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On the Approximation of an Analytic Function Represented by Laplace-Stieltjes Transformation

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Abstract. In the present paper, we have considered the approximation of analytic functions represented by Laplace-Stieltjes transformations using sequence of definite integrals. We have characterized their order and type in terms of the rate of decrease of $E_n(F,\beta)$ where $E_n(F,\beta)$ is the error in approximating of the function $F(s)$ by definite integral polynomials in the half plane $Res \le \beta < \alpha$.

Key Words: Laplace-Stieltjes transformation, analytic function, order, type, approximation error.
AMS Subject Classifications: 30D15, 32A15

1 Introduction

Consider the Laplace-Stieltjes transformation defined by

$$G(s)=\int_0^{\infty}\exp(-sx)d\alpha(x), \tag{1.1}$$

where $\alpha(x)$ is a function of bounded variation on any finite interval $[0,X]$, $(0<X<+\infty)$, $s=\sigma+it$, σ and t are real variables. We choose a monotonic increasing sequence of real numbers $\{\lambda_n\}$ satisfying the following conditions:

$$0=\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_n\uparrow+\infty, \tag{1.2a}$$

$$\lim_{n\to\infty}\sup(\lambda_{n+1}-\lambda_n)=\delta<+\infty, \quad \sup\frac{n}{\lambda_n}=D<+\infty. \tag{1.2b}$$

We put

$$K_n^*=\sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\Big|\int_{\lambda_n}^{x}e^{-ity}d\alpha(y)\Big|.$$

In [7], Yu Jiarong obtained the following Valiron-Knopp-Bohr formula:

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Theorem 1.1. *Suppose that the Laplace-Stieltjes transformation* (1.1) *satisfies*

$$\lim_{n\to\infty}\sup(\lambda_{n+1}-\lambda_n)<+\infty,\quad \lim_{n\to\infty}\sup\frac{\ln n}{\lambda_n}<+\infty,$$

and σ_μ^G *denotes the abscissa of uniform convergence of* (1.1). *Then*

$$\lim_{n\to\infty}\sup\frac{\ln K_n^*}{\lambda_n}\leq\sigma_\mu^G\leq\lim_{n\to\infty}\sup\frac{\ln K_n^*}{\lambda_n}+\lim_{n\to\infty}\sup\frac{\ln n}{\lambda_n}. \tag{1.3}$$

Suppose that

$$\lim_{n\to\infty}\sup\frac{\ln K_n^*}{\lambda_n}=0. \tag{1.4}$$

If $D=0$ then by (1.2b), (1.3) and (1.4), it follows that $\sigma_\mu^G=0$ and $G(s)$ is analytic in the right half plane $\sigma>0$. Kong and Yang [8] considered the Laplace-Stieltjes transformations given by (1.1) converging uniformly in the whole complex plane $Re(s)>-\infty$ and studied their growth properties.

In 2012, Luo Xi and Kong Yinying [5] defined Laplace-Stieltjes transformations in a different manner by taking positive exponents in the integral (1.1). Thus they defined Laplace-Stieltjes transformations as given below:

$$F(s)=\int_0^{+\infty}\exp(sy)d\alpha(y),(s=\sigma+it), \tag{1.5}$$

where $\alpha(y)$ satisfies the same conditions as stated earlier and the sequence $\{\lambda_n\}$ satisfies both conditions stated in (1.2b). We put

$$A_n^*=\sup_{\lambda_n<x\leq\lambda_{n+1},-\infty<t<+\infty}\Big|\int_{\lambda_n}^x e^{ity}d\alpha(y)\Big|.$$

A result similar to that of Theorem A can be proved easily for the integral (1.5) also. If the integral in (1.5) converges absolutely in the half plane $Res<\alpha$ $(-\infty<\alpha<\infty)$, then it represents an analytic function in $Res<\alpha$ and since (1.2a) holds we have

$$\lim_{n\to\infty}\inf\frac{\ln(A_n^*)^{-1}}{\lambda_n}=\alpha. \tag{1.6}$$

Definition 1.1. We define maximum modulus, the maximum term and the central index of the function $F(s)$ defined by (1.5) as

$$M(\sigma,F)=\sup_{-\infty<t<+\infty}|F(\sigma+it)|,$$

$$M_\mu(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\Big|\int_0^x e^{sy}d\alpha(y)\Big|,s=\sigma+it,\quad \sigma<\alpha,$$

$$\mu(\sigma,F)=\max_{1\leq n<N}\{A_n^*e^{\lambda_n\sigma}\},\quad \sigma<\alpha,$$

