Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 1 (2016), pp. 20-26

DOI: 10.4208/ata.2016.v32.n1.2

On Copositive Approximation in Spaces of Continuous Functions II: The Uniqueness of Best Copositive Approximation

Aref K. Kamal*

Department of Mathematics and Statistics, S. Q. University, P.O. Box 36 Al Khoudh 123 Muscat, Sultanate of Oman

Received 2 April 2015; Accepted (in revised version) 25 May 2015

Abstract. This paper is part II of "On Copositive Approximation in Spaces of Continuous Functions". In this paper, the author shows that if Q is any compact subset of real numbers, and M is any finite dimensional strict Chebyshev subspace of $C(Q)$, then for any admissible function $f\in C(Q)\backslash M$, the best copositive approximation to f from M is unique.

Key Words: Strict Chebyshev spaces, best copositive approximation, change of sign.

AMS Subject Classifications: 41A65

1 Introduction

If Q is a compact Hausdorff space, then $C(Q)$ denotes the Banach space of all continuous real valued functions on Q, together with the uniform norm, that is, $\|f\|=\max\{|f(x)|: x\in Q\}$. If M is a subspace of $C(Q)$, and $f\in C(Q)$, then $g\in M$ is said to be copositive with f on Q iff $f(x)g(x)\geq 0$ for all $x\in Q$. The element $g_0\in M$ is called a best copositive approximation to f from M iff g_0 is copositive with f on Q and $\|f-g_0\|=\inf\{\|f-g\|:g\in M$, and g is copositive with$f$ on $Q\}$. The set $\{g\in M: g$ is copositive with f on $Q\}$ is closed, so if the dimension of M is finite, then the best copositive approximation to each $f\in C(Q)$ from M is attained. If Q is a compact subset of real numbers, then the n-dimensional subspace M of $C(Q)$ is called Chebyshev subspace of $C(Q)$ if each $g\neq 0$ in M has at most $n-1$ zeros. The n-dimensional Chebyshev subspace M of $C(Q)$ is called a "Strict Chebyshev subspace" of $C(Q)$ if each $g\neq 0$ in M has at most $n-1$ changes of signs, that is, no $g\neq 0$ in M alternates strongly at $n+1$ points of Q, which means that there do not exist $n+1$ points, $x_0<x_2<\cdots<x_{n+1}$ in Q so that $g(x_i)g(x_{i+1})<0$ for all $i=1,2,\cdots,n$.

*Corresponding author. *Email address:* akamal@squ.edu.om (A. K. Kamal)

This paper is a continuation of the author's paper [1]. In this paper the author investigates the uniqueness of the best copositive approximation by elements of finite dimensional subspaces of $C(Q)$. Passow and Taylor [2] showed that when Q is any finite subset of real numbers, and M is a finite dimensional strict Chebyshev subspace of $C(Q)$ then the best copositive approximation to each $f\in C(Q)$ from M is unique. Zhong [3] proved the same result for the case when Q is a closed and bounded interval $[a,b]$ of the real numbers, and f does not vanish on any subinterval of $[a,b]$. In this paper it is shown that this fact is true for any compact subset of real numbers.

