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The Negative Spectrum of Schrödinger Operators with Fractal Potentials

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Abstract. Let $\Gamma\subset\mathbb{R}^2$ be a regular anisotropic fractal. We discuss the problem of the negative spectrum for the Schrödinger operators associated with the formal expression

$$H_\beta = id-\Delta+\beta tr_b^\Gamma,\quad \beta\in\mathbb{R},$$

acting in the anisotropic Sobolev space $W_2^{1,\alpha}(\mathbb{R}^2)$, where Δ is the Dirichlet Laplanian in $\mathbb{R}^2$ and tr_b^Γ is a fractal potential (distribution) supported by Γ.

Key Words: Anisotropic function space, anisotropic fractal, Schrödinger operators, negative eigenvalues.

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1 Introduction

Let $a(x,D)$ be a positive-definite, self-adjoint pseudodifferential operator in $L_2(\mathbb{R}^2)$, $W_1^2(\mathbb{R}^2)$ the classical Sobolev space. In general, we studied the operators of the type

$$H_\beta = a(x,D)+\beta V,\quad \beta\in\mathbb{R}, \tag{1.1}$$

where V is a generalized potential given by a distribution supported by a set $\Gamma\subset\mathbb{R}^2$ with Lebesgue measure $|\Gamma|=0$. Of interest is the number of eigenvalues, counted with respect to their multiplicities, in $(-\infty,0]$ in dependence on $\beta\to\infty$. Recall that $\sharp M$ denotes the number of a finite set M. Let $\sigma(H_\beta)$ be the spectrum of self-adjoint operator H_β in $L_2(\mathbb{R}^2)$, then

$$\sharp\big\{\sigma(H_\beta)\cap(-\infty,0]\big\}\le\sharp\big\{k\in\mathbb{N}:\sqrt{2}\beta e_k\big(a(x,D)^{-1}V\big)\ge 1\big\}. \tag{1.2}$$

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The study of $\sharp\{\sigma(H_\beta)\cap(-\infty,0]\}$ is sometimes called a problem of the negative spectrum.

The interest of the problem comes from quantum mechanics with the typical hydrogen-like operator

$$H=-\frac{h^2}{2m}\Delta+V(x), \tag{1.3}$$

where h is the Planck's constant, m is mass of the electron and $V(x)$ is the potential. Asking for (negative) eigenvalues one can divide H by $h^2/2m$ and obtains

$$H_\beta=-\Delta+\beta V(x), \tag{1.4}$$

where β is proportional to h^2. Of interest is the semi-classical limit where $h\to 0$, or, $\beta\to\infty$, hence the behaviour of the cardinal number in dependence on β.

In slight modification of (1.3), we are interested in the negative spectrum of

$$H_\beta=id-\Delta+\beta tr_b^\Gamma, \tag{1.5}$$

where tr_b^Γ is the interpretation of btr_Γ and $b\in L_r(\Gamma)$ is real with $0\le 1/r<1$, tr_Γ is the trace operator. Let N_β be the number of negative eigenvalues. If Γ is an (isotropic) d-set, we obtained satisfactory results, $N_\beta\sim\beta^1$.

For anisotropic fractals in $\mathbb{R}^2$, an anisotropic d-set Γ having deviation, $0\le a\le 1$ is, roughly speaking, a compact set which can be covered for any $j\in\mathbb{N}_0$ with $N_j\sim 2^{jd}$ rectangles $\{R_l^j: l=1,\cdots,N_j\}$, $c2^{-2j}\le vol R_l^j\le 2^{-2j}$, for some c, $0<c<1$, independent of j and l, having the sides parallel to the axes of coordinates and side lengths $2^{-a_1^{j,l}}$, $2^{-a_2^{j,l}}$ satisfying

$$(1-a)j\le a_1^{j,l}\le a_2^{j,l}\le(1+a)j$$

for any $l=1,\cdots,N_j$. In this case, one has for N_β only two side estimates of type

$$C_1\beta^{\omega_1}\le N_\beta\le C_2\beta^{\omega_2} \tag{1.6}$$

for appropriate positive numbers ω_1 and ω_2 with $\omega_1\le 1\le\omega_2$.

The aim of this paper is to show that sharp estimate of (1.6), restricting ourselves to the regular anisotropic fractals and the Schrödinger operators acting in the anisotropic Sobolev space $W_2^{1,\alpha}(\mathbb{R}^2)$. We will show that

$$N_\beta\sim\beta^1. \tag{1.7}$$

In Section 2 we present basic facts concerning anisotropic function spaces, regular anisotropic fractals and some preparatory facts for the proofs. The main results, containing the precise proof of (1.6), are presented in Section 3.

