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Some Inequalities for the Polynomial with S-Fold Zeros at the Origin

Ahmad Zireh[1,*] and Mahmood Bidkham[2]

[1] *Department of Mathematics, Shahrood University of Technology, Shahrood, Iran*
[2] *Department of Mathematics, University of Semnan, Semnan, Iran*

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Abstract. Let $p(z)$ be a polynomial of degree n, which has no zeros in $|z|<1$, Dewan et al. [K. K. Dewan and Sunil Hans, Generalization of certain well known polynomial inequalities, J. Math. Anal. Appl., 363 (2010), pp. 38–41] established

$$\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\le\frac{n}{2}\Big\{\Big(\Big|\frac{\beta}{2}\Big|+\Big|1+\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|p(z)|-\Big(\Big|1+\frac{\beta}{2}\Big|-\Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|p(z)|\Big\},$$

for any $|\beta|\le 1$ and $|z|=1$. In this paper we improve the above inequality for the polynomial which has no zeros in $|z|<k$, $k\ge 1$, except s-fold zeros at the origin. Our results generalize certain well known polynomial inequalities.

Key Words: Polynomial, s-fold zeros, inequality, maximum modulus, derivative.

AMS Subject Classifications: 30A10, 30C10, 30D15

1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree n, then according to a result known as Bernstein's inequality [3] on the derivative of a polynomial, we have

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1.1}$$

The result is best possible and equality holds for the polynomials having all its zeros at the origin.

If the polynomial $p(z)$ has all its zeros in $|z|\le 1$, then it was proved by Turan [10] that

$$\max_{|z|=1}|p'(z)|\ge\frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.2}$$

*Corresponding author. *Email addresses:* `azireh@shahroodut.ac.ir`, `azireh@gmail.com` (A. Zireh), `mdbidkham@gmail.om` (M. Bidkham)

With equality for those polynomials which have all their zeros at the origin.

For the class of polynomials having no zeros in $|z|<1$, the inequality (1.1) can be replaced by

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{1.3}$$

The inequality (1.3) was conjectured by Erdös and later proved by Lax [6].

As an extension of the inequality (1.2) Malik [7] proved that if $p(z)$ having all its zeros in $|z|\leq k$, $k\leq 1$, then

$$\max_{|z|=1}|p'(z)|\geq \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.4}$$

Govil [5] improved the inequality (1.4) and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then

$$\max_{|z|=1}|p'(z)|\geq \frac{n}{1+k}\Big\{\max_{|z|=1}|p(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|p(z)|\Big\}. \tag{1.5}$$

As a refinement of the inequality (1.4) Aziz and Zargar [2] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, with s-fold zeros at the origin, then

$$\max_{|z|=1}|p'(z)|\geq \frac{n+sk}{1+k}\max_{|z|=1}|p(z)|+\frac{n-s}{(1+k)k^s}\min_{|z|=k}|p(z)|. \tag{1.6}$$

Recently Dewan and Hans [4] obtained a refinement of inequalities (1.2) and (1.3). They proved that if $p(z)$ is a polynomial of degree n and has all its zeros in $|z|\leq 1$, then for every real or complex number β with $|\beta|\leq 1$,

$$\min_{|z|=1}\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\geq n\Big|1+\frac{\beta}{2}\Big|\min_{|z|=1}|p(z)|, \tag{1.7}$$

and in the case that $p(z)$ having no zeros in $|z|<1$, they proved that

$$\begin{aligned}&\max_{|z|=1}\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\\ \leq&\frac{n}{2}\Big\{\Big(\Big|1+\frac{\beta}{2}\Big|+\Big|\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|p(z)|-\Big(\Big|1+\frac{\beta}{2}\Big|-\Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|p(z)|\Big\}.\end{aligned} \tag{1.8}$$

In this paper, we obtain an improvement and generalizations of the above inequalities. For this purpose we first present the following result which is a generalization and refinement of inequalities (1.5), (1.6) and (1.7).

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, with s-fold zeros at the origin where $0\leq s\leq n$, then for every $\beta\in\mathbb{C}$ with $|\beta|\leq 1$ and $|z|=1$,*

$$\Big|zp'(z)+\beta\frac{n+sk}{1+k}p(z)\Big|\geq k^{-n}\Big|n+\beta\frac{n+sk}{1+k}\Big|\min_{|z|=k}|p(z)|. \tag{1.9}$$

With equality for $p(z)=az^n$ where $a\in\mathbb{C}$.