$$N(\sigma,F)=\max\{n;\mu(\sigma,F)=A_n^*e^{\lambda_n\sigma}\},$$

respectively. Let R_α denote the class of all the functions $F(s)$ of the form (1.5) which are analytic in the half plane $Res<\alpha$ $(-\infty<\alpha<\infty)$ and the sequence $\{\lambda_n\}$ satisfies (1.2a) and (1.2b). If in (1.5),$A_n^*=0$ for $n\geq k+1$, and $A_k^*\neq 0$, then $F(s)$ will be called an exponential polynomial of degree k usually denoted by p_k, i.e., $p_k(s)=\int_0^{\lambda_k}\exp(sy)d\alpha(y)$. By a suitable choice of the function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in tems of $\exp(s\lambda_i)$. Hence we shall call the function $p_k(s)$ as an exponential polynomial. The class of exponential polynomials of degree k will be denoted by $\prod_k$. Following the definition of order for an analytic function represented by classical Dirichlet series the order of $F(s)$ is defined as

$$\rho=\lim_{\sigma\to\alpha}\sup\frac{\ln^+\ln^+M_\mu(\sigma,\mathrm{F})}{-\ln(1-\exp(\sigma-\alpha))}. \tag{1.7}$$

If $\rho\in(0,+\infty)$, the type of $F(s)$ is defined as

$$\tau=\lim_{\sigma\to\alpha}\sup\frac{\ln^+M_\mu(\sigma,\mathrm{F})}{(1-\exp(\sigma-\alpha))^{-\rho}}. \tag{1.8}$$

For $x\geq 0$, we put $\ln^+x=\ln x$, $x\geq 1$ and $\ln^+x=0$, $x<1$.

Let R_β, $-\infty<\beta<\infty$ be the class of functions $F(s)$ given by (1.5) and analytic in $\mathrm{Re}s\leq\beta$. For $F(s)\in R_\beta$ we define $E_n(F,\beta)$ the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F,\beta)=\inf_{p\in\prod_n}\|F-p\|_\beta,\quad n=0,1,2,\cdots,$$

where $\prod_n$ is defined as above and

$$\|F-p\|_\beta=\max_{-\infty<t<\infty}|F(\beta+it)-p(\beta+it)|.$$

It is easy to see that by suitable choice of the function $\alpha(y)$ in (1.1) or (1.5), the function $G(s)$ or $F(s)$ can be written as a Dirichlet series (see [8]). The approximation of analytic Dirichlet series through exponential polynomials was studied in [1]. In this paper we will study the approximation of Laplace-Stieltjes transform $F(s)$ in terms of the exponential polynomials as defined above and characterize the order and type of $F(s)$ in term of the rate of decay of the approximation error $E_n(F,\beta)$, $0<\beta<\alpha$.

2 Preliminary results

We now give some results which will be used in the sequel.

Lemma 2.1. *Let $F(s)\in R_\alpha$ and $-\infty<\beta<\alpha<\infty$. Then for all σ $(\sigma<\alpha)$ sufficiently close to α we have*

$$E_k(F,\beta)\leq CM_\mu(\sigma,F)\exp\{(\sigma-\beta)\lambda_{k+1}\},\quad k=0,1,2,\cdots, \tag{2.1}$$

where C is a constant independent of k and σ.

Proof. Let $F(s)\in R_\alpha$ be given by (1.5) and let

$$p_k(s)=\int_0^{\lambda_k}\exp(s\lambda_n)d\alpha(y).$$

Now from the definition of approximation error we have

$$\begin{aligned}E_k(F,\beta)&\le\|F-p_k\|_\beta\le|F(\beta+it)-p_k(\beta+it)|\\&=\Big|\int_0^{+\infty}\exp(sy)d\alpha(y)-\int_0^{\lambda_k}\exp(sy)d\alpha(y)\Big|=\Big|\int_{\lambda_k}^{\infty}\exp(sy)d\alpha(y)\Big|\\&=\Big|\sum_{n=k}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}}\exp((\beta+it)y)d\alpha(y)\Big|\le\sum_{n=k}^{\infty}\Big|\int_{\lambda_n}^{\lambda_{n+1}}\exp((\beta+it)y)d\alpha(y)\Big|\\&\le\sum_{n=k}^{\infty}\exp(\beta\lambda_{n+1})\Big|\int_{\lambda_n}^{\lambda_{n+1}}\exp(ity)d\alpha(y)\Big|,\end{aligned}$$

or

$$E_k(F,\beta)\le\sum_{n=k}^{\infty}A_n^*\exp(\beta\lambda_{n+1}).$$

From the definition of A_n^*, we have the Cauchy type inequality

$$A_n^*\exp(\sigma\lambda_n)\le\sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\Big|\int_{\lambda_n}^{x}e^{(\sigma+it)y}d\alpha(y)\Big|,$$

or

$$A_n^*\le M_\mu(\sigma,F)\exp(-\sigma\lambda_n)\quad\text{for }\ (\sigma>\beta).\tag{2.2}$$