2 Preliminaries

2.1 Anisotropic function spaces $B^{s,\alpha}_{pq}(\mathbb{R}^2)$

We call

$$\alpha=(\alpha_1,\alpha_2), \quad \text{with } 0<\alpha_1, \quad \alpha_2<\infty, \quad \alpha_1+\alpha_2=2,$$

an anisotropy in $\mathbb{R}^2$. For $t>0$ and $r\in\mathbb{R}$ we put

$$t^\alpha x=(t^{\alpha_1}x_1,t^{\alpha_2}x_2) \quad \text{and} \quad t^{r\alpha}x=(t^r)^\alpha x,$$

where $x=(x_1,x_2)\in\mathbb{R}^2$.

For $x=(x_1,x_2)\in\mathbb{R}^2$, $x\neq 0$, let $|x|_\alpha$ be the unique positive number t such that

$$\frac{x_1^2}{t^{2\alpha_1}}+\frac{x_2^2}{t^{2\alpha_2}}=1$$

and let $|0|_\alpha=0$. By [10], we know that $|\cdot|_\alpha$ is an anisotropic distance function in $C^\infty(\mathbb{R}^2\setminus\{0\})$.

Let $\varphi_0^\alpha\in S(\mathbb{R}^2)$ be

$$\varphi_0^\alpha(x)=\begin{cases} 1, & |x|_\alpha\le 1, \\ 0, & |x|_\alpha\ge 2, \end{cases}$$

for $x=(x_1,x_2)\in\mathbb{R}^2$, and

$$\varphi_j^\alpha(x)=\varphi_0^\alpha(2^{-j\alpha}x)-\varphi_0^\alpha(2^{-(j-1)\alpha}x), \quad j\in\mathbb{N}.$$

Then

$$\sum_{j=0}^\infty \varphi_j^\alpha(x)=1, \quad x\in\mathbb{R}^2,$$

is an anisotropic resolution of unity with

$$\text{supp}\varphi_0^\alpha\subset R_1^\alpha, \quad \text{supp}\varphi_j^\alpha\subset R_{j+1}^\alpha\setminus R_{j-1}^\alpha, \quad j\in\mathbb{N},$$

where R_j^α are rectangles

$$R_j^\alpha=\{x\in\mathbb{R}^2:|x|_\alpha\le 2^j\}.$$

Let $0<p\le\infty$, $0<q\le\infty$, $s\in\mathbb{R}$. The anisotropic Besov space $B^{s,\alpha}_{pq}(\mathbb{R}^2)$ consists of all tempered distributions $f\in S'(\mathbb{R}^2)$ for which the quasi-norm

$$\|f|B^{s,\alpha}_{pq}(\mathbb{R}^2)\|_{\varphi^\alpha}=\Big(\sum_{j=0}^\infty 2^{jsq}\|(\varphi_j^\alpha f^\wedge)^\vee|L_p(\mathbb{R}^2)\|^q\Big)^{1/q}$$

(with usual modification if $q=\infty$) is finite.

Directly from the definition we have

Theorem 2.1 (see [8]). *Let $0<p,\ u\le\infty$, $0<q_0\le\infty$, $0<q_1\le\infty$ and $-\infty<s_0<s_1<\infty$, $0<p_2<p_1<\infty$, then*

$$B^{s_1,\alpha}_{pq_1}(\mathbb{R}^2)\subset B^{s_0,\alpha}_{pq_0}(\mathbb{R}^2),\quad B^{s,\alpha}_{p_1u}(\mathbb{R}^2)\subset B^{s,\alpha}_{p_2u}(\mathbb{R}^2). \tag{2.1}$$

More results about this subject are given in [8].

2.2 The spaces $\mathbb{B}^{s,\alpha}_{pq}(\Gamma)$

Definition 2.1 (see [7]). Let Γ be a d-set in $\mathbb{R}^2$ with $0<d<2$, and α a given anisotropy. Let $s>0$, $0<p\le\infty$ and $0<q\le\infty$; then

$$\mathbb{B}^{s,\alpha}_{pq}(\Gamma)=tr_\Gamma B^{s+\frac{2-d}{p}}_{pq}(\mathbb{R}^2), \tag{2.2}$$

equipped with the quasi-norm

$$\|f|\mathbb{B}^{s,\alpha}_{pq}(\Gamma)\|=\inf\Big\|g\Big|B^{s+\frac{2-d}{p},\alpha}_{pq}(\mathbb{R}^2)\Big\|, \tag{2.3}$$

where the infimum is taken over all $g\in B^{s+\frac{2-d}{p},\alpha}_{pq}(\mathbb{R}^2)$ with $tr_\Gamma g=f$.