Remark 1.1. Clearly for $k=1$ and $s=0$ the inequality (1.9) reduces to the inequality (1.7).

According to Lemma 2.1, if $p(z)$ is a polynomial of degree n, having all its zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at the origin, then for $|z|=1$,

$$|zp'(z)|\ge \frac{n+sk}{1+k}|p(z)|,$$

then for every complex number β with $|\beta|\le 1$, by choosing suitable argument of β we have

$$\Big|zp'(z)+\beta\frac{n+sk}{1+k}p(z)\Big|=|zp'(z)|-|\beta|\frac{n+sk}{1+k}|p(z)|. \tag{1.10}$$

Combining (1.9) and (1.10) we have

$$|zp'(z)|-|\beta|\frac{n+sk}{1+k}|p(z)|\ge k^{-n}\Big|n+\beta\frac{n+sk}{1+k}\Big|\min_{|z|=k}|p(z)|,$$

or

$$|zp'(z)|-|\beta|\frac{n+sk}{1+k}|p(z)|\ge k^{-n}\Big(n-|\beta|\frac{n+sk}{1+k}\Big)\min_{|z|=k}|p(z)|,$$

equivalently

$$|zp'(z)|\ge|\beta|\frac{n+sk}{1+k}|p(z)|+k^{-n}\Big(n-|\beta|\frac{n+sk}{1+k}\Big)\min_{|z|=k}|p(z)|.$$

Making $|\beta|\to 1$, then

$$|p'(z)|\ge\frac{n+sk}{1+k}|p(z)|+\frac{n-s}{(1+k)k^{n-1}}\min_{|z|=k}|p(z)|.$$

Since for $0\le s<n$ and $k\le 1$, we have $\frac{1}{k^s}\le\frac{1}{k^{n-1}}$ and for $s=n$ we have $n-s=0$, therefore the following result is a refinement and extention of the inequality (1.6).

Corollary 1.1. If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at the origin, then we have

$$\min_{|z|=1}|p'(z)|\ge\frac{n+sk}{1+k}\min_{|z|=1}|p(z)|+\frac{n-s}{(1+k)k^{n-1}}\min_{|z|=k}|p(z)|, \tag{1.11a}$$

$$\max_{|z|=1}|p'(z)|\ge\frac{n+sk}{1+k}\max_{|z|=1}|p(z)|+\frac{n-s}{(1+k)k^{n-1}}\min_{|z|=k}|p(z)|. \tag{1.11b}$$

If we take $s=0$ in Corollary 1.1, then inequality (1.11b) reduce to inequality (1.5). Now if we take $\beta=-1$ in Theorem 1.1, we have the following result

Corollary 1.2. If $p(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at the origin, then

$$\Big|zp'(z)-\frac{n+sk}{1+k}p(z)\Big|\ge\frac{n-s}{(1+k)k^s}\min_{|z|=k}|p(z)|. \tag{1.12}$$

If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<k$, $k\ge 1$, except s-fold zeros at the origin, i.e., $p(z)=z^s h(z)$, where $h(z)$ is a polynomial of degree $(n-s)$ that does not vanish in $|z|<k$, $k\ge 1$, then the polynomial

$$q(z)=z^{n+s}\overline{p\Big(\frac{1}{\bar{z}}\Big)}=z^n\overline{h\Big(\frac{1}{\bar{z}}\Big)}=z^s\Big(z^{n-s}\overline{h\Big(\frac{1}{\bar{z}}\Big)}\Big)$$

is of degree n, having all its zeros in $|z|\le 1/k$, with s-fold zeros at the origin. Also

$$\min_{|z|=1/k}|q(z)|=\frac{1}{k^{n+s}}\min_{|z|=k}|p(z)|.$$

By applying Theorem 1.1 for the polynomial $q(z)$, we get the following result

Corollary 1.3. If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<k$, $k\ge 1$, except s-fold zeros at the origin, then for any $\beta\in\mathbb{C}$ with $|\beta|\le 1$ and $|z|=1$,

$$\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|\ge k^{-s}\Big|n+\beta\frac{nk+s}{1+k}\Big|\min_{|z|=k}|p(z)|, \tag{1.13}$$

where

$$q(z)=z^{n+s}\overline{p\Big(\frac{1}{\bar{z}}\Big)}.$$

Finally by using Corollary 1.3, we prove the following interesting result which is a generalization of the inequality (1.8).

