VI, respectively. First, we estimate III.

$$\begin{aligned}
III &= \sum_{k\le -1} |B_k|^{\alpha(0)q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l|\|a_l\|_{L^{p(\cdot)}}\Big)^q \\
&\lesssim \sum_{k\le -1} |B_k|^{\alpha(0)q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l||B_l|^{-\alpha_l/n}\Big)^q \\
&= \sum_{k\le -1} \Big(\sum_{l=k}^{-1}|\lambda_l|2^{-\alpha(0)l+\alpha(0)k} + \sum_{l=0}^{\infty}|\lambda_l|2^{-\alpha_\infty l+\alpha(0)k}\Big)^q \\
&\lesssim \sum_{k\le -1} \Big(\sum_{l=k}^{-1}|\lambda_l|2^{-\alpha(0)l+\alpha(0)k}\Big)^q + \sum_{k\le -1} \Big(\sum_{l=0}^{\infty}|\lambda_l|2^{-\alpha_\infty l+\alpha(0)k}\Big)^q \\
&\lesssim \sum_{k\le -1} \Big(\sum_{l=k}^{-1}|\lambda_l|^q 2^{(k-l)\alpha(0)q/2}\Big)\Big(\sum_{l=k}^{-1}2^{(k-l)\alpha(0)q'/2}\Big)^{q/q'} \\
&\qquad + \sum_{k\le -1} 2^{\alpha(0)k}\Big(\sum_{l=0}^{\infty}|\lambda_l|^q 2^{-\alpha_\infty lq/2}\Big)\Big(\sum_{l=0}^{\infty}2^{-\alpha_\infty lq'/2}\Big)^{q/q'} \\
&\lesssim \sum_{k\le -1}\sum_{l=k}^{-1}|\lambda_l|^q 2^{(k-l)\alpha(0)q/2} + \sum_{l=0}^{\infty}|\lambda_l|^q 2^{-\alpha_\infty lq/2} \\
&\lesssim \sum_{l\le -1}\sum_{k=\infty}^{l}|\lambda_l|^q 2^{(k-l)\alpha(0)q/2} + \sum_{l=0}^{\infty}|\lambda_l|^q \\
&\lesssim \sum_{l\le -1}|\lambda_l|^q + \sum_{l=0}^{\infty}|\lambda_l|^q \lesssim \sum_{l=-\infty}^{\infty}|\lambda_l|^q.
\end{aligned}$$

Second, we estimate IV. As in the argument for II, we can obtain that

$$IV = \sum_{k\le -1} |B_k|^{\alpha(0)q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|\|(G_N a_l)\chi_k\|_{L^{p(\cdot)}}\Big)^q$$

$$
\begin{aligned}
&\lesssim \sum_{k\le -1}|B_k|^{\alpha(0)q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|2^{l(m+1)-k(n+m+1)}\times|B_l|^{-\alpha_l/n}\|\chi_{B_l}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\Big)^q\\
&\lesssim \sum_{k\le -1}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)}\Big)^q\\
&\lesssim \sum_{k\le -1}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|2^{(l-k)(m+1+n\delta_2-\alpha(0))}\Big)^q\\
&\lesssim \sum_{k\le -1}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|^q2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2}\Big)\Big(\sum_{l=-\infty}^{k-1}2^{(l-k)(m+1+n\delta_2-\alpha(0))q'/2}\Big)^{q/q'}\\
&\lesssim \sum_{k\le -1}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|^q2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2}\Big)\\
&=\sum_{l\le -2}\sum_{k=l+1}^{-1}|\lambda_l|^q2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2}\lesssim \sum_{l\le -1}|\lambda_l|^q.
\end{aligned}
$$

Third, we estimate V.

$$
\begin{aligned}
V&=\sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l|\|a_l\|_{L^{p(\cdot)}}\Big)^q\\
&\lesssim \sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l||B_l|^{-\alpha_l/n}\Big)^q\\
&\lesssim \sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l|^q|B_l|^{-\alpha_l q/(2n)}\Big)\Big(\sum_{l=k}^{\infty}|B_l|^{-\alpha_l q'/(2n)}\Big)^{q/q'}\\
&\lesssim \sum_{k\ge 0}\sum_{l=k}^{\infty}|B_k|^{\alpha_\infty q/(2n)}|\lambda_l|^q|B_l|^{-\alpha_\infty q/(2n)}\\
&\lesssim \sum_{l\ge 0}|\lambda_l|^q\sum_{k=0}^{l}2^{(k-l)\alpha_\infty q/2}\lesssim \sum_{k\ge 0}|\lambda_l|^q.
\end{aligned}
$$

Finally, we estimate VI.

$$
\begin{aligned}
VI&=\sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|\|(G_Na_l)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&\lesssim \sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|2^{l(m+1)-k(n+m+1)}\times|B_l|^{-\alpha_l/n}\|\chi_{B_l}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\Big)^q\\
&\lesssim \sum_{k\ge 0}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|2^{(l-k)(m+1+n\delta_2)-l\alpha_l+k\alpha_\infty}\Big)^q
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \sum_{k\geq 0}\Big(\sum_{l=-\infty}^{-1}|\lambda_l|2^{(l-k)(m+1+n\delta_2)-l\alpha(0)+k\alpha_\infty}\Big)^q+\sum_{k\geq 0}\Big(\sum_{l=0}^{k-1}|\lambda_l|2^{(l-k)(m+1+n\delta_2-\alpha_\infty)}\Big)^q\\
&\lesssim \sum_{k\geq 0}2^{qk[\alpha_\infty-(m+1+n\delta_2)]}\Big(\sum_{l=-\infty}^{-1}|\lambda_l|2^{l(m+1+n\delta_2-\alpha(0))}\Big)^q\\
&\quad+\sum_{k\geq 0}\Big(\sum_{l=0}^{k-1}|\lambda_l|^q2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2}\Big)\Big(\sum_{l=0}^{k-1}2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q'/2}\Big)^{q/q'}\\
&\lesssim \Big(\sum_{l=-\infty}^{-1}|\lambda_l|^q2^{l(m+1+n\delta_2-\alpha(0))q/2}\Big)\Big(\sum_{l=-\infty}^{-1}2^{l(m+1+n\delta_2-\alpha(0))q'/2}\Big)^{q/q'}\\
&\quad+\sum_{k\geq 0}\sum_{l=0}^{k-1}|\lambda_l|^q2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2}\\
&\lesssim \sum_{l=-\infty}^{-1}|\lambda_l|^q2^{l(m+1+n\delta_2-\alpha(0))q/2}+\sum_{l\geq 0}\sum_{k=l+1}^{\infty}|\lambda_l|^q2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2}\\
&\lesssim \sum_{l=-\infty}^{-1}|\lambda_l|^q+\sum_{l\geq 0}|\lambda_l|^q=\sum_{l=-\infty}^{\infty}|\lambda_l|^q.
\end{aligned}$$

Thus, we finish the proof of Theorem 2.1. □

Remark 2.3. If $f\in H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}\cap\mathcal{S}(\mathbb{R}^n)$, we can replace $f^{(j)}$ by f in the proof of the necessity, and have

$$f(x)=\sum_{k=-\infty}^{\infty}\lambda_k a_k(x)+\mu_k b_k(x),$$

where

$$\begin{aligned}
&\|a_k\|_{L^{p(\cdot)}}\leq|B(0,2^{k+1})|^{-\alpha_{k+1}/n}, && \|b_k\|_{L^{p(\cdot)}}\leq|B(0,2^{k+2})|^{-\alpha_{k+2}/n},\\
&\mathrm{supp}\,a_k\subset\tilde{A}_{k,\varepsilon}, && \mathrm{supp}\,b_k\subset\tilde{A}_{k,\varepsilon}\cup\subset\tilde{A}_{k+1,\varepsilon},
\end{aligned}$$

and

$$0\leq\lambda_k,\quad \mu_k\lesssim 2^{\alpha_k k}\sum_{j=k-1}^{k+1}\|G_N(f)\chi_j\|_{L^{p(\cdot)}},$$

with $N>n+\alpha_k+1$.