The rest of this section will be used to cover some notation and results that will be used later in Section 2. As in Kamal [1], If Q is a compact subset of real numbers, and $x_1<x_2$ in Q then the "intervals" $(x_1,x_2),(x_1,x_2]$, $[x_1,x_2)$, and $[x_1,x_2]$ in Q are defined in the ordinary way, for example; $(x_1,x_2)=\{x\in Q:x_1<x<x_2\}$. If Q is not connected then none of those intervals need to be connected. The point z_0 in Q is called "a limit point from both sides" in Q if z_0 is an accumulation for the set $\{x\in Q:x<z_0\}$, and the set $\{x\in Q:x>z_0\}$. If z_0 is an accumulation point for the set $\{x\in Q:x<z_0\}$ or the set $\{x\in Q:x>z_0\}$ but not for both then z_0 is called "a limit point from one side" in Q. The function $f\in C(Q)$ is said to have at "least k changes of sign in Q" if there are $k+1$ point $t_1<t_2<\cdots,t_{k+1}$ in Q so that $f(t_i)f(t_{i+1})<0$ for all $i=1,2,\cdots,k$. The "number of changes of sign of f" is defined to be the $\sup\{k:f$ has at least k changes of sign$\}$. Assume that $f\neq 0$ in $C(Q)$, the point $z\in Q$ is said to be a "double zero" for f in Q if $f(z)=0$, and there are $x<z<y$ in Q so that $f(\alpha)f(\beta)>0$ for all $\alpha\neq z$, and $\beta\neq z$ in $[x,y]$. If $f(z)=0$, and z is not a double zero then z is called a "single zero" in Q (see [4]). Finally the function $f\in C(Q)$ is called admissible if f does not vanish on any infinite interval of Q.

The following Proposition presents some of the known propertis of strict Chebyshev subspaces.

Proposition 1.1. Assume that Q is a compact subset of real numbers containing at least $n+1$ points, and that M is an n-dimensional strict Chebyshev subspace of $C(Q)$. The following facts hold;

i). If $z_1<z_2<\cdots<z_{n-1}$ are $n-1$ points in Q, then there is $g\in M$, such that $g(x)=0$ for all $x\in\{z_1,z_2,\cdots,z_{n-1}\}$ and;

1). $g(x)>0$, if $x<z_1$,

2). $(-1)^{n-1}g(x)>0$ if $x>z_{n-1}$, and;

3). $(-1)^i g(x)>0$ if $x\in(z_i,z_{i+1})$, and $i=1,2,\cdots,n-1$.

ii). No $g\neq 0$ in M alternates weakly at $n+1$ points in Q, that is, there do not exist $x_1<x_2<\cdots<x_{n+1}$ in Q, and $g\neq 0$ in M such that $(-1)^i g(x_i)\geq 0$ for each $i=1,2,\cdots,n+1$.

iii). If $g\neq 0$ in M and k is the number of single zeros of g, and m is the number of double zeros of g then $k+2m\leq n-1$.

Part i) in Proposition 1.1 can be obtained from Lemma 6.5 in Zielke [4], part ii), is in [4, Lemma 3.1b], part iii) is [4, Lemma 6.2].

Lemma 1.1 (see [1]). *Assume that Q is an infinite compact subset of real numbers, M is an n-dimensional strict Chebyshev subspace of $C(Q)$, and q is a limit point from both sides in Q. If g and h are two elements in M and $h\neq 0$, then $\lim_{x\to q^-}\frac{g(x)}{h(x)}$ and $\lim_{x\to q^+}\frac{g(x)}{h(x)}$ both exist as extended real numbers.*

Lemma 1.2. *Assume that Q is an infinite compact subset of real numbers, and that M is an n-dimensional strict Chebyshev subspace of $C(Q)$. Let q be a limit point from both sides in Q and let g and h be two nonzero elements in M, such that $g(q)=h(q)=0$. If the number of zeros of g is $n-1$, then*

$$\lim_{x\to q^-}\frac{g(x)}{h(x)}\neq 0 \quad \text{and} \quad \lim_{x\to q^+}\frac{g(x)}{h(x)}\neq 0.$$