Substituting the above estimate of A_n^*, in the right hand series above, we get

$$E_k(F,\beta)\le M_\mu(\sigma,F)\sum_{n=k}^{\infty}\exp((\beta-\sigma)\lambda_n).\tag{2.3}$$

Now proceeding as in (see [1, Lemma 1]) we get

$$\sum_{n=k}^{\infty}\exp((\beta-\sigma)\lambda_n)\le C\exp\{(\sigma-\beta)\lambda_{n+1}\}.$$

On substituting above estimate in (2.3) we finally get (2.1). This proves Lemma 2.1. ☐

In our next result, we find a reverse estimate for the growth of $E_k(F,\beta)$. We prove

Lemma 2.2. *Let $F(s)\in R_\beta$ and $-\infty<\beta<\infty$. Then for $n\ge1$ we have*

$$A_n^*\exp(\beta\lambda_n)\le 2E_{n-1}(F,\beta).\tag{2.4}$$

Proof. For $F(s)\in R_\beta$ we have

$$A_n^*\exp(\beta\lambda_n)=\sup_{\substack{\lambda_n<x\le\lambda_{n+1}\\ -\infty<t<+\infty}}\Big|\int_{\lambda_n}^{x}e^{ity}d\alpha(y)\Big|e^{\beta\lambda_n}\le\sup_{\substack{\lambda_n<x\le\lambda_{n+1}\\ -\infty<t<+\infty}}\Big|\int_{\lambda_n}^{x}e^{(\beta+it)y}d\alpha(y)\Big|$$
$$\le\sup_{-\infty<t<+\infty}\Big|\int_{\lambda_n}^{\infty}e^{(\beta+it)y}d\alpha(y)\Big|.$$

Hence

$$A_n^*\exp(\beta\lambda_n)\le|F(\beta+it)-p_n(\beta+it)|\le\|F-p_n\|_\beta, \tag{2.5}$$

for any $p_k(s)\in\prod_{n-1}$. By definition of $E_n(F,\beta)$, there exists $\widetilde{p_k}(s)\in\prod_{n-1}$, such that

$$\|F-\widetilde{p_k}\|_\beta\le2E_{n-1}(F,\beta). \tag{2.6}$$

Taking in particular $p_k(s)=\widetilde{p_k}(s)$ in (2.5) and using (2.6), Lemma 2.2 follows. □

3 Main results

We now obtain the coefficient characterization for the order of analytic function $F(s)$.

Theorem 3.1. *If $F(s)\in R_\alpha$, and the condition* (1.2b) *is satisfied then the order ρ of $F(s)$ is given by*

$$\frac{\rho}{1+\rho}=\limsup_{n\to\infty}\frac{\ln^+(\ln^+\{A_n^*\exp(\alpha\lambda_n)\})}{\ln\lambda_n}. \tag{3.1}$$

Proof. Let

$$\limsup_{n\to\infty}\frac{\ln^+(\ln^+\{A_n^*\exp(\alpha\lambda_n)\})}{\ln\lambda_n}=P. \tag{3.2}$$

Since α is the abscissa of absolute convergence, from (1.6), it is seen that $0\le P\le1$. First let $0\le P<1$, then

$$P=\frac{\mu}{1+\mu},$$

where $0\le\mu<\infty$. First we prove that $\rho\ge\mu$. This inequality obviously holds if $\mu=0$. We suppose that $0<\mu<\infty$ and let $0<\varepsilon<\mu$. If $R=\mu-\varepsilon$, from (3.2) we have

$$\ln A_n^*>(\lambda_n)^{R/1+R}-\alpha\lambda_n \quad\text{for a sequence}\quad n_1,n_2,\cdots\to\infty.$$

From (2.2), we have

$$\ln M_\mu(\sigma,F)\ge\ln A_n^*+\sigma\lambda_n \quad\text{for all }\ \sigma<\alpha\ \text{ and for all }\ n.$$

Combining the two inequalities, we get

$$\ln M_\mu(\sigma,F)\ge(\lambda_n)^{R/1+R}+(\sigma-\alpha)\lambda_n.$$

Let $\{\sigma_k\}$ be the sequence defined by

$$\lambda_{n_k}=\Big\{(\alpha-\sigma_k)\frac{(1+R)}{R}\Big\}^{-(1+R)},\quad k=1,2,\cdots.$$

Then

$$\ln M_\mu(\sigma,F)\geq\Big\{(\alpha-\sigma)\frac{(1+R)}{R}\Big\}^{-R}-(\alpha-\sigma)\Big\{(\alpha-\sigma)\frac{(1+R)}{R}\Big\}^{-(1+R)},$$

or

$$\ln M_\mu(\sigma,F)\geq(\alpha-\sigma)^{-R}\frac{R^R}{(1+R)^{(1+R)}}.$$

Since $1-\exp(\sigma-\alpha)\simeq\alpha-\sigma$ as $\sigma\to\alpha$, we have

$$\frac{\ln\ln M_\mu(\sigma,F)}{\ln(1-\exp(\sigma-\alpha))^{-1}}\geq R+o(1).$$

Hence we have

$$\rho=\limsup_{\sigma\to\alpha}\frac{\ln\ln M_\mu(\sigma,\mathrm{F})}{\ln(1-\exp(\sigma-\alpha))^{-1}}\geq R=\mu-\varepsilon.$$