As an application of the atomic decompositions, we shall prove the following result.

Theorem 2.2. *Let* $0<q<\infty$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ *and non-negative integer* $s=\max\{[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}$, *where* $n\delta_2\leq\alpha(0)$, $\alpha_\infty<\infty$ *and* δ_2 *is defined in Lemma 1.2. If a sublinear operator T satisfies that*

(i) T is bounded on $L^{p(\cdot)}$;

(ii) There exists a constant $\delta>0$ such that $s+\delta>\max\{\alpha(0)-n\delta_2,\alpha_\infty-n\delta_2\}$, and for any compact support function f with

$$\int_{\mathbb{R}^n} f(x)x^\beta dx=0, \quad |\beta|\le s,$$

Tf satisfies the size condition

$$|Tf(x)|\le C(\mathrm{diam}(\mathrm{supp} f))^{s+\delta}|x|^{-(n+s+\delta)}\|f\|_1, \quad \textit{if } \mathrm{dist}(x,\mathrm{supp} f)\ge |x|/2. \tag{2.1}$$

Then there exists a constant C such that

$$\|Tf\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\le C\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}} \quad \textit{and} \quad \|Tf\|_{K^{\alpha(\cdot),q}_{p(\cdot)}}\le C\|f\|_{HK^{\alpha(\cdot),q}_{p(\cdot)}},$$

for $f\in H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $f\in HK^{\alpha(\cdot),q}_{p(\cdot)}$, respectively.

Proof. It suffices to prove the homogeneous case. Suppose $f\in H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$. By Theorem 2.1, $f=\sum_{j=-\infty}^{\infty}\lambda_j b_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, where each b_j is a central $(\alpha(\cdot),q(\cdot))-$atom with support contained in B_j and

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\approx \inf\Big(\sum_{j=-\infty}^{\infty}|\lambda_j|^q\Big)^{1/q}.$$

Therefore, we get

$$\begin{aligned}
\|Tf\|^q_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}&=\sum_{k\le -1}2^{\alpha(0)kq}\|(Tf)\chi_k\|^q_{L^{p(\cdot)}}+\sum_{k\ge 0}2^{\alpha_\infty kq}\|(Tf)\chi_k\|^q_{L^{p(\cdot)}}\\
&\lesssim \sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q+\sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&\quad+\sum_{k\ge 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q+\sum_{k\ge 0}2^{\alpha_\infty kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&=:I_1+I_2+I_3+I_4.
\end{aligned}$$

Next we shall estimate I_1, I_2, I_3, I_4, respectively.

First, we estimate I_1. By the boundedness of T in $L^{p(\cdot)}$, we have

$$\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\le \|b_j\|_{L^{p(\cdot)}}\le |B_j|^{-\alpha_j/n}=2^{-\alpha_j j}.$$

Therefore, when $0<q\le 1$, we get

$$\begin{aligned} I_1&=\sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\lesssim \sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|2^{-\alpha_j j}\Big)^q\\ &\lesssim \sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j=k}^{-1}|\lambda_j|^q 2^{-\alpha(0)jq}+\sum_{j\ge 0}^{\infty}|\lambda_j|^q 2^{-\alpha_\infty jq}\Big)\\ &\lesssim \sum_{k\le -1}\sum_{j=k}^{-1}|\lambda_j|^q 2^{\alpha(0)(k-j)q}+\sum_{k\le -1}2^{\alpha(0)kq}\sum_{j\ge 0}^{\infty}|\lambda_j|^q 2^{-\alpha_\infty jq}\\ &\lesssim \sum_{j\le -1}|\lambda_j|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}+\sum_{j\ge 0}^{\infty}|\lambda_j|^q\lesssim \sum_{j=-\infty}^{\infty}|\lambda_j|^q. \end{aligned}$$

When $1<q<\infty$, let $1/q+1/q'=1$ and we obtain

$$\begin{aligned} I_1&=\sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\lesssim \sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|2^{-\alpha_j j}\Big)^q\\ &\lesssim \sum_{k\le -1}\Big(\sum_{j=k}^{-1}|\lambda_j|2^{\alpha(0)(k-j)}\Big)^q+\sum_{k\le -1}2^{\alpha(0)kq}\Big(\sum_{j\ge 0}^{\infty}|\lambda_j|2^{-\alpha_\infty j}\Big)^q\\ &\lesssim \sum_{k\le -1}\Big(\sum_{j=k}^{-1}|\lambda_j|^q 2^{\alpha(0)(k-j)q/2}\Big)\Big(\sum_{j=k}^{-1}2^{\alpha(0)(k-j)q'/2}\Big)^{q/q'}\\ &\quad+\Big(\sum_{j\ge 0}^{\infty}|\lambda_j|^q 2^{-\alpha_\infty jq/2}\Big)\Big(\sum_{j\ge 0}^{\infty}2^{-\alpha_\infty jq'/2}\Big)^{q/q'}\\ &\lesssim \sum_{j\le -1}|\lambda_j|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q/2}+\sum_{j\ge 0}|\lambda_j|^q\lesssim \sum_{j=-\infty}^{\infty}|\lambda_j|^q. \end{aligned}$$

So, we have

$$I_1\lesssim \sum_{j=-\infty}^{\infty}|\lambda_j|^q.$$

Second, we estimate I_2. By (2.1) and Lemma 1.1, we get

$$\begin{aligned} |Tb_j(x)|&\lesssim |x|^{-(n+s+\delta)}2^{j(s+\delta)}\int_{B_j}|b_j(y)|dy\\ &\lesssim 2^{-k(n+s+\delta)}2^{j(s+\delta)}\|b_j\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\\ &\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}. \end{aligned}$$

So by Lemma 1.1 and 1.2, we have

$$\begin{aligned}\|(Tb_j)\chi_k\|_{L^{p(\cdot)}} &\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\\ &\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta)}2^{-kn}\left(|B_k|\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}}\right)\|\chi_{B_j}\|_{L^{p'(\cdot)}}\\ &\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}}\\ &\lesssim 2^{(s+\delta+n\delta_2)(j-k)-j\alpha_j}.\end{aligned}$$