Proof. It will be shown that $\lim_{x\to q^-}\frac{g(x)}{h(x)}\neq 0$. With the same method one can prove that $\lim_{x\to q^+}\frac{g(x)}{h(x)}\neq 0$. By Lemma 1.1, $\lim_{x\to q^-}\frac{g(x)}{h(x)}$ exists as an extended real number. Assume that $\lim_{x\to q^-}\frac{g(x)}{h(x)}=0$, and let $x_1<x_2<\cdots<x_{n-1}$ be the zeros of g. For each $k=1,2,\cdots,n-2$, let $I_k=(x_k,x_{k+1})$. Let $I_0=\{x\in Q: x<x_1\}$, and $I_{n-1}=\{x\in Q: x>x_{n-1}\}$. By Proposition 1.1, all the zeros of g are single zeros. Thus one can assume without loss of generality that $(-1)^k g(x)>0$ for all $x\in I_k$, and $k=0,1,2,\cdots,n-1$. The proof will be given first for the case at which $I_k\neq\phi$ for all k. In this case for each k, choose $t_k\in I_k$. Then g alternates strongly at the n points $t_0<t_1<\cdots<t_{n-1}$ in Q. Since $q=x_{i_0}$ for some i_0, then $t_{i_0-1}<x_{i_0}<t_{i_0}$. It is clear that g does not changes sign in neither $[t_{i_0-1},x_{i_0}]$ nor in $[x_{i_0},t_{i_0}]$. Let $c>0$ be chosen so that $c\|h\|<\min\{|g(t_0)|,|g(t_1)|,\cdots,|g(t_{n-1})|\}$. Since $\lim_{x\to q^-}\frac{g(x)}{h(x)}=0$, then there is $y_0\neq x_{i_0}$ in (t_{i_0-1},x_{i_0}), so that $|g(y_0)|<c|h(y_0)|$. If $g(t_{i_0-1})h(y_0)>0$, then let $\psi=g-ch$, and if $g(t_{i_0-1})h(y_0)<0$, then let $\psi=g+ch$. In both cases $\psi\neq 0$, and $\psi(t_{i_0-1})\psi(y_0)<0$. Therefore, ψ alternates weakly at the $n+2$ points of the set $\{t_0,t_1,\cdots,t_{i_0-1},y_0,x_{i_0},t_{i_0},\cdots,t_{n-1},\}$, which contradicts Proposition 1.1.

Second, assume that some of the intervals $I_0,I_1,\cdots,I_{n-1}$ are empty. The proof will be given by strong induction. Assume that the number of empty intervals among $I_0,I_1,\cdots,I_{n-1}$ is k. Then $0\le k<n$. The hypothesis is true for $k=0$. Now let $k\ge 0$, and assume that the hypothesis is true for all $0\le i\le k$. It will be shown that it is true for $k+1$. Assume that the number of empty intervals is $k+1$, and let $I_j=(x_j,x_{j+1})$ be one of those empty intervals. Since x_{i_0} is a limit point from both sides in Q then $I_j\neq I_{i_0-1}$ and $I_j\neq I_{i_0}$. Q is infinite, so one can find a natural number $\alpha\in\{0,1,2,\cdots,n-1\}$, so that I_α is infinite. Let s be any point in I_α such that $\{x\in I_\alpha: x<s\}\neq\phi$ and $\{x\in I_\alpha: x>s\}\neq\phi$, and let g_0 be a non zero element in M having $n-1$ zeros at $[\{x_1,x_2,\cdots,x_{n-1}\}\setminus\{x_{j+1}\}]\cup\{s\}$. The zeros of g_0 includes $q=x_{i_0}$, and if $J_0,J_1,\cdots,J_{n-1}$ are the intervals between its zeros then the number of empty intervals among them is no more than k. By induction $\lim_{x\to q^-}\frac{g_0(x)}{h(x)}\neq 0$. But $\lim_{x\to q^-}\frac{g(x)}{h(x)}=0$. So $\lim_{x\to q^-}\frac{g(x)}{g_0(x)}=0$. Let t_1 be any element in I_{i_0-1}, and t_2 be any element in I_{i_0}, then $x_{i_0-1}<t_1<x_{i_0}<t_2<x_{i_0+1}$. Choose $c>0$ be so that $c\|g_0\|<\min\{|g(t_1)|,|g(t_2)|\}$. Since $\lim_{x\to q^-}\frac{g(x)}{g_0(x)}=0$, then there is $y_0\neq x_{i_0}$ in (t_1,x_{i_0}) such that $|g(y_0)|<c|g_0(y_0)|$. If