Remark 2.1. There are some embeddings on spaces $\mathbb{B}^{s,\alpha}_{pq}(\Gamma)$ similar to Theorem 2.1.

2.3 Regular anisotropic d-sets in $\mathbb{R}^2$

Definition 2.2 (see [4]). Let Q be a cube in $\mathbb{R}^2$ with side length 1, $0<d<2$, $\alpha=(\alpha_1,\alpha_2)$ a given anisotrophy and $c_1,c_2>0$ given numbers. Let $N_0=1$, and for any $j\in\mathbb{N}$, N_j be a natural number satisfying

$$c_1 2^{jd}\le N_j\le c_2 2^{jd}.$$

A compact set $\Gamma\subset\mathbb{R}^2$ is called a regular anisotropic d-set (with respect to the anisotropy α) if for any $j\in\mathbb{N}_0$ there exists a finite sequence of open rectangles $\{R^j_l: l=1,\cdots,N_j\}$ having sides parallel to axes; the side length of the rectangle R^j_l with respect to the x_i-axe is denoted by $2^{-a^{j,l}_i}$, where $i=1,2$. Moreover $R^0_1=\mathring{Q}$ with the following properties:

(i)
$$R^j_l\cap R^j_m=\varnothing\quad\text{for } l\neq m,\quad j\in\mathbb{N}.$$

(ii) for any rectangle R^{j+1}_l there exists a rectangle R^j_m, $m=m(l)$, such that

$$R^{j+1}_l\subset R^j_m,\quad j\in\mathbb{N}_0.$$

(iii)
$$(volR^j_l)^{\frac{d}{2}}=\sum_{R^{j+1}_m\subset R^j_l}(volR^{j+1}_m)^{\frac{d}{2}}$$

for any $j\in\mathbb{N}_0$ and $l=1,\cdots,N_j$.

(iv) there exists a constant $1\le c_0<2$ such that for all $j\in\mathbb{N}_0$ and all $l=1,\cdots,N_j$

$$2^{-j\alpha_i}\le 2^{-a_i^{j,l}}\le (c_0 2^{-j})^{\alpha_i},\quad i=1,2.$$

(v) There is

$$\Gamma=\bigcap_j\Big(\bigcup_{l=1}^{N_j}\overline{R_l^j}\Big).$$

Theorem 2.2 (see [7]). *Let Γ be a regular anisotropic d-set with respect to the anisotropy $\alpha=(\alpha_1,\alpha_2)$ and let $\{R_l^j:l=1,\cdots,N_j\}$ be the rectangles as in the Definition 2.2. Then there exists a Radon measure μ in $\mathbb{R}^2$ uniquely determined with $supp\mu=\Gamma$ and*

$$\mu(\Gamma\cap R_l^j)=(volR_l^j)^{d/2},\quad j\in\mathbb{N}_0,\quad l=1,\cdots,N_j. \tag{2.4}$$

2.4 Basic results

Let h be a C^∞ function in $\mathbb{R}^2$ with

$$\text{supp }h\subset B=\{y:|y|<1\},$$

and let α be an anisotropy. Then the anisotropic version of the local means is given by

$$h^\alpha(t,f)(x)=\int_{\mathbb{R}^2}h(y)f(x+t^\alpha y)dy=t^{-2}\int_{\mathbb{R}^2}h(t^{-\alpha}(y-x))f(y)dy,\quad 0<t<1,$$

for $f\in S'(\mathbb{R}^2)$.

Theorem 2.3 (see [8]). *Let α be an anisotropy in $\mathbb{R}^2$, h_0 and h be two C^∞ functions in $\mathbb{R}^2$ with*

$$\text{supp }h_0\subset B,\quad \text{supp }h\subset B,$$

$h_0^\vee(0)\neq 0$, $h_0^\vee$ is the inverse Fourier transform of h_0, and for some $\varepsilon>0$ and $\kappa>0$,

$$h^\vee(\xi)\neq 0,\quad 0<|\xi|\le\varepsilon,$$

and

$$(D^\beta h^\vee)(0)=0,\quad \beta\alpha<\kappa.$$

Let $0<p\le 0$, $0<q\le 0$, $s\in\mathbb{R}$ and $\kappa>\max(s,\sigma_p)+\sigma_p$, where $\sigma_p=2(\frac{1}{p}-1)_+$. Then

$$\|h_0(1,f)|L_p(\mathbb{R}^2)\|+\Big(\sum_{j=1}^\infty 2^{jsq}\|h^\alpha(2^{-j},f)|L_p(\mathbb{R}^2)\|^q\Big)^{1/q}, \tag{2.5}$$

(usual modification if $q=\infty$) is an equivalent quasi-norm in $B_{pq}^{s,\alpha}(\mathbb{R}^2)$.