Therefore, when $0<q\leq 1$, by $n\delta_2\leq\alpha(0)<s+\delta+n\delta_2$ we get

$$\begin{aligned}I_2&=\sum_{k\leq -1}2^{\alpha(0)kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\ &\lesssim\sum_{k\leq -1}2^{\alpha(0)kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q}\Big)\\ &=\sum_{j\leq -2}|\lambda_j|^q\sum_{k=j+1}^{-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q}\\ &\lesssim\sum_{j\leq -1}|\lambda_j|^q.\end{aligned}$$

When $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2\leq\alpha(0)<s+\delta+n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}I_2&=\sum_{k\leq -1}2^{\alpha(0)kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\ &\lesssim\sum_{k\leq -1}2^{\alpha(0)kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|2^{(s+\delta+n\delta_2)(j-k)-j\alpha(0)}\Big)^q\\ &\lesssim\sum_{k\leq -1}2^{\alpha(0)kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\Big)\Big(\sum_{j=-\infty}^{k-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q'/2}\Big)^{q/q'}\\ &\lesssim\sum_{k\leq -1}2^{\alpha(0)kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\Big)\\ &=\sum_{j\leq -2}|\lambda_j|^q\sum_{k=j+1}^{-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\\ &\lesssim\sum_{j\leq -1}|\lambda_j|^q.\end{aligned}$$

So, we have

$$I_2\lesssim\sum_{j\leq -1}|\lambda_j|^q.$$

Third, we estimate I_3. When $0<q\leq 1$, by the boundedness of T in $L^{p(\cdot)}$, we have

$$\begin{aligned}I_3&=\sum_{k\geq 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\&\leq\sum_{k\geq 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|^q\|(Tb_j)\chi_k\|^q_{L^{p(\cdot)}}\Big)\\&\lesssim\sum_{k\geq 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|^q\|(Tb_j)|B_j|^{-\alpha_j q/n}\Big)\\&=\sum_{j\geq 0}|\lambda_j|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty q}\lesssim\sum_{j\geq 0}|\lambda_j|^q.\end{aligned}$$

When $1<q<\infty$, by the boundedness of T in $L^{p(\cdot)}$ and Hölder's inequality, we have

$$\begin{aligned}I_3&=\sum_{k\geq 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\&\leq\sum_{k\geq 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|^q\|(Tb_j)\chi_k\|^{q/2}_{L^{p(\cdot)}}\Big)\times\Big(\sum_{j=k}^{\infty}\|(Tb_j)\chi_k\|^{q'/2}_{L^{p(\cdot)}}\Big)^{q/q'}\\&\lesssim\sum_{k\geq 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|^q\|b_j\|^{q/2}_{L^{p(\cdot)}}\Big)\Big(\sum_{j=k}^{\infty}\|b_j\|^{q'/2}_{L^{p(\cdot)}}\Big)^{q/q'}\\&\lesssim\sum_{k\geq 0}2^{\alpha_\infty kq}\Big(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-\alpha_j q/(2n)}\Big)\Big(\sum_{j=k}^{\infty}|B_j|^{-\alpha_j q'/(2n)}\Big)^{q/q'}\\&\lesssim\sum_{k\geq 0}2^{\alpha_\infty kq/2}\Big(\sum_{j=k}^{\infty}|\lambda_j|^q|B_j|^{-\alpha_j q/(2n)}\Big)\\&=\sum_{j\geq 0}|\lambda_j|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty q/2}\lesssim\sum_{j\geq 0}|\lambda_j|^q.\end{aligned}$$

So, we have

$$I_3\lesssim\sum_{j\geq 0}|\lambda_j|^q.$$

Last, we estimate I_4. By (2.1) and Lemma 1.1, we get

$$\begin{aligned}|Tb_j(x)|&\lesssim|x|^{-(n+s+\delta)}2^{j(s+\delta)}\int_{B_j}|b_j(y)|dy\\&\lesssim 2^{-k(n+s+\delta)}2^{j(s+\delta)}\|b_j\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\\&\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}.\end{aligned}$$

So by Lemmas 1.1 and 1.2, we have

$$\begin{aligned}\|(Tb_j)\chi_k\|_{L^{p(\cdot)}} &\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\\
&\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta)}2^{-kn}\Big(|B_k|\|\chi_{B_k}\|^{-1}_{L^{p'(\cdot)}}\Big)\|\chi_{B_j}\|_{L^{p'(\cdot)}}\\
&\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}}\\
&\lesssim 2^{(s+\delta+n\delta_2)(j-k)-j\alpha_j}.\end{aligned}$$

Therefore, when $0<q\le 1$, by $n\delta_2\le\alpha(0)$, $\alpha_\infty<s+\delta+n\delta_2$ we get

$$\begin{aligned}I_4&=\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&\lesssim\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|^q2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_j]q}\Big)\\
&=\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=-\infty}^{-1}|\lambda_j|^q2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q}\Big)+\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=0}^{k-1}|\lambda_j|^q2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_\infty]q}\Big)\\
&\lesssim\sum_{k\ge0}2^{[\alpha_\infty-(s+\delta+n\delta_2)]kq}\sum_{j=-\infty}^{-1}|\lambda_j|^q2^{(s+\delta+n\delta_2-\alpha(0))jq}+\sum_{j\ge0}|\lambda_j|^q\sum_{k=j+1}^{\infty}2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q}\\
&\lesssim\sum_{j=-\infty}^{-1}|\lambda_j|^q+\sum_{j\ge0}|\lambda_j|^q=\sum_{j=-\infty}^{\infty}|\lambda_j|^q.\end{aligned}$$

When $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2\le\alpha(0)$, $\alpha_\infty<s+\delta+n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}I_4&=\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\
&\lesssim\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=-\infty}^{k-1}|\lambda_j|2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_j]}\Big)^q\\
&\lesssim\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=-\infty}^{-1}|\lambda_j|2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]}\Big)^q+\sum_{k\ge0}2^{\alpha_\infty kq}\Big(\sum_{j=0}^{k-1}|\lambda_j|2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_\infty]}\Big)^q\\
&\lesssim\sum_{k\ge0}2^{[\alpha_\infty-(s+\delta+n\delta_2)]kq}\Big(\sum_{j=-\infty}^{-1}|\lambda_j|2^{(s+\delta+n\delta_2-\alpha(0))j}\Big)^q+\sum_{k\ge0}\Big(\sum_{j=0}^{k-1}|\lambda_j|2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)}\Big)^q\\
&\lesssim\Big(\sum_{j=-\infty}^{-1}|\lambda_j|^q2^{(s+\delta+n\delta_2-\alpha(0))jq/2}\Big)\times\Big(\sum_{j=-\infty}^{-1}2^{(s+\delta+n\delta_2-\alpha(0))jq'/2}\Big)^{q/q'}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k\geq 0}\Big(\sum_{j=0}^{k-1}|\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q/2}\Big)\times\Big(\sum_{j=0}^{k-1}2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q'/2}\Big)^{q/q'}\\
\lesssim&\sum_{j=-\infty}^{-1}|\lambda_j|^q 2^{(s+\delta+n\delta_2-\alpha(0))jq/2}+\sum_{k\geq 0}\sum_{j=0}^{k-1}|\lambda_j|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q/2}\\
\lesssim&\sum_{j=-\infty}^{-1}|\lambda_j|^q+\sum_{j\geq 0}|\lambda_j|^q\sum_{k=j+1}^{\infty}2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q/2}\\
\lesssim&\sum_{j=-\infty}^{-1}|\lambda_j|^q+\sum_{j\geq 0}|\lambda_j|^q=\sum_{j=-\infty}^{\infty}|\lambda_j|^q.
\end{aligned}
$$