$g(t_1)g_0(y_0)>0$, then let $\psi=g-cg_0$, and if $g(t_1)g_0(y_0)<0$, then let $\psi=g+cg_0$. In both cases $\psi\neq 0$, and $\psi(t_1)\psi(y_0)<0$, and since $g(t_1)g(t_2)<0$, it follows that $\psi(t_2)\psi(y_0)>0$. Therefore, ψ alternates weakly at the points $t_1<y_0<x_{i_0}<t_2$. But $g(x_k)=g_0(x_k)=0$ for all $k\neq j+1$. So $\psi(x_k)=0$ for all $k\neq j+1$. Thus ψ alternates weakly at the $n+1$ points of the set $[\{x_1,x_2,\cdots,x_{n-1}\}\backslash\{x_{j+1}\}]\cup\{t_1,t_2,y_0\}$, which contradicts Proposition 1.1. $\square$

2 The main results

This section is devoted to show that the best copositive approximation is unique. Let n be any natural number, Q be any compact subset of the real numbers containing more than $n+1$ points, and let M be any n-dimensional strict Chebyshev subspace of $C(Q)$.

Let f be any element in $C(Q)$. If f has more than $n-1$ changes of sign then there are $n+1$ points $t_1<t_2<\cdots<t_{n+1}$ in Q so that $f(t_i)f(t_{i+1})<0$ for all $i=1,2,\cdots,n$. If g is any best copositive approximation to f from M then $g(t_i)g(t_{i+1})\leq 0$ for all $i=1,2,\cdots,n$. Therefore by Proposition 1.1. g must be zero. Hence $g=0$ is the unique best copositive approximation to f from M. So in this section the function f will have no more than $n-1$ changes of sign.

As in Kamal [1], if Q is a compact subset of real numbers containing at least $n+1$ points, and f is an admissible function in $C(Q)$ having no more than $n-1$ changes of sign. Define $X_0(f)=\{z_1,z_2,\cdots,z_m\}$ to be the set of all $z\in Q$ such that z is a limit point from both sides in Q, and that f changes sign at z. If M is an n-dimensional strict Chebyshev subspace of $C(Q)$, then for each $g\neq 0$ in M, copositive with f, define;

$$X_1(f,g)=\{x\in Q:|f(x)-g(x)|=\|f-g\|\}\cup\{x\in Q:f(x)\neq 0 \text{ and } g(x)=0\},$$
$$X_2(f,g)=\{x\in Q:g(x)=0,f(x)=0, \text{ and } x \text{ is not an isolated point in} Q\}.$$

Let $X(f,g)=X_1(f,g)\cup X_2(f,g)$, and define M_0 to be $\{g\in M:g(z)=0 \text{ for all } z\in X_0(f)\}$. It is clear that M_0 is an $(n-m)$-dimensional subspace of M, and that if $g\in M$ is copositive with f on Q, then $g\in M_0$.

The function $\theta\neq 0$ in M is said to be "copositive with f around the elements of $X_0(f)$" if for each $z\in X_0(f)$, there is a neighborhood U_z around z such that $f(x)\theta(x)\geq 0$ for all $x\in U_z$. It is clear that $\theta(z)=0$ for all $z\in X_0(f)$. For such function, define $X_3(f,g,\theta)$ to be

$$\Big\{z\in X_0(f):\lim_{x\to z^-}\frac{g(x)}{\theta(x)}=0\Big\}$$

and $X_4(f,g,\theta)$ to be

$$\Big\{z\in X_0(f):\lim_{x\to z^+}\frac{g(x)}{\theta(x)}=0\Big\}.$$

Lemma 2.1 (see [1]). *Assume that f is admissible function in $C(Q)\backslash M$ having no more than $n-1$ changes of sign. If g is a best copositive approximation to f from M then there is a non-zero function $\varphi\in M_0$ copositive with f around the elements of $X_0(f)$, such that the number of elements in $[X(f,g)\backslash X_0(f)]\cup X_3(f,g,\varphi)\cup X_4(f,g,\varphi)$ is more than or equal to $n-m+1$.*