Let Q^{α}_{vm} be the rectangle in $\mathbb{R}^2$ with sides parallel to the axes of coordinates, centred at

$$2^{-v\alpha}m=(2^{-v\alpha_1}m_1,2^{-v\alpha_2}m_2),\quad m\in\mathbb{Z}^2,\quad v\in\mathbb{N}_0,$$

and with side lengths $2^{-(v-1)\alpha_1}$, $2^{-(v-1)\alpha_2}$. If Q is the rectangle in $\mathbb{R}^2$ and $r>0$, then rQ is the rectangle in $\mathbb{R}^2$ concentric with Q and with side lengths r times the side lengths of Q. In particular, Q^{α}_{00} is the cube with side length 1 centered at the origin and $2^{\alpha}Q^{\alpha}_{00}$ the rectangle concentric with Q^{α}_{00} and with side lengths respectively 2^{α_1}, 2^{α_2}.

Theorem 2.4 (Subatomic Decomposition Theorem, see [3])**.** *Let $0<p\le\infty$, $0<q\le\infty$ and $s>\sigma_p=2(\frac{1}{p}-1)_+$. Then there exists a number $\kappa>0$ with the following property: for $r>\kappa$; $f\in S'(\mathbb{R}^2)$ belongs to $B^{s,\alpha}_{pq}(\mathbb{R}^2)$, if and only if it can be represented as*

$$f=\sum_{\beta\in\mathbb{N}_0^2}\sum_{v}^{\infty}\sum_{m\in\mathbb{Z}^2}\lambda^{\beta}_{vm}(\beta qu)^{\alpha}_{vm}(x),\tag{2.6}$$

convergence being in $S'(\mathbb{R}^2)$, and

$$\sup_{\beta\in\mathbb{N}_0^2}2^{r\alpha\beta}\|\lambda^{\beta}|b_{pq}\|<\infty,\tag{2.7}$$

where

$$\begin{aligned}
&\psi\in S(\mathbb{R}^2),\quad \mathrm{supp}\psi\subset 2^{\alpha}Q^{\alpha}_{00}\quad \textit{and}\quad \sum_{k\in\mathbb{Z}^2}\psi(x-k)=1\quad \textit{if}\;\; x\in\mathbb{R}^2,\\
&(\beta qu)^{\alpha}_{vm}(x)=2^{-v(s-\frac{2}{p})}\psi^{\beta}(2^{v\alpha}x-m),\\
&\psi^{\beta}(x)=x^{\beta}\psi(x),\quad x^{\beta}=x_1^{\beta_1}x_2^{\beta_2},\quad \beta=(\beta_1,\beta_2)\in\mathbb{N}_0^2,\quad |\beta|=\beta_1+\beta_2,\\
&b_{pq}=\Big\{\lambda=\{\lambda_{vm}\}_{v\in\mathbb{N}_0,m\in\mathbb{Z}^2}\subset\mathbb{C}:\|\lambda|b_{pq}\|=\Big(\sum_{v=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^2}|\lambda_{vm}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}<\infty\Big\}.
\end{aligned}$$

Let A and B be two quasi-Banach spaces and let $P:A\to B$ be a linear operator. Just as for the Banach space case, P is said to be bounded operator, if

$$\|P\|:=\sup\{\|Px|B\|:\ x\in A,\ \|x|A\|\le 1\}<\infty.$$

The set of all such P is denoted by $L(A,B)$. Then for all $k\in\mathbb{N}$, the kth entropy number $e_k(P)$ of P is defined by

$$e_k(P)=\inf\Big\{\varepsilon>0:P(U_A)\subset\bigcup_{j=1}^{2^{k-1}}(b_j+\varepsilon U_B)\text{ for some }b_1,\cdots,b_{2^{k-1}}\in B\Big\},$$

where $U_A=\{a\in A:\|a|A\|\le 1\}$ stands for the unit ball in A. Let $\{\lambda_k(P):k\in\mathbb{N}\}$ be the sequence of all non-zero eigenvalues of P, repeated according to algebraic multiplicity and ordered so that

$$|\lambda_1(P)|\ge|\lambda_2(P)|\ge\cdots\ge 0.$$

If P has only $m(<\infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities, we put $\lambda_k(P)=0$ for $k>M$.

Theorem 2.5 (see [1, 2]). *Let A, B, C be quasi-Banach spaces, $P\in L(A,B)$ and $R\in L(B,C)$. Then*

$$e_{k+l-1}(P\circ R)\le e_k(P)e_l(R), \tag{2.8a}$$

$$|\lambda_k(P)|\le\sqrt{2}e_k(P), \tag{2.8b}$$

for all $k,l\in\mathbb{N}$.