So, we have

$$I_4\lesssim\sum_{j=-\infty}^{\infty}|\lambda_j|^q.$$

Combining the estimates of I_1-I_4, we have

$$\|Tf\|^q_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\lesssim\|f\|^q_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}.$$

This finishes the proof of Theorem 2.2. □

Definition 2.3. Let $K(x,y)$ be a locally integrable function on $\mathbb{R}^n\times\mathbb{R}^n\setminus\{x=y\}$. K is a standard kernel if there exists $\delta\in(0,1]$ and $C>0$, such that

$$
\begin{aligned}
&|K(x,y)|\leq\frac{C}{|x-y|^n}, && x\neq y,\\
&|K(x,y)-K(x,y')|\leq C\frac{|y-y'|^\delta}{|x-y|^{n+\delta}}, && |y-y'|\leq\frac{1}{2}|x-y|,\\
&|K(x,y)-K(x',y)|\leq C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}}, && |x-x'|\leq\frac{1}{2}|x-y|.
\end{aligned}
$$

A linear operator $T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ is called a Calderón-Zygmund operator with standard kernel K if

(i) T can be extended to be a L^2-boundedness operator;

(ii) for any $f\in L^2$ with compact support and almost everywhere $x\notin\operatorname{supp}f$,

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy.$$

It is well known that a Calderón-Zygmund operator is also bounded in $L^{p(\cdot)}$ for any $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, for example, see [5].

Theorem 2.3. *Let T be an Calderón-Zygmund operator with standard kernel $K(x,y)$ and $0<\delta\le 1$ is a constant associated with the standard kernel. If $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ and $n\delta_2\le\alpha(0)$, $\alpha_\infty<n\delta_2+\delta$, where δ_2 is defined in Lemma 1.2, then for any $0<q<\infty$, there exists a constant C such that*

$$\|Tf\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}}\le C\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}} \quad \text{and} \quad \|Tf\|_{K_{p(\cdot)}^{\alpha(\cdot),q}}\le C\|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}},$$

for $f\in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}$ and $f\in HK_{p(\cdot)}^{\alpha(\cdot),q}$, respectively.

Proof. It is easy to know that $s=\max\{[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}=0$ when $n\delta_2\le\alpha(0)$, $\alpha_\infty<n\delta_2+\delta$. The proof here can be directly obtained by Theorem 2.2. □

3 The molecular characterizations and their applications

In this section, we shall consider the molecular decompositions of $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}$ and $HK_{p(\cdot)}^{\alpha(\cdot),q}$. Before stating our result, we give the notation of molecule firstly.

Definition 3.1. Let $0<q<\infty$, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ and non-negative integer $s\ge\max\{[\alpha(0)-n\delta_1],[\alpha_\infty-n\delta_1]\}$, where $n\delta_1\le\alpha(0),\alpha_\infty<\infty$ and δ_1 is defined in Lemma 1.2. Set $\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\}$ and $b=1-\delta_1+\varepsilon$. Moreover, for any $l\in\mathbb{Z}$, when $l<0$, $\alpha_l:=\alpha(0)$ and $a:=1-\delta_1-\alpha(0)/n+\varepsilon$; when $l\ge 0$, $\alpha_l:=\alpha_\infty$ and $a:=1-\delta_1-\alpha_\infty/n+\varepsilon$.

(i) A function $M_l\in L^{p(\cdot)}$ with $l\in\mathbb{Z}$ is said to be a dyadic central $(\alpha(\cdot),p(\cdot);s,\varepsilon)_l$-molecule if it satisfies

(i_1) $\|M_l\|_{L^{p(\cdot)}}\le 2^{-l\alpha_l}$;

(i_2) $\mathcal{R}_{p(\cdot)}(M_l)=\|M_l\|_{L^{p(\cdot)}}^{a/b}\||\cdot|^{nb}M_l\|_{L^{p(\cdot)}}^{1-a/b}<\infty$;

(i_3) $\int_{\mathbb{R}^n}M_l(x)x^\beta dx=0$, for any multi index β with $|\beta|\le s$.

(ii) A function $M_l\in L^{p(\cdot)}$ with $l\in\mathbb{N}_0$ is said to be a dyadic central $(\alpha(\cdot),p(\cdot);s,\varepsilon)_l$-molecule of restricted type if it satisfies above (i_2) and (i_3) and

(i'_1) $\|M_l\|_{L^{p(\cdot)}}\le 2^{-l\alpha_\infty}$.

Definition 3.2. Let $0<q<\infty$, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ and non-negative integer $s\ge\max\{[\alpha(0)-n\delta_1],[\alpha_\infty-n\delta_1]\}$, where $n\delta_1\le\alpha(0)$, $\alpha_\infty<\infty$ and δ_1 is defined in Lemma 1.2. Set

$$\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\} \quad \text{and} \quad b=1-\delta_1+\varepsilon.$$

Moreover, for any function $M\in L^{p(\cdot)}$, when $\|M\|_{L^{p(\cdot)}}>1$, $a:=1-\delta_1-\alpha(0)/n+\varepsilon$; when $\|M\|_{L^{p(\cdot)}}\le 1$, $a:=1-\delta_1-\alpha_\infty/n+\varepsilon$.

(i) M is said to be a central $(\alpha(\cdot),p(\cdot);s,\varepsilon)$-molecule, if it satisfies

(i_4) $\mathcal{R}_{p(\cdot)}(M)=\|M\|_{L^{p(\cdot)}}^{a/b}\||\cdot|^{nb}M\|_{L^{p(\cdot)}}^{1-a/b}<\infty$;

(i_5) $\int_{\mathbb{R}^n} M(x)x^\beta dx=0$, for any β with $|\beta|\le s$.

(ii) M is said to be a central $(\alpha(\cdot),p(\cdot);s,\varepsilon)$-molecule of restricted, if it satisfies (i_4), (i_5) above and

(i_6) $\|M\|_{L^{p(\cdot)}}\le 1$.

Next lemma shows that molecules are a generalization of atoms.