Since $\varepsilon>0$, is arbitrary, we have

$$\rho\geq\mu. \tag{3.3}$$

Now we will prove the reverse part. Let $\varepsilon>0$, then by (3.2) we have for all $n>n_o$

$$\ln A_n^*<(\lambda_n)^{k/1+k}-\alpha\lambda_n,\quad \text{where } k=\mu+\varepsilon.$$

Then from the definition of $M_\mu(\sigma,F)$, we have for $\sigma<\alpha$,

$$M_\mu(\sigma,F)\leq\sum_{n=1}^{\infty}A_n^*\exp(\sigma\lambda_n),$$

or

$$M_\mu(\sigma,F)<\mathcal{O}(1)+\sum_{n=n_o+1}^{\infty}\exp((\lambda_n)^c+(\sigma-\alpha)\lambda_n),\quad \text{where } c{=}k/(k+1). \tag{3.4}$$

Let us put $g(x)=x^c+(\sigma-\alpha)x$. Then the maximum value of $g(x)$ is attained at a point

$$x_0=\Big[\frac{(\alpha-\sigma)}{c}\Big]^{1/(c-1)},$$

and

$$g(x_0)=(\alpha-\sigma)^{c/(c-1)}(\beta^{-c/(c-1)}-\beta^{-1/(c-1)}).$$

We choose a positive integer

$$N=[(D+\varepsilon')\{2/(\alpha-\sigma)\}^{1+k}],\quad \varepsilon'>0.$$

Then for σ close to $\alpha, N > n_0$ Now from (3.4), we have

$$M_\mu(\sigma,F) < \mathcal{O}(1) + N\exp\{(\alpha-\sigma)^{c/(c-1)}(c^{-c/(c-1)} - c^{-1/(c-1)})\} + \sum_{N+1}^{\infty} \exp((\lambda_n)^c + (\sigma-\alpha)\lambda_n),$$

or

$$M_\mu(\sigma,F) < \mathcal{O}(1) + N\exp\{K(c)(\alpha-\sigma)^{c/c-1}\} + \sum_{N+1}^{\infty} \exp((\lambda_n)^c + (\sigma-\alpha)\lambda_n),$$

where $K(c) = (c^{-c/(c-1)} - c^{-1/(c-1)})$. Now following as in the proof of Theorem 1 of [6],we obtain

$$M_\mu(\sigma,F) < \mathcal{O}(1) + N\exp\{K(c)(\alpha-\sigma)^{-k}\}(1+o(1)),$$

or

$$\frac{\ln\ln M_\mu(\sigma,F)}{-\ln(1-\exp(\sigma-\alpha))} \le k\frac{\ln(\alpha-\sigma)}{\ln(1-\exp(\sigma-\alpha))} + o(1).$$

On proceeding to the limits we have

$$\rho = \lim_{\sigma\to\alpha} \frac{\ln\ln M_\mu(\sigma,F)}{-\ln(1-\exp(\sigma-\alpha))} \le k = \mu+\varepsilon$$

and since $\varepsilon > 0$, we get

$$\rho \le \mu. \tag{3.5}$$

On combining (3.4) and (3.5), we get $\rho=\mu$ and so (3.1) holds if $0 \le P < 1$. If $P=1$, then using an arbitrary large number in place of $\mu-\varepsilon$ in the first part of the proof of this theorem, it follows that $\rho=\infty$. This complete the proof of Theorem 3.1. □

Our next result characterizes the type of $F(s)$. We prove

Theorem 3.2. *If $F(s) \in R_\alpha$,* (1.2b) *is satisfied and $F(s)$ is of order ρ $(0<\rho<\infty)$, then the type τ $(0 \le \tau \le \infty)$ of $F(s)$ is given by*

$$\tau = \lim_{n\to\infty} \sup \Big(\frac{\rho}{\lambda_n}\Big)^\rho \Big[\frac{ln^+(A_n^*\exp(\alpha\lambda_n))}{(\rho+1)}\Big]^{(\rho+1)}. \tag{3.6}$$

Proof. Let

$$\lim_{n\to\infty} \sup \Big(\frac{\rho}{\lambda_n}\Big)^\rho \Big[\frac{ln^+(A_n^*\exp(\alpha\lambda_n))}{(\rho+1)}\Big]^{(\rho+1)} = v. \tag{3.7}$$