Let $d>0$, $\delta\le 0$, and let $(M_j)_{j\in\mathbb{N}_0}$ be a subsequence of natural numbers. We assume that there exists two positive numbers $c_1>0$ and $c_2>0$ with

$$c_1 2^{jd}\le M_j\le c_2 2^{jd},\quad j\in\mathbb{N}_0.$$

Let $0<p\le\infty$ and $0<q\le\infty$. Then $l_q(2^{j\delta}l_p^{M_j})$ denotes the linear space of all complex sequences $x=\{x_{jl}:j\in\mathbb{N}_0,\ l=1,\cdots,M_j\}$ endowed with the quasi-norm

$$\|x|l_q(2^{j\delta}l_p^{M_j})\|=\Big(\sum_{j=0}^{\infty}\Big(\sum_{l=1}^{M_j}2^{j\delta p}|x_{jl}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}},$$

(obvious modification if $p=\infty$ or $q=\infty$). Let, in addition, $\mu\le 0$, $0<u\le\infty$, $l_u[l_q(2^{j\delta}l_p^{M_j})]$ be the linear space of all $l_q(2^{j\delta}l_p^{M_j})$-valued sequences $x=\{x^m:m\in\mathbb{N}_0\}$ endowed with the quasi-norm

$$\big\|x|l_u[l_q(2^{j\delta}l_p^{M_j})]\big\|=\Big(\sum_{m=0}^{\infty}\|x^m|l_q(2^{j\delta}l_p^{M_j})\|^u\Big)^{\frac{1}{q}}$$

with the usual modification if $u=\infty$.

Theorem 2.6 (see [7]). *Let $d>0$, $\delta>0$, $\mu>0$, $M_j\in\mathbb{N}$, $M_j\sim 2^{jd}$, and p_1, p_2, q_1, q_2, u_1, u_2 satisfy*

$$o<p_1\le p_2\le\infty,\quad 0<q_1\le\infty,\quad 0<q_2\le\infty,\quad 0<u_1\le\infty,\quad 0<u_2\le\infty.$$

Let

$$id:l_{u_1}[2^{\mu m}l_{q_1}(2^{j\delta}l_{p_1}^{M_j})]\to l_{u_2}[l_{q_2}(l_{p_2}^{M_j})] \tag{2.9}$$

be the identity mapping. Then

$$e_k(id)\sim k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}},\quad k\in\mathbb{N}. \tag{2.10}$$

3 Main results

If $1\le p\le\infty$ and $f^{\Gamma}\in L_p(\Gamma)$ is interpreted as a tempered distribution $f\in S'(\mathbb{R}^2)$ given by:

$$f(\varphi)=\int_{\Gamma} f^{\Gamma}(\gamma)(\varphi|\Gamma)(\gamma)\mu(\gamma), \quad \varphi\in S(\mathbb{R}^2), \tag{3.1}$$

where $\varphi|\Gamma$ is the restriction of φ to Γ, and μ is the respective measure on Γ.

Theorem 3.1 (see [4]). *Let $0<d<2$, and Γ be a regular anisotropic d-set, α an anisotropy. If $1<p\le\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, then*

$$L_p(\Gamma)\subset B_{p\infty}^{-\frac{2-d}{p'},\alpha}(\mathbb{R}^2) \tag{3.2}$$

in the sense of (3.1).

Theorem 3.2 (see [4]). *Let $0<d<2$, and let Γ be a regular anisotropic d-set, α an anisotropy. If $\frac{d}{2}<p\le\infty$ and $0<q\le\min(1,p)$, then*

$$tr_{\Gamma}B_{pq}^{\frac{2-d}{p},\alpha}(\mathbb{R}^2)=L_p(\Gamma).$$

From Definition 2.1, we can get $\mathbb{B}_{pq}^{0,\alpha}(\Gamma)=L_p(\Gamma)$.

Theorem 3.3. *Let $0<d<2$, and let Γ a regular anisotropic d-set, α be an anisotropy. Let $0<p_1\le\infty$, $0<p_2\le\infty$, $0<q_1\le\infty$, $0<q_2\le\infty$, $0\le s_2<s_1<\infty$ and*

$$\delta_+=s_1-s_2-d\Big(\frac{1}{p_1}-\frac{1}{p_2}\Big)_+>0. \tag{3.3}$$

Then the embedding of $\mathbb{B}_{p_1q_1}^{s_1,\alpha}(\Gamma)$ into $\mathbb{B}_{p_2q_2}^{s_2,\alpha}(\Gamma)$ is compact and there is a constant $c>0$ such that for the related entropy numbers

$$e_k\big(id:\mathbb{B}_{p_1q_1}^{s_1,\alpha}(\Gamma)\to\mathbb{B}_{p_2q_2}^{s_2,\alpha}(\Gamma)\big)\le ck^{-\frac{s_1-s_2}{d}}. \tag{3.4}$$