Lemma 3.1. *Let* $0<q<\infty$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ *and non-negative integer* $s\ge\max\{[\alpha(0)-n\delta_1],[\alpha_\infty-n\delta_1],[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}$, *where* $\max\{n\delta_1,n\delta_2\}\le\alpha(0)$, $\alpha_\infty<\infty$ *and* δ_1, δ_2 *is defined in Lemma 1.2. Set* $\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\}$ *and* $b:=1-\delta_1+\varepsilon$. *Moreover, for any function* $M\in L^{p(\cdot)}$, *when* $\|M\|_{L^{p(\cdot)}}>1$, $a:=1-\delta_1-\alpha(0)/n+\varepsilon$; *when* $\|M\|_{L^{p(\cdot)}}\le 1$, $a:=1-\delta_1-\alpha_\infty/n+\varepsilon$.

(i) If M is a central $(\alpha(\cdot),p(\cdot))$*-atom, then M is a central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)$*-molecule such that* $\mathcal{R}_{p(\cdot)}(M)\le C$ *with C independent of M.*

(ii) If M is a central $(\alpha(\cdot),p(\cdot))$*-atom of restricted type, then M is a central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)$*-molecule of restricted type such that* $\mathcal{R}_{p(\cdot)}(M)\le C$ *with C independent of M.*

Proof. We only prove (i). (ii) can be proved in the similar way.

Let M is a $(\alpha(\cdot),p(\cdot))$-atom with support on a ball $B(0,r)$, then we get

$$\|M\|_{L^{p(\cdot)}}^{a/b}\||\cdot|^{nb}M\|_{L^{p(\cdot)}}^{1-a/b}\le r^{nb(1-a/b)}\|M\|_{L^{p(\cdot)}}\lesssim r^{\alpha_r}r^{-\alpha_r}\lesssim 1. \qquad \square$$

Now we give the molecular decompositions of Herz type Hardy spaces with variable exponents.

Theorem 3.1. *Let* $0<q<\infty$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ *and non-negative integer* $s\ge\max\{[\alpha(0)-n\delta_1],[\alpha_\infty-n\delta_1],[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}$, *where* $\max\{n\delta_1,n\delta_2\}\le\alpha(0)$, $\alpha_\infty<\infty$ *and* δ_1, δ_2 *as in Lemma 1.2. Set* $\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\}$ *and* $b:=1-\delta_1+\varepsilon$. *Moreover, for any* $k\in\mathbb{Z}$, *when* $k<0$, $\alpha_k:=\alpha(0)$ *and* $a:=1-\delta_1-\alpha(0)/n+\varepsilon$; *when* $k\ge 0$, $\alpha_k:=\alpha_\infty$ *and* $a:=1-\delta_1-\alpha_\infty/n+\varepsilon$.

(i) $f\in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}$ *if and only if f can be represented as* $f=\sum_{k=-\infty}^{\infty}\lambda_k M_k$ *in the sense of* $\mathcal{S}'(\mathbb{R}^n)$, *where each* M_k *is a dyadic central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)_k$*-molecule, and* $\sum_{k=-\infty}^{\infty}|\lambda_k|^q<\infty$. *Moreover*

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}}\approx\inf\Big(\sum_{k=-\infty}^{\infty}|\lambda_k|^q\Big)^{1/q},$$

where the infimum is taken over all above decompositions of f.

(ii) $f\in HK_{p(\cdot)}^{\alpha(\cdot),q}$ *if and only if* $f=\sum_{k=0}^{\infty}\lambda_k M_k$ *in the sense of* $\mathcal{S}'(\mathbb{R}^n)$, *where each* M_k *is a dyadic central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)_k$*-molecule of restricted type, and* $\sum_{k=0}^{\infty}|\lambda_k|^q<\infty$. *Moreover*

$$\|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}}\approx\inf\Big(\sum_{k=0}^{\infty}|\lambda_k|^q\Big)^{1/q},$$

where the infimum is taken over all above decompositions of f.

Theorem 3.2. *Let* $0<q\le 1$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ *and non-negative integer* $s\ge\max\{[\alpha(0)-n\delta_1],[\alpha_\infty-n\delta_1],[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}$, *where* $\max\{n\delta_1,n\delta_2\}\le\alpha(0)$, $\alpha_\infty<\infty$ *and* δ_1, δ_2 *is defined in Lemma 1.2. Set* $\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\}$ *and* $b:=1-\delta_1+\varepsilon$. *Moreover, for any function* $M\in L^{p(\cdot)}$, *when* $\|M\|_{L^{p(\cdot)}}>1$, $a:=1-\delta_1-\alpha(0)/n+\varepsilon$; *when* $\|M\|_{L^{p(\cdot)}}\le 1$, $a:=1-\delta_1-\alpha_\infty/n+\varepsilon$.

(i) $f\in H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}$ *if and only if* f *can be represented as* $f=\sum_{k=1}^\infty\lambda_kM_k$ *in the sense of* $\mathcal{S}'(\mathbb{R}^n)$, *where each* M_k *is a central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)$*-molecule, and* $\sum_{k=1}^\infty|\lambda_k|^q<\infty$.

Moreover

$$\|f\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}}\approx\inf\Big(\sum_{k=1}^\infty|\lambda_k|^q\Big)^{1/q},$$

where the infimum is taken over all above decompositions of f.

(ii) $f\in HK_{p(\cdot)}^{\alpha(\cdot),q}$ *if and only if* $f=\sum_{k=1}^\infty\lambda_kM_k$ *in the sense of* $\mathcal{S}'(\mathbb{R}^n)$, *where each* M_k *is a central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)$*-molecule of restricted type, and* $\sum_{k=1}^\infty|\lambda_k|^q<\infty$.

Moreover

$$\|f\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}}\approx\inf\Big(\sum_{k=1}^\infty|\lambda_k|^q\Big)^{1/q},$$

where the infimum is taken over all above decompositions of f.

By Theorem 2.1, Remark 2.3 and Lemma 3.1, we know that Theorems 3.1 and 3.2 can be obtained from the following lemma.

Lemma 3.2. *Let* $0<q<\infty$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ *and non-negative integer* $s\ge\max\{[\alpha(0)-n\delta_1],[\alpha_\infty-n\delta_1],[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}$, *where* $\max\{n\delta_1,n\delta_2\}\le\alpha(0)$, $\alpha_\infty<\infty$ *and* δ_1, δ_2 *is defined in Lemma 1.2. Set* $\varepsilon>\max\{s/n,\alpha(0)/n+\delta_1-1,\alpha_\infty/n+\delta_1-1\}$ *and* $b:=1-\delta_1+\varepsilon$. *Moreover, for any function* $M\in L^{p(\cdot)}$ *and* $l\in\mathbb{Z}$, *when* $\|M\|_{L^{p(\cdot)}}>1$ *or* $l<0$, *let* $\alpha_l:=\alpha(0)$ *and* $a:=1-\delta_1-\alpha(0)/n+\varepsilon$; *when* $\|M\|_{L^{p(\cdot)}}\le 1$ *or* $l\ge 0$, *let* $\alpha_l:=\alpha_\infty$ *and* $a:=1-\delta_1-\alpha_\infty/n+\varepsilon$.

(i) If $0<q\le 1$, *there exists a constant* C *such that for any central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)$*-molecule* M, *and any central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)$*-molecule of restricted type* M,

$$\|M\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}}\le C\quad\text{and}\quad\|M\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}}\le C,$$

respectively.