Theorem 1.2. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<k$, $k\ge 1$, except s-fold zeros at the origin, then for every complex number β with $|\beta|\le 1$,*

$$\begin{aligned}&\max_{|z|=k^2}\Big|zp'(z)+\beta\frac{nk+s}{1+k}p(z)\Big|\\ \le&\frac{1}{2}\Big[\Big\{k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|+k^s\Big|s+\beta\frac{nk+s}{1+k}\Big|\Big\}\max_{|z|=k}|p(z)|\\ &-\Big\{k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|-k^s\Big|s+\beta\frac{nk+s}{1+k}\Big|\Big\}\min_{|z|=k}|p(z)|\Big].\end{aligned} \tag{1.14}$$

If we take $k=1$ in (1.14) we have

Corollary 1.4. If $p(z)$ is a polynomial of degree n, having no zeros in $|z|<1$, except s-fold zeros at the origin, then for every complex number β with $|\beta|\le 1$,

$$\begin{aligned}&\max_{|z|=1}\Big|zp'(z)+\beta\frac{n+s}{2}p(z)\Big|\\ \le&\frac{1}{2}\Big[\Big\{\Big|n+\beta\frac{n+s}{2}\Big|+\Big|s+\beta\frac{n+s}{2}\Big|\Big\}\max_{|z|=1}|p(z)|\\ &-\Big\{\Big|n+\beta\frac{n+s}{2}\Big|-\Big|s+\beta\frac{n+s}{2}\Big|\Big\}\min_{|z|=1}|p(z)|\Big].\end{aligned} \tag{1.15}$$

For $s=0$ the inequality (1.15) reduces to the inequality (1.8).

2 Lemmas

For the proof of these theorems, we need the following lemmas. The first lemma is due to Aziz and Shah [1].

Lemma 2.1. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z|\le k$, $k\le 1$, with s-fold zeros at the origin, then for $|z|=1$,*

$$|zp'(z)|\ge\frac{n+sk}{1+k}|p(z)|. \tag{2.1}$$

Lemma 2.2. *Let $F(z)$ be a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$ and $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|f(z)|\le|F(z)|$ for $|z|=k$, $k\le 1$, and $F(z)$, $f(z)$ have common s- fold zeros at the origin, then for every real or complex number β with $|\beta|\le 1$ and $|z|=1$,*

$$\Big|zf'(z)+\beta\frac{n+sk}{1+k}f(z)\Big|\le\Big|zF'(z)+\beta\frac{n+sk}{1+k}F(z)\Big|. \tag{2.2}$$

Proof. Let α be a complex number with $|\alpha|<1$, then $|\alpha f(z)|<|F(z)|$ for $|z|=k$. It is concluded from Rouche's Theorem, the polynomial $\alpha f(z)-F(z)$ has as many zeros in $|z|<k$ as $F(z)$ and so has all of its zeros in $|z|<k$, with s-fold zeros at the origin. On applying Lemma 2.1, we have for $|z|=1$,

$$|\alpha zf'(z)-zF'(z)|\ge\frac{n+sk}{1+k}|\alpha f(z)-F(z)|.$$

Therefore for any real or complex number β with $|\beta|<1$, the polynomial

$$T(z)=\alpha zf'(z)-zF'(z)+\beta\frac{n+sk}{1+k}(\alpha f(z)-F(z))\neq 0,$$

for $|z|=1$.