Lemma 2.2. *Assume that f is admissible function in $C(Q)\backslash M$ having no more than $n-1$ changes of sign, g is a best copositive approximation to f from M, and let φ be any element in M_0 copositive with f around the elements of $X_0(f)$. For any $h_0\in M_0$, if there are $\xi_1,\xi_2,\cdots,\xi_\eta$ in $X(f,g)\backslash X_0(f)$, and $y_1,y_2,\cdots,y_r$ in $X_3(f,g,\varphi)\cup X_4(f,g,\varphi)$, such that $\eta+r=n-m+1$, $h_0(\xi_i)=0$ for all $i=1,2,\cdots,\eta$, and for all $j=1,2,\cdots,r$, either*

$$\lim_{x\to(y_j)^+}\frac{h_0(x)}{\varphi(x)}=0 \quad or \quad \lim_{x\to(y_j)^-}\frac{h_0(x)}{\varphi(x)}=0,$$

then $h_0=0$.

Proof. By contradiction, assume that there is $h_0\in M_0$ with the given properties, and that $h_0\neq 0$. Since h_0 has zeros at the points of the two distinct sets $\{\xi_1,\xi_2,\cdots,\xi_\eta\}$ and $\{z_1,z_2,\cdots,z_m\}$, then $\eta+m\le n-1$. If Q is finite then $r=0$, so $\eta=n-m+1$. Thus $\eta+m=n+1$. But then h_0 has more than $n-1$ zeros, which contradict the fact that M is a strict n-dimensional Chebyshev space. So one may assume that Q is infinite, and that $r>0$. By Proposition 1.1, let h_1 be any nonzero element in M having $n-1$ zeros, including $\{\xi_1,\xi_2,\cdots,\xi_\eta\}\cup\{z_1,z_2,\cdots,z_m\}$, and choose the location of the extra zeros so that h_1, and h_0 have the same sign in some neighborhood around y_j for all $j=1,2,\cdots r$. This can be done by replacing each double zero of h_0 by two very close single zeros for h_1. By Proposition 1.1, the number of zeros of h_1 may still less than $n-1$. To make this number equal $n-1$, one can add extra zeros after z_m or before z_1. For each $j=1,2,\cdots,r$, choose e_j in Q so that if $\lim_{x\to(y_j)^-}\frac{h_0(x)}{\varphi(x)}=0$, then $e_j<y_j$, and if $\lim_{x\to(y_j)^+}\frac{h_0(x)}{\varphi(x)}=0$ then $e_j>y_j$, and with the properties that, if I_j is the open interval between e_j and y_j in Q, then I_j does not intersect $\{\xi_1,\xi_2,\cdots,\xi_\eta\}$, neither h_0, nor h_1 change sign or have zeros in I_j, and $h_0(e_j)\neq 0$. Let $\lambda_0>0$, so that