(ii) There exists a constant C *such that for any dyadic central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)_l$*-molecule* M_l, $l\in\mathbb{Z}$, *and any dyadic central* $(\alpha(\cdot),p(\cdot);s,\varepsilon)_l$*-molecule of restricted type* M_l, $l\in\mathbb{N}_0$,

$$\|M_l\|_{H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}}\le C\quad\text{and}\quad\|M_l\|_{HK_{p(\cdot)}^{\alpha(\cdot),q}}\le C,$$

respectively.

Proof. We only prove (i). (ii) can be proved in the similar way.

Suppose that M be a central $(\alpha(\cdot),p(\cdot);s,\varepsilon)$-molecule. Let

$$\sigma:=\begin{cases} \|M\|_{L^{p(\cdot)}}^{-1/\alpha(0)}, & \sigma<1,\\ \|M\|_{L^{p(\cdot)}}^{-1/\alpha_\infty}, & \sigma\geq 1,\end{cases}$$

$E_0:=\{x:|x|\leq\sigma\}$, $E_{k,\sigma}:=\{x:2^{k-1}\sigma<|x|\leq 2^k\sigma\}$, $k\in\mathbb{N}_0$ and $B_{k,\sigma}:=\{x:|x|\leq 2^k\sigma\}$. We denote by $\chi_{k,\sigma}$ the characteristic function of $E_{k,\sigma}$ when $k\in\mathbb{N}$ and denote by $\chi_{0,\sigma}$ the characteristic function of E_0 when $k=0$. So we have

$$M(x)=\sum_{k=0}^{\infty}M(x)\chi_{k,\sigma}(x).$$

Let $M_k(x):=M(x)\chi_{k,\sigma}(x)$. We denote by $\mathcal{P}_m$ the class of all real polynomials of degree less than m. Let $P_{E_{k,\sigma}}M_k\in\mathcal{P}_m$ be the unique polynomial satisfying

$$\int_{E_{k,\sigma}}(M_k(x)-P_{E_{k,\sigma}}M_k(x))x^\beta dx=0,\quad |\beta|\leq s. \tag{3.1}$$

Let $Q_k(x):=(P_{E_{k,\sigma}}M_k)(x)\chi_{k,\sigma}(x)$. We claim that

(a) There exists a constant $C>0$ and a sequences of numbers $\{\lambda_k\}_{k\in\mathbb{N}_0}$ such that

$$\sum_{k=0}^{\infty}|\lambda_k|^q<\infty,\quad M_k-Q_k=\lambda_k a_k,$$

where each a_k is an $(\alpha(\cdot),p(\cdot))$-atom;

(b) $\sum_{k=0}^{\infty}Q_k$ has an $(\alpha(\cdot),p(\cdot))$-atom decomposition.

Then the conclusion of the lemma can be deduced directly by the claim. We first show (a). With loss of generality, we can suppose that $\mathcal{R}_{p(\cdot)}(M)=1$, which implies that

$$\||\cdot|^{nb}M\|_{L^{p(\cdot)}}=\|M\|_{L^{p(\cdot)}}^{-a/(b-a)}=\sigma^{na}.$$

Choose $\{\varphi_l^k:|l|\leq s\}\subset\mathcal{P}_s(\mathbb{R}^n)=1$ such that

$$\langle\varphi_\mu^k,\varphi_\nu^k\rangle_{E_{k,\sigma}}=\frac{1}{|E_{k,\sigma}|}\int_{E_{k,\sigma}}\varphi_\mu^k(x)\varphi_\nu^k(x)dx=\delta_{\mu\nu}.$$

It is easy to see that

$$Q_k(x)=\sum_{|l|\leq s}\langle M_k,\varphi_l^k\rangle_{E_{k,\sigma}}\varphi_l^k(x),\quad \text{if } x\in E_{k,\sigma}, \tag{3.2}$$

and

$$|Q_k(x)|\lesssim\frac{1}{|E_{k,\sigma}|}\int_{E_{k,\sigma}}|M_k(x)|dx.$$

Thus for any $k\in\mathbb{N}_0$, by Lemma 1.3 and let $b-a=\alpha_\sigma/n$ we have

$$\begin{aligned}\|M_k-Q_k\|_{L^{p(\cdot)}}&\le\|M_k\|_{L^{p(\cdot)}}+\|Q_k\|_{L^{p(\cdot)}}\\&\lesssim\|M_k\|_{L^{p(\cdot)}}+\frac{1}{|E_{k,\sigma}|}\|M_k\|_{L^{p(\cdot)}}\|\chi_{E_{k,\sigma}}\|_{L^{p'(\cdot)}}\|\chi_{E_{k,\sigma}}\|_{L^{p(\cdot)}}\\&\lesssim\|M_k\|_{L^{p(\cdot)}}\lesssim\||\cdot|^{nb}M\|_{L^{p(\cdot)}}|2^k\sigma|^{-nb}\\&=|2^k\sigma|^{-nb}\sigma^{na}=2^{-kna}|B_{k,\sigma}|^{-\alpha_\sigma/n}.\end{aligned}$$

We see that $M_k-Q_k=\lambda_k a_k$, with $\lambda_k\approx2^{-kna}$ and a_k is a central $(\alpha(\cdot),p(\cdot))$-atom supported in $B_{k,\sigma}$. Obviously, $\sum_{k=0}^\infty|\lambda_k|^q<\infty$.

Next we shall show (b). Let $\{\psi_l^k:|l|\le s\}\subset\mathcal{P}_s(\mathbb{R}^n)$ be the dual basis of $\{x^{\alpha_\sigma}:|\alpha_\sigma|\le s\}$ with respect to the weight $\frac{1}{|E_{k,\sigma}|}$ on $E_{k,\sigma}$, that is

$$\langle\psi_l^k,x^{\alpha_\sigma}\rangle=\frac{1}{|E_{k,\sigma}|}\int_{E_{k,\sigma}}\psi_l^k(x)x^{\alpha_\sigma}dx=\delta_{l\alpha_\sigma}.$$

If set $\varphi_l^k(x):=\sum_{|\nu|\le s}\beta_{\nu l}^k x^\nu$ and $\psi_l^k(x):=\sum_{|\nu|\le s}\tau_{\nu l}^k\varphi_\nu^k(x)$, then we have

$$\tau_{\nu l}^k=\langle\psi_l^k,\varphi_\nu^k\rangle=\sum_{|\gamma|\le s}\beta_{\nu\gamma}^k\langle\psi_l^k,x^\gamma\rangle=\sum_{|\gamma|\le s}\beta_{\nu\gamma}^k\delta_{l\gamma}=\beta_{\nu l}^k.$$