Suppose $0 < v < \infty$ and let $0 < \varepsilon < 1$, then from (3.7) we have

$$\ln A_n^* > (1+\rho)(v-\varepsilon)^{\frac{1}{1+\rho}} \Big(\frac{\lambda_n}{\rho}\Big)^{\frac{\rho}{1+\rho}} - \alpha\lambda_n \quad \text{for a sequence } n_1, n_2, \cdots \to \infty. \tag{3.8}$$

Using (2.2), we have for all $\sigma<\alpha$ and $n=n_1,n_2,\cdots$,

$$\ln M_\mu(\sigma,F)\geq(1+\rho)(\nu-\varepsilon)^{\frac{1}{1+\rho}}\Big(\frac{\lambda_n}{\rho}\Big)^{\frac{\rho}{1+\rho}}+(\sigma-\alpha)\lambda_n.$$

Let $\{\sigma_k\}$ be the sequence defined by

$$\lambda_{n_k}=\frac{(\upsilon-\varepsilon)\rho}{\{-(\sigma_k-\alpha)\}^{1+\rho}},\quad k=1,2,\cdots.$$

Then

$$\ln M_\mu(\sigma,F)\geq\frac{(\upsilon-\varepsilon)}{\{-(\sigma-\alpha)\}^{\rho}},\quad \sigma=\sigma_k,\quad k=1,2,3,\cdots.$$

From the above inequality, since $1-\exp(\sigma_k-\alpha)\equiv\alpha-\sigma_k$, we have

$$\limsup_{\sigma\to\alpha}\frac{\ln M_\mu(\sigma,\mathrm{F})}{(1-\exp(\sigma-\alpha))^{-\rho}}\geq\lim_{k\to\infty}\frac{\ln M_\mu(\sigma_k,\mathrm{F})}{(1-\exp(\sigma_k-\alpha))^{-\rho}}\geq\upsilon-\varepsilon.$$

Since this holds for every $\varepsilon>0$, we have

$$\tau\geq\upsilon. \tag{3.9}$$

If $\upsilon=0$, above inequality is obvious. If $\upsilon=\infty$, taking an arbitrarily large number in place of $\nu-\varepsilon$ in (3.8) and proceeding as above, we get $\tau=\infty$.

Now we show that $\tau\leq\upsilon$. If $\upsilon=\infty$, there is nothing to prove. Hence we assume that $\upsilon<\infty$. Then by (3.7), we have for a given $\varepsilon>0$ and for all $n>n_o(\varepsilon)$,

$$\ln A_n^*<(1+\rho)(\nu+\varepsilon)^{\frac{1}{1+\rho}}\Big(\frac{\lambda_n}{\rho}\Big)^{\frac{\rho}{1+\rho}}-\alpha\lambda_n.$$

Then we have for $\sigma<\alpha$,

$$M_\mu(\sigma,F)<\mathcal{O}(1)+\sum_{n=n_o+1}^{\infty}A_n^*\exp(\sigma\lambda_n)$$

or

$$M_\mu(\sigma,F)<\mathcal{O}(1)+\sum_{n=n_o+1}^{\infty}\exp\Big((1+\rho)(\nu+\varepsilon)^{\frac{1}{1+\rho}}\rho^{-(\frac{\rho}{1+\rho})}(\lambda_n)^{\frac{\rho}{1+\rho}}+(\sigma-\alpha)\lambda_n\Big).$$

Now we put $H(x)=Qx^{\frac{\rho}{1+\rho}}+(\sigma-\alpha)x$. Then the maximum value of $H(x)$ is attained for a value of x given by

$$x=\lambda_n=\Big[\frac{(\alpha-\sigma)(1+\rho)}{Q\rho}\Big]^{-(1+\rho)}$$

the maximum value being $(\alpha-\sigma)^{-\rho}(\upsilon+\varepsilon)$. We choose a positive integer N as

$$N=[(D+\varepsilon')(1+\rho)^{(1+\rho)}(\nu+\varepsilon)\rho^{-\rho}2^{(1+\rho)}(\alpha-\sigma)^{-(1+\rho)}],\quad \varepsilon'>0,$$

where $[x]$ denotes the integer part of x. For σ close to α, $N>n_0$. Hence we have

$$\begin{aligned}M_\mu(\sigma,F)<&\mathcal{O}(1)+N\exp\{(\nu+\varepsilon)(\alpha-\sigma)^{-\rho}\}\\&+\sum_{n=N+1}^{\infty}\exp\Big((1+\rho)(\nu+\varepsilon)^{\frac{1}{1+\rho}}\rho^{-(\frac{\rho}{1+\rho})}(\lambda_n)^{\frac{\rho}{1+\rho}}+(\sigma-\alpha)\lambda_n\Big).\end{aligned}\tag{3.10}$$