$$\lambda_0\|h_1\|<\min\{|h_0(e_1)|,|h_0(e_2)|,\cdots,|h_0(e_r)|\},$$

and let $h_2=h_0-\lambda_0 h_1$. It is clear that $h_2\neq 0$, and that $h_2(x)=0$ for all $x\in\{\xi_1,\xi_2,\cdots,\xi_\eta\}\cup\{z_1,z_2,\cdots,z_m\}$, and that $h_2(e_j)h_0(e_j)>0$ for all j. For each $1\le j\le r$, either $\lim_{x\to(y_j)^-}\frac{h_0(x)}{\varphi(x)}=0$, or $\lim_{x\to(y_j)^+}\frac{h_0(x)}{\varphi(x)}=0$. Assume first that $\lim_{x\to(y_j)^-}\frac{h_0(x)}{\varphi(x)}=0$. Since h_1 has $n-1$ zeros and Q is infinite and y_j is a limit point from both sides in Q, then by Lemma 1.2 $\lim_{x\to(y_j)^-}\frac{h_1(x)}{\varphi(x)}\neq 0$. So $\lim_{x\to(y_j)^+}\frac{h_0(x)}{h_1(x)}=0$. Since h_1 and h_0 have the same sign at e_j, and $h_2(e_j)h_0(e_j)>0$, then $h_2(e_j)h_1(e_j)>0$. On the other hand $\lim_{x\to(y_j)^-}\frac{h_2(x)}{h_1(x)}=-\lambda_0$. Thus there is a point u_j in Q such that $e_j<u_j<y_j$ and that $h_2(u_j)h_1(u_j)<0$. Since h_1 has a constant sign in $[e_j,y_j)$ and $h_2(e_j)h_1(e_j)>0$ then $h_2(e_j)h_2(u_j)<0$. In the same manner, if $\lim_{x\to(y_j)^+}\frac{h_0(x)}{\varphi(x)}=0$, then one can find point u_j in Q such that $y_j<u_j<e_j$ and that $h_2(u_j)h_2(e_j)<0$. Let $\{s_1,s_2,\cdots,s_{\eta+m}\}=\{\xi_1,\xi_2,\cdots,\xi_\eta\}\cup\{z_1,z_2,\cdots,z_m\}$, then $h_2(s_i)=0$ for all $i=1,2,\cdots,\eta+m$, and if $s_i=y_j$ for some j, then the two points u_j, e_j lie between s_i and s_{i-1} or s_i, and s_{i+1}. Furthermore $h_2(u_j)h_2(e_j)<0$. Thus one can choose $t_j\in\{u_j,e_j\}$ so that h_2 alternates weakly at the points of $\{s_1,s_2,\cdots,s_{\eta+m}\}\cup\{t_1,t_2,\cdots,t_r\}$. But $\eta+m+r=(n-m+1)+m=n+1$. So h_2 alternates weakly at $n+1$ points of Q. This is a contradiction. $\square$

Theorem 2.1. *Assume that Q is a compact subset of real numbers having at least $n+1$ points, and that M is an n-dimensional strict Chebyshev subspace of $C(Q)$. If f is an admissible function in $C(Q)\backslash M$, then the best copositive approximation to f from M is unique.*

Proof. If f has more than $n-1$ changes of sign then as the argument at the start of this section, the best copositive approximation to f from M is unique. So one may assume that f have no more than $n-1$ changes of sign. By contradiction, assume that g_1 and g_2 are two distinct best copositive approximations to f from M. Let $g*=\frac{g_1+g_2}{2}$, $g_0=g_1-g_2$, then $g_0\neq 0$ and $g*$ is another best copositive approximation to f from M. By Lemma 2.1, there is a non-zero function $\varphi\in M_0$ copositive with f around the elements of $X_0(f)$ such that the number of the elements in $[X(f,g*)\backslash X_0(f)]\cup X_3(f,g*,\varphi)\cup X_4(f,g*,\varphi)$ is more than or equal to $n-m+1$. Thus let $\xi_1,\xi_2,\cdots,\xi_\eta$ be elements in $X(f,g*)\backslash X_0(f)$, and let $y_1,y_2,\cdots,y_r$ be elements in $X_3(f,g*,\varphi)\cup X_4(f,g*,\varphi)$ such that $\eta+r=n-m+1$.

It will be shown that $g_0(\xi_i)=0$ for all $i=1,2,\cdots,\eta$, and for all $j=1,2,\cdots,r$, either $\lim_{x\to(y_j)^+}\frac{g_0(x)}{\varphi(x)}=0$ or $\lim_{x\to(y_j)^-}\frac{g_0(x)}{\varphi(x)}=0$. If this is true, then by Lemma 2.2, $g_0=0$, which is a contradiction.