Equivalently

$$T(z)=\alpha\Big\{zf'(z)+\beta\frac{n+sk}{1+k}f(z)\Big\}-\Big\{zF'(z)+\beta\frac{n+sk}{1+k}F(z)\Big\}\neq 0, \tag{2.3}$$

for $|z|=1$. This concludes that

$$\Big|zf'(z)+\beta\frac{n+sk}{1+k}f(z)\Big|\leq\Big|zF'(z)+\beta\frac{n+sk}{1+k}F(z)\Big|, \tag{2.4}$$

for $|z|=1$. If the inequality (2.4) is not true, then there is a point $z=z_0$ with $|z_0|=1$ such that

$$\Big|z_0f'(z_0)+\beta\frac{n+sk}{1+k}f(z_0)\Big|>\Big|z_0F'(z_0)+\beta\frac{n+sk}{1+k}F(z_0)\Big|.$$

Now take

$$\alpha=-\frac{z_0F'(z_0)+\beta\frac{n+sk}{1+k}F(z_0)}{z_0f'(z_0)+\beta\frac{n+sk}{1+k}f(z_0)},$$

then $|\alpha|<1$ and with this choice of α, we have from (2.3), $T(z_0)=0$ for $|z_0|=1$. But this contradicts the fact that $T(z)\neq 0$ for $|z|=1$. For β with $|\beta|=1$, the inequality (2.4) follows by continuity. This is equivalent to the desired result. □

If we take $F(z)=M(\frac{z}{k})^n$ in Lemma 2.2, where $M=\max_{|z|=k}|p(z)|$, then we have:

Lemma 2.3. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin, then for any $\beta\in\mathbb{C}$ with $|\beta|\leq 1$, $k\leq 1$ and $|z|=1$,*

$$\Big|zp'(z)+\beta\frac{n+sk}{1+k}p(z)\Big|\leq k^{-n}\Big|n+\beta\frac{n+sk}{1+k}\Big|\max_{|z|=k}|p(z)|. \tag{2.5}$$

Lemma 2.4. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin and $k\geq 1$, then for any $\beta\in\mathbb{C}$ with $|\beta|\leq 1$ and $|z|=1$,*

$$\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|\leq k^{-s}\Big|n+\beta\frac{nk+s}{1+k}\Big|\max_{|z|=k}|p(z)|, \tag{2.6}$$

where

$$q(z)=z^{n+s}\overline{p\Big(\frac{1}{\overline{z}}\Big)}.$$

Proof. Let $p(z)=z^sh(z)$, where $h(z)$ is a polynomial of degree $n-s$. Then the polynomial

$$q(z)=z^{n+s}\overline{p\Big(\frac{1}{\overline{z}}\Big)}=z^n\overline{h\Big(\frac{1}{\overline{z}}\Big)}=z^s\Big(z^{n-s}\overline{h\Big(\frac{1}{\overline{z}}\Big)}\Big)$$

is of degree n with s-fold zeros at the origin. Also

$$\max_{|z|=\frac{1}{k}}|q(z)|=\frac{1}{k^{n+s}}\max_{|z|=k}|p(z)|.$$

By applying Lemma 2.3 for the polynomial $q(z)$, we get the result. □

Lemma 2.5. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin and $k\geq 1$, then for any $\beta\in\mathbb{C}$ with $|\beta|\leq 1$ and $|z|=1$,*

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)\Big|+k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|\\ \leq&\Big\{k^s\Big|s+\beta\frac{nk+s}{1+k}\Big|+k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|\Big\}\max_{|z|=k}|p(z)|,\end{aligned}\tag{2.7}$$

where

$$q(z)=z^{n+s}\overline{p\Big(\frac{1}{\overline{z}}\Big)}.$$

Proof. Let $M=\max_{|z|=k}|p(z)|$, then for every complex number α with $|\alpha|>1$, it follows by Rouche's Theorem that the polynomial $G(z)=p(z)-\alpha M(\frac{z}{k})^s$ has no zeros in $|z|<k$, except s-fold zeros at the origin. Correspondingly the polynomial

$$H(z)=z^{n+s}\overline{G\Big(\frac{1}{\overline{z}}\Big)}=q(z)-\overline{\alpha}k^{-s}Mz^n,$$

has all its zeros in $|z|\leq 1/k$ with s-fold zeros at the origin and

$$\Big|\frac{1}{k^{n+s}}G(k^2z)\Big|=|H(z)|$$

for $|z|=1/k$. Therefore, by applying Lemma 2.2 to polynomials $G(k^2z)$ and $k^{n+s}H(z)$, we have for $|\beta|\leq 1$, $1/k\leq 1$ and $|z|=1$,

$$\Big|zk^2G'(k^2z)+\beta\frac{nk+s}{1+k}G(k^2z)\Big|\leq k^{n+s}\Big|zH'(z)+\beta\frac{nk+s}{1+k}H(z)\Big|,$$

or

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)-\alpha\Big(s+\beta\frac{nk+s}{1+k}\Big)k^sMz^s\Big|\\ \leq&\Big|k^{n+s}\Big(zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big)-\overline{\alpha}k^n\Big(n+\beta\frac{nk+s}{1+k}\Big)Mz^n\Big|.\end{aligned}\tag{2.8}$$