Again it is easy to see that if σ is sufficiently close to α and $n>N$ then

$$(1+\rho)(\nu+\varepsilon)^{\frac{1}{1+\rho}}\Big(\frac{\lambda_n}{\rho}\Big)^{\frac{\rho}{1+\rho}}+(\sigma-\alpha)\lambda_n<(\sigma-\alpha)\frac{\lambda_n}{2}.$$

Hence we have

$$\begin{aligned}&\sum_{n=N+1}^{\infty}\exp\Big\{(1+\rho)(\nu+\varepsilon)^{\frac{1}{1+\rho}}\Big(\frac{\lambda_n}{\rho}\Big)^{\frac{\rho}{1+\rho}}+(\sigma-\alpha)\lambda_n\Big\}\\\le&\sum_{n=N+1}^{\infty}\exp\Big\{\frac{(\sigma-\alpha)\lambda_n}{2}\Big\}\le\sum_{n=1}^{\infty}\exp\Big\{\frac{(\sigma-\alpha)\lambda_n}{2}\Big\}.\end{aligned}$$

Now using (1.2b), we can find integer $n_o'(\varepsilon)$ such that $n<(D+\varepsilon)\lambda_n$ for all $n>n_0'$. Hence

$$\sum_{n=1}^{\infty}\exp\Big\{\frac{(\sigma-\alpha)\lambda_n}{2}\Big\}\le\mathcal{O}(1)+\sum_{n>n_0'}\exp\Big\{\frac{(\sigma-\alpha)n}{2(D+\varepsilon)}\Big\}\le\mathcal{O}(1),$$

the series on right hand side being convergent since $\sigma<\alpha$. Hence using (3.10), we obtain

$$\ln M_\mu(\sigma,F)<(\nu+\varepsilon)\{-(\sigma-\alpha)\}^{-\rho}(1+o(1))$$

or

$$\frac{\ln M_\mu(\sigma,F)}{(1-\exp(\sigma-\alpha))^{-\rho}}<\frac{(\nu+\varepsilon)(\alpha-\sigma)^{-\rho}}{(1-\exp(\sigma-\alpha))^{-\rho}}+o(1).$$

After passing to the limits as $\sigma\to\alpha$, we have

$$\tau\le\nu.\tag{3.11}$$

On combining (3.10) and (3.11), we get (3.6) and Theorem 3.2 follows. □

In the next result, we characterize the extension of analytic function in terms of the approximation error. We have

Theorem 3.3. *Let $F(s)\in R_\beta$, $-\infty<\beta<\infty$ be Laplace-Stieltjes transformations then $F(s)\in R_\alpha$, $\beta<\alpha<\infty$, if and only if*

$$\lim_{n\to\infty}\sup\{\ln E_n^{-1}(F,\beta)/\lambda_{n+1}\}=\alpha-\beta.$$

Proof. Suppose that $F(s)\in R_\alpha$. Then we have

$$\lim_{n\to\infty}\sup\frac{\ln A_n^*}{\lambda_n}=-\alpha.$$

Now by Lemma 2.2, we have

$$\ln A_n^*\le\ln 2+\ln E_{n-1}(F,\beta)-\beta\lambda_n,$$

or

$$\lim_{n\to\infty}\sup\frac{\ln A_n^*}{\lambda_n}\le\lim_{n\to\infty}\sup\frac{\ln E_{n-1}(F,\beta)-\beta\lambda_n}{\lambda_n},$$

or

$$\beta-\alpha\le\lim_{n\to\infty}\sup\frac{\ln E_n(F,\beta)}{\lambda_{n+1}}. \tag{3.12}$$

To obtain the reverse inequality, from Lemma1,we have

$$\ln E_n(F,\beta)\le\ln M_\mu(\sigma,F)+(\beta-\alpha)\lambda_{n+1}+\ln C,$$

or

$$\frac{\ln E_n(F,\beta)}{\lambda_{n+1}}\le(\beta-\alpha)+\frac{\ln M_\mu(\sigma,F)+\ln C}{\lambda_{n+1}}.$$

Hence

$$\lim_{n\to\infty}\sup\frac{\ln E_n(F,\beta)}{\lambda_{n+1}}\le\beta-\alpha. \tag{3.13}$$

From (3.12) and (3.13), we get the desired result of Theorem 3.3. □

In our next result, we characterize the growth of an auxiliary function. We have

Theorem 3.4. *Let $F(s)\in R_\alpha$ and $-\infty<\beta<\alpha<\infty$ and let*

$$F_\beta(s)=\sum_{n=1}^{\infty}\{E_{n-1}(F,\beta)\exp(-\beta\lambda_n)\}\exp(s\lambda_n).$$

Then $F_\beta(s)\in R_\alpha$. Further the order and type of $F_\beta(s)$ are given by

$$\rho_\beta=\lim_{n\to\infty}\sup\frac{\ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))}{\ln\lambda_{n+1}-\ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))} \tag{3.14}$$

and

$$\tau_\beta=\lim_{n\to\infty}\sup\Big(\frac{\rho_\beta}{\lambda_n}\Big)^{\rho_\beta}\Big[\frac{\ln^+(E_{n-1}(F,\beta)\exp((\alpha-\beta)\lambda_n))}{(\rho_\beta+1)}\Big]^{(\rho_\beta+1)}, \tag{3.15}$$

respectively.