So $\psi_l^k(x)=\sum_{|\nu|\le s}\beta_{\nu l}^k\varphi_\nu^k(x)$. For any $x\in E_{k,\sigma}$, we have

$$\langle M_k,\varphi_l^k\rangle_{E_{k,\sigma}}\varphi_l^k(x)=\Big\langle M_k,\sum_{|\nu|\le s}\beta_{\nu l}^k x^\nu\Big\rangle_{E_{k,\sigma}}\varphi_l^k(x)=\sum_{|\nu|\le s}\langle M_k,x^\nu\rangle_{E_{k,\sigma}}\beta_{\nu l}^k\varphi_l^k(x),$$

which together with (3.2) implies that

$$Q_k(x)=\sum_{|l|\le s}\langle M_k,x^l\rangle_{E_{k,\sigma}}\psi_l^k(x),\quad\text{if }\ x\in E_{k,\sigma}.\tag{3.3}$$

We denote $E:=\{x\in\mathbb{R}^n:1\le|x|\le2\}$, $F:=\{x\in\mathbb{R}^n:|x|\le1\}$, $\{e_l:|l|\le s\}\subset\mathcal{P}_s(\mathbb{R}^n)$ satisfying

$$\frac{1}{|E|}\int_E e_l(x)x^{\alpha_\sigma}dx=\delta_{l\alpha_\sigma}\quad\text{and}\quad\{\tilde{e}_l:|l|\le s\}\subset\mathcal{P}_s(\mathbb{R}^n),$$

satisfying

$$\frac{1}{|F|}\int_F\tilde{e}_l(x)x^{\alpha_\sigma}dx=\delta_{l\alpha_\sigma}.$$

Noting that

$$\delta_{l\alpha_\sigma}=\frac{1}{|E_{k,\sigma}|}\int_{E_{k,\sigma}}\psi_l^k(x)x^{\alpha_\sigma}dx=\frac{1}{|E|}\int_E(2^{k-1}\sigma)^{|\alpha_\sigma|}\psi_l^k(2^{k-1}\sigma y)y^{\alpha_\sigma}dy,$$

we get $e_l(y)=(2^{k-1}\sigma)^{|l|}\psi_l^k(2^{k-1}\sigma y)$. This in turn leads to that

$$\psi_l^k(x)=(2^{k-1}\sigma)^{-|l|}e_l\Big(\frac{x}{2^{k-1}\sigma}\Big),\quad x\in E_{k,\sigma}.$$

By the similar way, we have

$$\psi_l^k(x)=(2^{k-1}\sigma)^{-|l|}\tilde{e}_l\Big(\frac{x}{\sigma}\Big),\quad x\in F.$$

By taking $C:=\sup_{l:|l|\le s}\{\|e_l\|_{L^\infty(\mathbb{R}^n)},\|\tilde{e}_l\|_{L^\infty(\mathbb{R}^n)}\}$ we have

$$|\psi_l^k(x)|\le C(2^{k-1}\sigma)^{-|l|}. \tag{3.4}$$

Now we can conclude (b). Let

$$N_l^k:=\sum_{j=k}^{\infty}|E_{j,\sigma}|\langle M_j,x^l\rangle_{E_{j,\sigma}},\quad k\in\mathbb{N}_0.$$

It is easy to see that

$$N_l^0=\sum_{j=0}^{\infty}|E_{j,\sigma}|\langle M_j,x^l\rangle_{E_{j,\sigma}}=\sum_{j=0}^{\infty}\int_{E_{j,\sigma}}M(x)x^l dx=\int_{E_{\mathbb{R}^n}}M(x)x^l dx=0,$$

and for any $k\in\mathbb{N}$, there exists $E_\sigma\subset E_{j,\sigma}$ such that $|E_\sigma|<\infty$ and $\|E_\sigma\|_{L^{p(\cdot)}}<\infty$. By Lemmas 1.2 and 1.3, we have

$$\begin{aligned}|N_l^k|&\le\sum_{j=k}^{\infty}\int_{E_{j,\sigma}}|M_j(x)x^l|dx\lesssim\sum_{j=k}^{\infty}\|M_j|\cdot|^l\|_{L^{p(\cdot)}}\|\chi_{E_{j,\sigma}}\|_{L^{p'(\cdot)}}\\&\lesssim\sum_{j=k}^{\infty}\|M_j|\cdot|^l\|_{L^{p(\cdot)}}|E_{j,\sigma}|\frac{\|\chi_{E_\sigma}\|_{L^{p(\cdot)}}}{\|\chi_{E_{j,\sigma}}\|_{L^{p(\cdot)}}}\frac{1}{\|\chi_{E_\sigma}\|_{L^{p(\cdot)}}}\\&\lesssim\sum_{j=k}^{\infty}(2^j\sigma)^{|l|-nb}\||\cdot|^{nb}M_j\|_{L^{p(\cdot)}}|E_{j,\sigma}|\Big(\frac{|E_\sigma|}{|E_{j,\sigma}|}\Big)^{\delta_1}\\&\lesssim\sum_{j=k}^{\infty}\sigma^{|l|-\alpha_\sigma+n(1-\delta_1)}2^{j(|l|-n\varepsilon)}\lesssim\sigma^{|l|-\alpha_\sigma+n(1-\delta_1)}2^{k(|l|-n\varepsilon)}.\end{aligned}$$

By (3.4) we have

$$|E_{j,\sigma}|^{-1}|N_l^k\psi_l^k(x)\chi_{k,\sigma}(x)|\lesssim\sigma^{-\alpha_\sigma-n\delta_1}2^{-kn(1+\varepsilon)}\to 0,\quad k\to\infty. \tag{3.5}$$

Using Abel's transform, from (3.3) and (3.5) we get

$$\begin{aligned}\sum_{k=0}^{\infty}Q_k(x)&=\sum_{k=0}^{\infty}\sum_{|l|\le s}\langle M_k,x^l\rangle_{E_{k,\sigma}}\psi_l^k(x)\\&=\sum_{|l|\le s}\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k}|E_{j,\sigma}|\langle M_j,x^l\rangle_{E_{j,\sigma}}\Big)\\&\quad\times\Big\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\Big\}\\&=\sum_{|l|\le s}\sum_{k=0}^{\infty}(-N_l^{k+1})\Big\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\Big\}.\end{aligned}$$

Because of

$$\begin{aligned}&\Big|N_l^{k+1}\Big\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\Big\}\Big|\\&\lesssim|N_l^{k+1}||E_{k+1,\sigma}|^{-1}|\psi_l^{k+1}(x)|\lesssim 2^{-kna}|E_{k+1,\sigma}|^{-\alpha_\sigma/n-\delta_1},\end{aligned}$$

we denote $\lambda_{lk}\approx 2^{-kna}$,

$$a_{lk}=\lambda_{lk}^{-1}(-N_l^{k+1})\Big\{|E_{k,\sigma}|^{-1}\psi_l^k(x)\chi_{k,\sigma}(x)-|E_{k+1,\sigma}|^{-1}\psi_l^{k+1}(x)\chi_{k+1,\sigma}(x)\Big\}$$

and we have

$$\sum_{k=0}^{\infty}Q_k(x)=\sum_{|l|\le s}\sum_{k=0}^{\infty}\lambda_{lk}a_{lk},$$

where a_{lk} is an $(\alpha(\cdot),p(\cdot))$-atom, and $\sum_{|l|\le s}\sum_{k=0}^{\infty}|\lambda_{lk}|^q<\infty$. Thus we finish the proof of Lemma 3.2. □

If the operator T in Theorem 2.3 is of convolution type, we then can obtain the following stronger conclusion.