Now by applying the inequality (2.6) and choosing a suitable argument of α, we have

$$\begin{aligned}&\Big|k^{n+s}\Big(zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big)-\overline{\alpha}k^n\Big(n+\beta\frac{nk+s}{1+k}\Big)Mz^n\Big|\\ =&|\alpha|k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|M-k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|.\end{aligned}\tag{2.9}$$

By combining inequalities (2.8) and (2.9), we obtain

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)\Big|-|\alpha|\Big|s+\beta\frac{nk+s}{1+k}\Big|k^sM\\ \leq&|\alpha|k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|M-k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|.\end{aligned}\tag{2.10}$$

Or

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)\Big|+k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|\\&\le|\alpha|\Big\{k^s\Big|s+\beta\frac{nk+s}{1+k}\Big|+k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|\Big\}M.\end{aligned}\tag{2.11}$$

Making $|\alpha|\to 1$ we have the result. □

The following lemma is due to Zireh [11].

Lemma 2.6. *If*

$$p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$$

is a polynomial of degree n, having all its zeros in $|z|<k$, $(k>0)$, then $m<k^n|a_n|$, where $m=\min_{|z|=k}|p(z)|$.

3 Proofs of the theorems

Proof of Theorem 1.1. If $p(z)$ has a zero on $|z|=k$, then $\min_{|z|=k}|p(z)|=0$ and the inequality (1.9) is true. Therefore we suppose that $p(z)$ has all its zeros in $|z|<k$ with s-fold zeros at the origin. We consider $p(z)=z^sh(z)$, where $h(z)$ is a polynomial of degree $(n-s)$ has all its zeros in $|z|<k$ and $h(0)\neq 0$. Let $m=\min_{|z|=k}|p(z)|$ and $m_1=\min_{|z|=k}|h(z)|$ then $m=k^sm_1>0$ and

$$|p(z)|\ge m\Big|\Big(\frac{z}{k}\Big)\Big|$$

for $|z|=k$, hence

$$|h(z)|\ge m_1\Big|\Big(\frac{z}{k}\Big)^{n-s}\Big|$$

for $|z|=k$. Therefore, if $|\lambda|<1$ then it follows by Rouche's Theorem that the polynomial

$$G(z)=p(z)-\lambda m\Big(\frac{z}{k}\Big)^n=z^s\Big(h(z)-\lambda m_1\Big(\frac{z}{k}\Big)^{n-s}\Big)$$

has all its zeros in $|z|<k$ with s-fold zeros at the origin. Also by using Lemma 2.6 the polynomial

$$G(z)=p(z)-\lambda m\Big(\frac{z}{k}\Big)^n$$

is of degree n, for $|\lambda|<1$. On applying Lemma 2.1 to the polynomial $G(z)$ of degree n, we get

$$|zG'(z)|\ge\frac{n+sk}{1+k}|G(z)|,$$

i.e.,

$$\Big|zp'(z)-\lambda mn\Big(\frac{z}{k}\Big)^n\Big| \geq \frac{n+sk}{1+k}\Big|p(z)-\lambda m\Big(\frac{z}{k}\Big)^n\Big|,$$

where $|z|=1$.

Therefore for β with $|\beta|<1$, it can be easily verified that the polynomial

$$T(z)=\Big(zp'(z)-\lambda mn\Big(\frac{z}{k}\Big)^n\Big)+\beta\frac{n+sk}{1+k}\Big\{p(z)-\lambda m\Big(\frac{z}{k}\Big)^n\Big\},$$

i.e.,

$$T(z)=\Big(zp'(z)+\beta\frac{n+sk}{1+k}p(z)\Big)-\lambda m\Big(\frac{z}{k}\Big)^n\Big(n+\beta\frac{n+sk}{1+k}\Big)$$

will have no zeros on $|z|=1$. As $|\lambda|<1$ we have for β with $|\beta|<1$ and $|z|=1$,

$$\Big|zp'(z)+\beta\frac{n+sk}{1+k}p(z)\Big| > m\Big|\lambda\Big(\frac{z}{k}\Big)^n\Big|\Big|n+\beta\frac{n+sk}{1+k}\Big|,$$

i.e.,

$$\Big|zp'(z)+\beta\frac{n+sk}{1+k}p(z)\Big| \geq mk^{-n}\Big|n+\beta\frac{n+sk}{1+k}\Big|. \tag{3.1}$$