Proof. Using Theorem 3.3, we have

$$\liminf_{n\to\infty}\frac{\ln[E_{n-1}(F,\beta)\exp(-\beta\lambda_n)]^{-1}}{\lambda_n}=\alpha.$$

Hence $F_\beta(s)$ is analytic in the half plane $Re(s)<\alpha$. Now using the coefficient formula for the order of an analytic function represented by Dirichlet series [6, Theorem 1], we get

$$\frac{\rho_\beta}{\rho_\beta+1}=\lim_{n\to\infty}\sup\frac{\ln^+[\ln^+(E_n(F,\beta)+(\alpha-\beta)\lambda_{n+1})}{\ln\lambda_{n+1}},$$

which is equivalent to (3.14). Similarly, using the formula for type (see [6, Theorem 2]), we obtain (3.15). □

Now we obtain the characterization of order of $F(s)$ in terms of the approximation error. We prove

Theorem 3.5. *Let $F(s)\in R_\alpha$ be of order ρ and $-\infty<\beta<\alpha<\infty$. Then*

$$\rho=\lim_{n\to\infty}\sup\frac{\ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))}{\ln\lambda_{n+1}-\ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))}.$$

Proof. Using the definition of $M_\mu(\sigma,F)$ and Lemma 2.2, we have

$$M_\mu(\sigma,F)\le\sum_{n=1}^{\infty}A_n^*\exp(\sigma\lambda_n)\le A_o^*+2\sum_{n=1}^{\infty}E_{n-1}(F,\beta)\exp((\sigma-\beta)\lambda_n)$$

or

$$M_\mu(\sigma,F)\le A_o^*+2M_\mu(\sigma,F_\beta),$$

where F_β is as defined in Theorem 3.4 and $F_\beta(s)$ belongs to R_α. Hence we get $\rho\le\rho_\beta$ where ρ_β is the order of $F_\beta(s)$ given by (3.14). Now to prove the reverse inequality let

$$\lim_{n\to\infty}\sup\frac{\ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))}{\ln^+\lambda_{n+1}-\ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))}=\xi.$$

Obviously $0\le\xi\le\infty$, first suppose that $0<\xi<\infty$ and choose ξ', $0<\xi'<\xi$. Then there exists a sequence $\{n_k\}$ of positive integers tending to ∞ such that

$$\frac{\ln^+\ln^+(E_{n_k}(F,\beta)\exp((\alpha-\beta)\lambda_{n_{k+1}}))}{\ln^+\lambda_{n_{k+1}}-\ln^+\ln^+(E_{n_k}(F,\beta)\exp((\alpha-\beta)\lambda_{n_{k+1}}))}>\xi',$$

or

$$(1+\xi')\ln\ln(E_{n_k}(F,\beta)\exp((\alpha-\beta)\lambda_{n_{k+1}}))>\xi'\ln\lambda_{n_{k+1}},$$

or

$$\ln[E_{n_k}(F,\beta)\exp((\alpha-\beta)\lambda_{n_{k+1}})]>(\lambda_{n_{k+1}})^{\xi'/(1+\xi')},$$

or

$$\ln E_{n_k}(F,\beta) > (\lambda_{n_{k+1}})^{\xi'/(1+\xi')} - ((\alpha-\beta)\lambda_{n_{k+1}}), \tag{3.16}$$

for $k=1,2,3,\cdots$, Now from Lemma 2.1, we have

$$M_\mu(\sigma,F) \geq E_{n_k}(F,\beta)\exp\{(\sigma-\beta)\lambda_{n_{k+1}}\}/C \quad \text{for } k=1,2,3,\cdots.$$

Using (3.16), we get

$$\ln M_\mu(\sigma,F) \geq (\lambda_{n_{k+1}})^{\xi'/(1+\xi')} + (\sigma-\alpha)\lambda_{n_{k+1}} - \ln C, \tag{3.17}$$

for the sequence $\{n_k\}$ and for all $\sigma(\sigma<\alpha)$ sufficiently close to α. Let $\{\sigma_k\}$ be a sequence defined as

$$\sigma_k = \alpha - (\xi'/1+\xi')(1/\lambda_{n_{k+1}})^{1/(1+\xi')},$$

then $\sigma_k \to \alpha$ as $k\to\infty$. Now for all sufficiently large values of k we have by (3.17)

$$\begin{aligned}\ln M_\mu(\sigma_k,F) &\geq \frac{1}{1+\xi'}\Big\{\Big(\frac{1+\xi'}{\xi'}\Big)(\alpha-\sigma_k)\Big\}^{-\xi'} - \ln C\\ &= \frac{(\xi')^{\xi'}}{(1+\xi')^{1+\xi'}}(\alpha-\sigma_k)^{-\xi'} - \ln C.\end{aligned}$$