Proof. Step 1. Let $p_1\le p_2$. With

$$\sigma_1=s_1+\frac{2-d}{p_1}, \quad \sigma_2=s_2+\frac{2-d}{p_2} \quad \text{and} \quad \delta=\delta_+, \tag{3.5}$$

we have

$$\sigma_1-\frac{2}{p_1}=s_1-\frac{d}{p_1}=\delta+s_2-\frac{d}{p_2}=\delta+\sigma_2-\frac{2}{p_2}. \tag{3.6}$$

Let $f\in\mathbb{B}_{p_1q_1}^{s_1,\alpha}(\Gamma)$, then by (2.3) there is a (non-linear) bounded extension operator $g=extf$ such that

$$tr_{\Gamma}g=f \quad \text{and} \quad \|g|B_{p_1q_1}^{\sigma_1,\alpha}(\mathbb{R}^2)\|\le 2\|f|\mathbb{B}_{p_1q_1}^{s_1,\alpha}(\Gamma)\|. \tag{3.7}$$

To fix the imagination we may assume that g is equal to zero outside a fixed neighborhood of Γ. By the subatomic decomposition theorem, we have

$$g=\sum_{\beta\in\mathbb{N}_0}\sum_{v=0}^{\infty}\sum_{m\in\mathbb{Z}^2}\lambda_{vm}^{\beta}2^{-v(\sigma_1-\frac{2}{p_1})}\psi^{\beta}(2^{v\alpha}x-m), \tag{3.8}$$

and

$$\sup_{\beta\in\mathbb{N}_0^2}2^{r_1\alpha\beta}\|\lambda^{\beta}|b_{p_1q_1}\|\le c\|g|B_{p_1q_1}^{\sigma,\alpha}(\mathbb{R}^2)\|, \tag{3.9}$$

with $r_1>0$ large enough, where

$$\lambda^{\beta}=\{\lambda_{vm}^{\beta}:\ v\in\mathbb{N}_0,\ m\in\mathbb{Z}^2\}.$$

For $v\in\mathbb{N}_0$ and $m\in\mathbb{Z}^2$, let Q_{vm}^{α} be the rectangles defined above. Let

$$\lambda^{\beta,\Gamma}=\{\lambda_{vm}^{\beta}:\ v\in\mathbb{N}_0,\ m\in\mathbb{Z}^2,\ CQ_{vm}^{\alpha}\cap\Gamma\neq\emptyset\}, \tag{3.10}$$

where we may assume that $C>1$ is fixed and sufficiently large such that all what follows is justified.

For a fixed $v\in\mathbb{N}_0$ let M_v be the number of the rectangles Q_{vm}^{α} such that $CQ_{vm}^{\alpha}\cap\Gamma\neq\emptyset$. By Definition 2.2(iv), we have $c_1 2^{vd}\le M_v\le c_2 2^{vd}$, where c_1, c_2 are two positive constants. Now we factorize id by

$$id\big(\mathbb{B}_{p_1q_1}^{s_1,\alpha}(\Gamma)\to\mathbb{B}_{p_2q_2}^{s_2,\alpha}(\Gamma)\big)=tr_{\Gamma}\circ V\circ id\circ U\circ ext \tag{3.11}$$

with

$$\begin{aligned}
&ext:\mathbb{B}_{p_1q_1}^{s_1,\alpha}(\Gamma)\to B_{p_1q_1}^{\sigma_1,\alpha}(\mathbb{R}^2),\\
&U:B_{p_1q_1}^{\sigma_1,\alpha}(\mathbb{R}^2)\to l_{\infty}[2^{r_1\alpha\beta}l_{q_1}(2^{v\delta}l_{p_1}^{M_v})],\\
&id:l_{\infty}[2^{r_1\alpha\beta}l_{q_1}(2^{v\delta}l_{p_1}^{M_v})\to l_{\infty}[2^{r_2\alpha\beta}l_{q_2}(l_{p_2}^{M_v})],\\
&V:l_{\infty}[2^{r_2\alpha\beta}l_{q_2}(l_{p_2}^{M_v})]\to B_{p_2q_2}^{\sigma_2,\alpha}(\mathbb{R}^2),\\
&tr_{\Gamma}:B_{p_2q_2}^{\sigma_2,\alpha}(\mathbb{R}^2)\to\mathbb{B}_{p_2q_2}^{s_2,\alpha}(\Gamma).
\end{aligned}$$

We introduce the operator U, V as follows.