For each $i=1,2,\cdots,\eta,\xi_i\in X(f,g*)\backslash X_0(f)=[X_1(f,g*)\cup X_2(f,g*)]\backslash X_0(f)$. So either $\xi_i\in X_1(f,g*)$, or $\xi_i\in X_2(f,g*)\backslash X_0(f)$. If $\xi_i\in X_1(f,g*)$, and $|(f-g*)(\xi_i)|=\|f-g*\|$, then since

$$\|f-g*\|=\|f-g_1\|=\|f-g_2\| \quad \text{and} \quad |(f-g*)(\xi_i)|=\left|\frac{f-g_1}{2}(\xi_1)+\frac{f-g_2}{2}(\xi_1)\right|,$$

it follows that

$$(f-g_1)(\xi_i)=(f-g_2)(\xi_i)=\pm(f-g*)(\xi_i).$$

Therefore,

$$g_0(\xi_i)=(g_2-g_1)(\xi_i)=(f-g_1)(\xi_i)-(f-g_2)(\xi_i)=0.$$

If $\xi_i\in X_1(f,g*)$, and $f(\xi_i)\neq 0$, but $g*(\xi_i)=0$, then since $g*$, g_1, and g_2 are copositive with f on Q, and $g*(\xi_i)=\frac{g_1(\xi_i)}{2}+\frac{g_2(\xi_i)}{2}$, it follows that $g_1(\xi_i)=g_2(\xi_i)=0$. Therefore $g_0(\xi_i)=0$. If $\xi_i\in X_2(f,g*)$, then $g*(\xi_i)=0$ and ξ_i is a limit point either from both sides or from one side in Q. Since $g*$, g_1 and g_2 are all continuous on Q, and copositive with the admissible function f, then $g_1(\xi_i)=g_2(\xi_i)=g*(\xi_i)=0$. Thus $g_0(\xi_i)=0$.

Finally, it will be shown that for each $j=1,2,\cdots,r$, either $\lim_{x\to(y_j)^+}\frac{g_0(x)}{\varphi(x)}=0$ or $\lim_{x\to(y_j)^-}\frac{g_0(x)}{\varphi(x)}=0$. Since $y_j\in X_3(f,g*,\varphi)\cup X_4(f,g*,\varphi)$, then either $\lim_{x\to(y_j)^-}\frac{g^*(x)}{\varphi(x)}=0$ or $\lim_{x\to(y_j)^+}\frac{g^*(x)}{\varphi(x)}=0$. Assume first that $\lim_{x\to(y_j)^-}\frac{g^*(x)}{\varphi(x)}=0$. Since $g*$, g_1, g_2 are all continuous on Q, and copositive with the admissible function f, then $\lim_{x\to(y_j)^-}\frac{g_1(x)}{\varphi(x)}=0$, and $\lim_{x\to(y_j)^-}\frac{g_2(x)}{\varphi(x)}=0$. So

$$\lim_{x\to(y_j)^-}\frac{g_0(x)}{\varphi(x)}=\lim_{x\to(y_j)^-}\frac{g_2(x)}{\varphi(x)}-\lim_{x\to(y_j)^-}\frac{g_1(x)}{\varphi(x)}=0.$$

In the same method one can show that if $\lim_{x\to(y_j)^+}\frac{g^*(x)}{\varphi(x)}=0$ then $\lim_{x\to(y_j)^+}\frac{g_0(x)}{\varphi(x)}=0$. $\square$

Acknowledgments

The author want to thank Dr. AL. Brown for his patience in reading the manuscript and all his suggestions and corrections which made the paper readable.

References

[1] A. K. Kamal, On copositive approximation in spaces of continuous functions I, the alternation property of copositive approximation, Anal. Theory Appl., 31 (2015), 354–372.

[2] E. Passow and G. D, Taylor, An alternation theory for copositive approximation, J. Approx. Theory, 19 (1977), 123–134.

[3] J. Zhong, Best copositive approximation, J. Approx. Theory, 72 (1993), 210–233.

[4] R. Zielke, Discontinuous Chebyshev system, Lecture Notes in Mathematics, Vol. 707, Spriger-Verlag, Berlin-Heidelberg-New York, 1979.