Since $(1-\exp(\sigma_k-\alpha)) \approx (\alpha-\sigma_k)$ as $k\to\infty$, the above inequality gives

$$\limsup_{\sigma\to\alpha} \frac{\ln\ln M_\mu(\sigma,F)}{-\ln(1-\exp(\sigma-\alpha))} \geq \lim_{k\to\infty} \frac{\ln\ln M_\mu(\sigma_k,F)}{-\ln(1-\exp(\sigma_k-\alpha))} \geq \xi'.$$

Since $\xi'<\xi$ is arbitrary, we get from above inequality

$$\rho \geq \xi = \limsup_{n\to\infty} \frac{\ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))}{\ln\lambda_{n+1} - \ln^+\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))}. \tag{3.18}$$

The inequality (3.18) holds if $\xi=0$. For $\xi=\infty$, by choosing an arbitrary large number in place of ξ', and following above method, we can show that $\rho=\infty$. Combining (3.18) with the earlier result $\rho\leq\rho_\beta$, we get the desired result and Theorem 3.5 follows. □

Theorem 3.6. *Let $F(s)=\int_0^{+\infty}\exp(sy)d\alpha(y)$, belongs to the class R_α $(-\infty<\beta<\alpha<\infty)$ having order $\rho(0<\rho<\infty)$. Then $F(s)$ is of type τ if and only if*

$$\tau = \frac{\rho^\rho}{(1+\rho)^{1+\rho}} \limsup_{n\to\infty} \frac{(\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1})))^{\rho+1}}{(\lambda_{n+1})^\rho}.$$

Proof. From the definition of τ we get

$$\ln M_\mu(\sigma,F)\le(\tau+\varepsilon)(1-\exp(\sigma-\alpha))^{-\rho}$$

for $\sigma_o<\sigma<\alpha$. Now using Lemma 2.1, we have

$$\ln M_\mu(\sigma,F)\ge\ln E_n(F,\beta)+(\sigma-\beta)\lambda_{n+1}-\ln C.$$

From above two inequalities we get

$$\ln\{E_n(F,\beta)\exp(\alpha-\beta)\lambda_{n+1}\}\le(\tau+\varepsilon)(1-\exp(\sigma-\alpha))^{-\rho}+\lambda_{n+1}(\alpha-\sigma)+\ln C, \tag{3.19}$$

for all n and for all σ sufficiently close to α. We choose a sequence $\{\sigma_n\}$ as

$$(1-\exp(\sigma_n-\alpha))=((\tau+\varepsilon)\rho/\lambda_{n+1})^{1/(\rho+1)}, \tag{3.20}$$

then $\sigma_n\to\alpha$ as $n\to\infty$. Using (3.19) and (3.20), we get

$$\ln\{E_n(F,\beta)\exp(\alpha-\beta)\lambda_{n+1}\}\le\frac{(\tau+\varepsilon)^{1/(\rho+1)}}{\rho^{\rho/(\rho+1)}}(\lambda_{n+1})^{\rho/(\rho+1)}(1+\rho+o(1)),$$

for all sufficiently large values of n. On proceeding to limits as $n\to\infty$, we get

$$\limsup_{n\to\infty}\frac{[\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1}))]^{\rho+1}}{(\lambda_{n+1})^\rho}\le\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(\tau+\varepsilon),$$

and since $\varepsilon>0$ is arbitrary, we get

$$\tau\ge\frac{\rho^\rho}{(1+\rho)^{1+\rho}}\limsup_{n\to\infty}\frac{(\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1})))^{\rho+1}}{(\lambda_{n+1})^\rho}. \tag{3.21}$$

It follows from Theorem 3.4 and Theorem 3.5 that order of $F_\beta(s)$ is equal to order of $F(s)$. Hence from the relation

$$M_\mu(\sigma,F)\le A_o^*+2M_\mu(\sigma,F_\beta),$$

it follows that $\tau\le\tau_\beta$, the type of $F_\beta(s)$. Hence from Theorem 3.4, we get (on taking $\rho=\rho_\beta$)

$$\tau\le\frac{\rho^\rho}{(1+\rho)^{1+\rho}}\limsup_{n\to\infty}\frac{(\ln^+(E_n(F,\beta)\exp((\alpha-\beta)\lambda_{n+1})))^{\rho+1}}{(\lambda_{n+1})^\rho}. \tag{3.22}$$

From (3.21) and (3.22), we obtain the desired formula for τ. ◻

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