Theorem 3.3. *Let* $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ *and*

$$Tf(x)=\text{p.v.}\int_{\mathbb{R}^n}K(x-y)f(y)dy$$

be a bounded operator on $L^{p(\cdot)}$ *for some* $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ *with kernel* K *satisfies*

$$|K(x-y)-K(x)|\le C\frac{|y|^\delta}{|x|^{n+\delta}},\quad |x|\ge|y|,$$

where C *is a positive constant and* $0<\delta\le 1$. *Then for any* $0<q<\infty$, $\max\{n\delta_1,n\delta_2\}\le\alpha_r<\min\{n\delta_1+\delta,n\delta_2+\delta\}$ *with* α_r *as in Definition 2.2, the operator* T *is a bounded operator on* $H\dot{K}_{p(\cdot)}^{\alpha(\cdot),q}$ *and* $HK_{p(\cdot)}^{\alpha(\cdot),q}$.

Proof. The idea is similar to Theorem 6.2.3 in [30] for the constant exponents case. We only prove homogeneous case and non-homogeneous can be handled by the similar way.

We first consider the case that $0<q\le 1$. Let a be a central $(\alpha(\cdot),p(\cdot))$-atom which supported in $B(0,r)$. If Ta is a central $(\alpha(\cdot),p(\cdot);0,\varepsilon)$-molecule for some $\varepsilon>\alpha_r/n+\delta_1-1$ and by Lemma 3.2, we have $\|Ta\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\le C$ and $\|Ta\|_{HK^{\alpha(\cdot),q}_{p(\cdot)}}\le C$. Then for homogeneous case

$$\|Tf\|^q_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\le\sum_{l=-\infty}^{\infty}|\lambda_l|^q\|Ta\|^q_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}<\infty$$

and non-homogeneous

$$\|Tf\|^q_{HK^{\alpha(\cdot),q}_{p(\cdot)}}\le\sum_{l=0}^{\infty}|\lambda_l|^q\|Ta\|^q_{HK^{\alpha(\cdot),q}_{p(\cdot)}}<\infty.$$

Thus, T is a bounded operator on $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $HK^{\alpha(\cdot),q}_{p(\cdot)}$ by taking supremum of above formula. Now it suffices to show Ta is a central $(\alpha(\cdot),p(\cdot);0,\varepsilon)$-molecule for some $\varepsilon>\alpha_r/n+\delta_1-1$. To prove it, let $a=1-\delta_1-\alpha_r/n+\varepsilon$, $b=1-\delta_1+\varepsilon$. It is easy to see that we only need to verify the size condition for molecules

$$\mathcal{R}_{p(\cdot)}(Ta)=\|Ta\|^{a/b}_{L^{p(\cdot)}}\||\cdot|^{nb}(Ta)\|^{1-a/b}_{L^{p(\cdot)}}<\infty.$$

First we estimate $\||\cdot|^{nb}(Ta)\|^{1-a/b}_{L^{p(\cdot)}}$. In fact, we have

$$\||\cdot|^{nb}(Ta)\|_{L^{p(\cdot)}(|\cdot|\le 2r)}\lesssim r^{nb}\|Ta\|_{L^{p(\cdot)}}\lesssim r^{nb-\alpha_r}.$$

On the other hand, for any x with $|x|>2r$, the vanishing moment of a and the regularity of K, we have

$$\begin{aligned}|Ta(x)|&=\Big|\int_{\mathbb{R}^n}(K(x-y)-K(x))a(y)dy\Big|\\&\lesssim\int_{\mathbb{R}^n}\frac{|y|^\delta}{|x-y|^{n+\delta}}|a(y)|dy\\&\lesssim r^{n+\delta}|x|^{-(n+\delta)}\frac{1}{|B(0,r)|}\int_{B(0,r)}|a(y)|dy\\&\lesssim r^{n+\delta}|x|^{-(n+\delta)}\mathcal{M}a(x)\\&\le\mathcal{M}a(x)\end{aligned}$$

and

$$\||\cdot|^{nb}(Ta)\|_{L^{p(\cdot)}(|\cdot|>2r)}\lesssim\||\cdot|^{nb}\mathcal{M}a\|_{L^{p(\cdot)}(|\cdot|>2r)}\lesssim r^{nb}\||a\|_{L^{p(\cdot)}}\lesssim r^{nb-\alpha_r}.$$

Thus, we get

$$\begin{aligned}\mathcal{R}_{p(\cdot)}(Ta)=&\|Ta\|_{L^{p(\cdot)}}^{a/b}\||\cdot|^{nb}(Ta)\|_{L^{p(\cdot)}}^{1-a/b}\\ \lesssim&\|a\|_{L^{p(\cdot)}}^{a/b}r^{(nb-\alpha_r)(1-a/b)}\\ \lesssim&r^{-\alpha_r a/b+(nb-\alpha_r)(1-a/b)}\\ =&1.\end{aligned}$$

We now consider the case that $1<q<\infty$. From the previous case we know that $Ta(x)$ is an integrable function and satisfies the vanishing moment condition if $a(x)$ is a central $(\alpha(\cdot),p(\cdot))$-atom which supported in $B(0,r)$. Write

$$\begin{aligned}\|G_N Tf\|^q_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}=&\sum_{k\le -1}2^{\alpha(0)kq}\|(G_N Tf)\chi_k\|^q_{L^{p(\cdot)}}+\sum_{k\ge 0}2^{\alpha_\infty kq}\|(G_N Tf)\chi_k\|^q_{L^{p(\cdot)}}\\ =&\sum_{k\le -1}|B_k|^{\alpha(0)q/n}\|(G_N Tf)\chi_k\|^q_{L^{p(\cdot)}}+\sum_{k\ge 0}2^{\alpha_\infty q/n}\|(G_N Tf)\chi_k\|^q_{L^{p(\cdot)}}\\ \lesssim&\sum_{k\le -1}|B_k|^{\alpha(0)q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l|\|a_l\|_{L^{p(\cdot)}}\Big)^q+\sum_{k\le -1}|B_k|^{\alpha(0)q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|\|(G_N Ta_l)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\ &+\sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\Big(\sum_{l=k}^{\infty}|\lambda_l|\|a_l\|_{L^{p(\cdot)}}\Big)^q+\sum_{k\ge 0}|B_k|^{\alpha_\infty q/n}\Big(\sum_{l=-\infty}^{k-1}|\lambda_l|\|(G_N Ta_l)\chi_k\|_{L^{p(\cdot)}}\Big)^q\\ =:&III'+IV'+V'+VI'.\end{aligned}$$

By the similar way of III, IV, V, VI, we have

$$\|G_N Tf\|^q_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\lesssim\sum_{l=-\infty}^{\infty}|\lambda_l|^q.$$

Thus

$$\|Tf\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}\lesssim\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}}.$$

Therefore we finish the proof of Theorem 3.3. □

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