For β with $|\beta|=1$, (3.1) follows by continuity. This completes the proof of Theorem 1.1. □

Proof of the Theorem 1.2. Let $m=\min_{|z|=k}|p(z)|$. By hypothesis the polynomial $p(z)$ has no zeros in $|z|<k$, except s-fold zeros at the origin. Correspondingly the polynomial

$$q(z)=z^{n+s}\overline{p\Big(\frac{1}{\overline{z}}\Big)}$$

has all its zeros in $|z|\leq 1/k$ with s-fold zeros at the origin and

$$\frac{1}{k^{n+s}}|p(k^2z)|=|q(z)|$$

for $|z|=1/k$. Then by applying Lemma 2.2 to the polynomials $p(k^2z)$ and $k^{n+s}q(z)$, we have for $|z|=1$,

$$\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)\Big| \leq k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|. \tag{3.2}$$

If $m=0$, by combining inequalities (3.2) and (2.7), Theorem 1.2 follows.

Therefore we suppose that $m\neq 0$ then for every complex number λ with $|\lambda|<1$, we have

$$|\lambda m|<m\leq |p(z)|,$$

where $|z|=k$. Hence by Rouche's Theorem the polynomial

$$G(z)=p(z)-\lambda m\Big(\frac{z}{k}\Big)^s$$

has no zero in $|z|<k$ except s-fold zeros at the origin. Therefore the polynomial

$$H(z)=z^{n+s}\overline{G(1/\overline{z})}=q(z)-\overline{\lambda}k^{-s}mz^n,$$

will have all its zeros in $|z|\le 1/k$ with s-fold zeros at the origin. Also $|G(k^2z)|=k^{n+s}|H(z)|$ for $|z|=1/k$.

On applying Lemma 2.2 for $G(k^2z)$ and $k^{n+s}H(z)$, we have

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)-\Big(s+\beta\frac{nk+s}{1+k}\Big)\lambda k^s mz^s\Big|\\ \le&\Big|k^{n+s}\Big(zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big)-k^n\Big(n+\beta\frac{nk+s}{1+k}\Big)\overline{\lambda}mz^n\Big|.\end{aligned}\tag{3.3}$$

By using the inequality (1.13), for an appropriate choice of the argument of λ, we have

$$\begin{aligned}&\Big|k^{n+s}\Big(zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big)-k^n\Big(n+\beta\frac{nk+s}{1+k}\Big)\overline{\lambda}mz^n\Big|\\ =&k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|-k^n\Big|n+\beta\frac{nk+s}{1+k}\Big||\lambda|m.\end{aligned}\tag{3.4}$$

By combining (3.3) and (3.4), we get for $|z|=1$ and $|\beta|\le 1$,

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)\Big|-k^s\Big|s+\beta\frac{nk+s}{1+k}\Big||\lambda|m\\ \le&k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|-k^n\Big|n+\beta\frac{nk+s}{1+k}\Big||\lambda|m.\end{aligned}$$

Equivalently

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)\Big|\\ \le&k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|-\Big\{k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|-k^s\Big|s+\beta\frac{nk+s}{1+k}\Big|\Big\}|\lambda|m.\end{aligned}$$

As $|\lambda|\to 1$, we have

$$\begin{aligned}&\Big|zk^2p'(k^2z)+\beta\frac{nk+s}{1+k}p(k^2z)\Big|\\ \le&k^{n+s}\Big|zq'(z)+\beta\frac{nk+s}{1+k}q(z)\Big|-\Big\{k^n\Big|n+\beta\frac{nk+s}{1+k}\Big|-k^s\Big|s+\beta\frac{nk+s}{1+k}\Big|\Big\}m.\end{aligned}$$

This is a conjunction with inequality (2.7), which completes the proof of Theorem 1.2. $\square$

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