$$U:B_{p_1q_1}^{\sigma_1,\alpha}(\mathbb{R}^2)\to l_{\infty}[2^{r_1\alpha\beta}l_{q_1}(2^{v\delta}l_{p_1}^{M_v})] \tag{3.12}$$

is given by

$$Ug=\eta=\{\eta^{\beta,\Gamma}:\beta\in\mathbb{N}_0^2\},\quad g\in B_{p_1q_1}^{\sigma_1,\alpha}(\mathbb{R}^2), \tag{3.13}$$

with

$$\eta^{\beta,\Gamma}=\{2^{-v\delta}\lambda_{vm}^{\beta}:\ v\in\mathbb{N}_0,\ m\in\mathbb{Z}^2,\ CQ_{vm}^{\alpha}\cap\Gamma\neq\emptyset\}. \tag{3.14}$$

By Theorem 2.4, U is a bounded mapping. We define

$$V: l_\infty[2^{r_2\alpha\beta} l_{q_2}(l_{p_2}^{M_v})] \to B^{\sigma_2,\alpha}_{p_2q_2}(\mathbb{R}^2) \tag{3.15}$$

as

$$V\theta(\gamma) = \sum_{\beta\in\mathbb{N}_0}\sum_{v=0}^{\infty}\sum_{m\in\mathbb{Z}^2} \theta^\beta_{vm} 2^{\frac{vd}{p_2}} \psi^\beta(2^{v\alpha}\gamma - m), \quad \gamma\in\Gamma, \tag{3.16}$$

where $\theta = \{\theta^{\beta,\Gamma}_{vm} : \beta\in\mathbb{N}_0\}$, and the sum over m in (3.16) is taken according to

$$\theta^{\beta,\Gamma} = \{\theta^\beta_{vm} : v\in\mathbb{N}_0,\ m\in\mathbb{Z}^2,\ CQ^\alpha_{vm}\cap\Gamma\neq\emptyset\}. \tag{3.17}$$

Then it follows by Theorem 2.4 that V is linear and bounded. Using Theorem 2.6 applied to

$$id: l_\infty[2^{r_1\alpha\beta} l_{q_1}(2^{v\delta} l_{p_1}^{M_v}) \to l_\infty[2^{r_2\alpha\beta} l_{q_2}(l_{p_2}^{M_v})],$$

where $p_1 \le p_2$.

Thus

$$e_k(id: l_\infty[2^{r_1\alpha\beta} l_{q_1}(2^{v\delta} l_{p_1}^{M_v}) \to l_\infty[2^{r_2\alpha\beta} l_{q_2}(l_{p_2}^{M_v})]) \le ck^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}. \tag{3.18}$$

Using Theorem 2.5 and 3.18 we obtain (3.4) when $p_1 \le p_2$.

Step 2. Let $p_1 > p_2$. From Comments below Definition 2.1, we claim

$$\mathbb{B}^{s_2,\alpha}_{p_1q_2}(\Gamma) \subset \mathbb{B}^{s_2,\alpha}_{p_2q_2}(\Gamma). \tag{3.19}$$

Now (3.4) is a consequence of (3.19) and Step 1 applied to $p_1 = p_2$. Thus, the proof is completed. □

Theorem 3.4. *Let Γ be a regular anisotropic d-set with respect to the anisotropy $\alpha = (\alpha_1,\alpha_2)$. Let $b\in Ł_r(\Gamma)$ be real with $0<1/r<1$, and let tr^Γ_b be the operator in the interpretation of* (1.4) *with $\frac{2}{p} = 1-\frac{1}{r}$ and $\frac{2}{q} = 1+\frac{1}{r}$. Then B,*

$$B = (id-\Delta)^{-1}\circ tr^\Gamma_b \tag{3.20}$$

is compact, non-negative, self adjoint in $W^{1,\alpha}_2(\mathbb{R}^2)$ and has null space

$$N(B) \supset \{f\in W^{1,\alpha}_2(\mathbb{R}^2) : tr_\Gamma f = 0\}. \tag{3.21}$$

Let

$$G_\beta = id + \beta B \quad \text{with } \beta > 0, \tag{3.22}$$

and let

$$N_\beta = \sharp\{\sigma(G_\beta)\cap(-\infty,0]\} \tag{3.23}$$

be the number of non-positive eigenvalues of G_β. Then there exists constants $c_1>0$, $c_2>0$ such that

$$c_2\beta^1 \le N_\beta \le c_1\beta^1. \tag{3.24}$$

Proof. Step 1. To estimate the eigenvalues B by their magnitude we decompose

$$W_2^{1,\alpha}(\mathbb{R}^2)=\{f\in W_2^{1,\alpha}(\mathbb{R}^2):\ tr_\Gamma f=0\}\oplus H^{\frac{d}{2},\alpha}(\Gamma), \tag{3.25}$$

where $H^{\frac{d}{2},\alpha}(\Gamma)=\mathbb{B}_{22}^{\frac{d}{2},\alpha}$, see [7]. By (3.21), the eigenvalues of B coincide with the eigenvalues of the restriction of B on $H^{\frac{d}{2},\alpha}(\Gamma)$, denoted by B^Γ. we factorize B^Γ by

$$B^\Gamma=id_3\circ tr_\Gamma\circ(id-\Delta)^{-1}\circ id_2\circ b\circ id_1, \tag{3.26}$$

with

$$id_1:\ H^{\frac{d}{2},\alpha}(\Gamma)\to L_p(\Gamma),\quad \text{where}\quad \frac{1}{p}=\frac{1}{2}\Big(1-\frac{1}{r}\Big), \tag{3.27a}$$

$$b:\ L_p(\Gamma)\to L_q(\Gamma),\quad \text{where}\ \frac{1}{q}=\frac{1}{p}+\frac{1}{r}, \tag{3.27b}$$

$$id_2:\ L_q(\Gamma)\to B_{q\infty}^{-\frac{2-d}{q'},\alpha}(\mathbb{R}^2), \tag{3.27c}$$

$$(id-\Delta)^{-1}:\ B_{q\infty}^{-\frac{2-d}{q'},\alpha}(\mathbb{R}^2)\to B_{q\infty}^{2-\frac{2-d}{q'},\alpha}(\mathbb{R}^2), \tag{3.27d}$$

$$tr_\Gamma;\ B_{q\infty}^{2-\frac{2-d}{q'},\alpha}(\mathbb{R}^2)\to \mathbb{B}_{q\infty}^{d,\alpha}(\Gamma), \tag{3.27e}$$

$$id_3:\ \mathbb{B}_{q\infty}^{d,\alpha}(\Gamma)\to H^{\frac{d}{2},\alpha}(\Gamma). \tag{3.27f}$$

By Theorem 3.3 and $d/2>0$ we have id_1 is compact and

$$e_k(id_1)\le ck^{-\frac{1}{2}}. \tag{3.28}$$

Since $d>d/2$, using Theorem 3.3, we have id_3 is compact and

$$e_k(id_3)\le ck^{-\frac{1}{2}}. \tag{3.29}$$

The multiplication with b is simply Hölder's inequality. The continuity of id_2 and tr_Γ follows from Theorem 3.1 and Definition 2.1. By (2.8a), we obtain

$$e_k(B^\Gamma)\le ck^{-1}. \tag{3.30}$$

Finally, from (1.2), we estimate N_β by asking for the largest $k\in\mathbb{N}$ such that

$$\beta k^{-1}\ge c>0, \tag{3.31}$$

and obtain the right side of (3.24).

Step 2. We consider the quadratic form belonging to the self-adjoint operator G_β in (3.22) in $W_2^{1,\alpha}(\mathbb{R}^2)$.

$$Q_\beta[f]=\|f|W_2^{1,\alpha}(\mathbb{R}^2)\|^2+\beta\int_\Gamma|tr_\Gamma f(\gamma)|^2\mu(d\gamma). \tag{3.32}$$

If

$$\phi_{jl}=\phi\Big(\frac{2(x_1-x_1^{j,l})}{2^{-a_1^{j,l}}},\frac{2(x_2-x_2^{j,l})}{2^{-a_2^{j,l}}}\Big), \tag{3.33}$$

then $\mathrm{supp}\phi_{jl}\subset R_{jl}$. We may assume that for any fixed $j\in\mathbb{N}$.

$$\Big\{\phi_{jl}(x),\quad l=1,\cdots,M_j\Big\},\quad \text{where } M_j\sim 2^{jd}, \tag{3.34}$$

is an orthonormal system in $W_2^{1,\alpha}(\mathbb{R}^2)$ and that

$$\int_\Gamma |\phi_{jl}(\gamma)|^2\mu(d\gamma)\sim 2^{-jd}. \tag{3.35}$$

Let $\beta=c2^{jd}$ then we have for a suitably chosen $c>0$, and all $j\in\mathbb{N}$ and $l=1,\cdots,M_j$,

$$Q_\beta[\phi_{jl}]\le 1-c'<0. \tag{3.36}$$

Hence the quadratic form Q_β with $\beta=c2^{jd}$ is negative on the span of functions $\phi_{jl}(x)$. By the Max-Min principle, see [2], it follows

$$N_\beta\ge 2^{jd}\quad \text{with } \beta=c2^{jd}. \tag{3.37}$$

Now, we arrive at the left side of (3.24). □

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