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Toeplitz Type Operator Associated to Singular Integral Operator with Variable Kernel on Weighted Morrey Spaces

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Abstract. Suppose $T^{k,1}$ and $T^{k,2}$ are singular integrals with variable kernels and mixed homogeneity or $\pm I$ (the identity operator). Denote the Toeplitz type operator by

$$T^b=\sum_{k=1}^{Q} T^{k,1}M^bT^{k,2},$$

where $M^bf=bf$. In this paper, the boundedness of T^b on weighted Morrey space are obtained when b belongs to the weighted Lipschitz function space and weighted BMO function space, respectively.

Key Words: Toeplitz type operator, singular integral operator, variable Calderón-Zygmund kernel, weighted BMO function, weighted Lipschitz function, weighted Morrey space.

AMS Subject Classifications: 42B20, 40B35

1 Introduction

The classical Morrey spaces, introduced by Morrey [1] in 1938, have been studied intensively by various authors, and it, together with weighted Lebesgue spaces play an important role in the theory of partial differential equations, see [2,3]. The boundedness of the Hardy-Littlewood maximal operator, singular integral operator, fractional integral operator and commutator of these operators in Morrey spaces have been studied by Chiarenza and Frasca in [4]. Komori and Shirai [5] introduced a version of the weighted Morrey space $L^{p,\kappa}(\omega)$, which is a natural generalization of the weighted Lebesgue space $L^p(\omega)$.

As the development of singular integral operators, their commutators have been well studied [6–8]. In [7], the authors proved that the commutators $[b,T]$, which generated by

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Calderón-Zygmund singular integral operator and BMO functions, are bounded on L^p for $1<p<\infty$. The commutator generated by the Calderón-Zygmund operator T and a locally integrable function b can be regarded as a special case of the Toeplitz operator

$$T^b=\sum_{k=1}^{Q} T^{k,1}M^bT^{k,2}, \tag{1.1}$$

where $T^{k,1}$ and $T^{k,2}$ are the Calderón-Zygmund operators or $\pm I$ (the identity operator), $M^bf=bf$. When $b\in BMO$, the L^p-boundedness of T^b was discussed, see [9,10]. In [11,12], the authors studied the boundedness of T^b on Morrey spaces.

Let $K(x,\xi)$: $\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be a variable kernel with mixed homogeneity. The singular integral operator is defined by

$$Tf(x)=p.v.\int_{\mathbb{R}^n}K(x,x-y)f(y)dy. \tag{1.2}$$

The variable kernel $K(x,\xi)$ depends on some parameter x and possesses good properties with respect to the second variable ξ, which was firstly introduced by Fabes and Rieviéve in [13]. They generalized the classical Calderón-Zygmund kernel and the parabolic kernel studied by Jones in [14]. By introducing a new metric ρ, Fabes and Rieviéve studied (1.2) in $L^p(\mathbb{R}^n)$, where $\mathbb{R}^n$ was endowed with the topology induced by ρ and defined by ellipsoids.

By using this metric ρ, Softova in [15] obtained that the integral operator (1.2) and its commutator were continuous in generalized Morrey space $L^{p,\omega}(\mathbb{R}^n)$, $1<p<\infty$, ω satisfying suitable conditions. Ye and Zhu in [16] discussed the continuity of (1.2) and its multilinear commutator in the weighted Morrey spaces $L^{p,\kappa}(\omega)$, $1<p<\infty$, $0<\kappa<1$, and ω is A_p weight.

Suppose $T^{k,1}$ and $T^{k,2}$ are singular integrals whose kernels are variable kernel with mixed homogeneity or $\pm I$ (the identity operator). In this paper, we study the boundedness of Toeplitz operators T^b as (1.1) in weighted Morrey spaces when b belongs to weighted Lipschitz spaces and weighted BMO spaces, respectively. The main results are as follows.

Theorem 1.1. *Suppose that T^b is a Toeplitz type operator associated to singular integral operator with variable kernel, $\omega\in A_1$, and $b\in Lip_{\beta,\omega}$. Let $0<\kappa<p/q$, $1<p<n/\beta$ and $1/q=1/p-\beta/n$. If $T^1(f)=0$ for any $f\in L^{p,\kappa}(\omega)$, then there exists a constant $C>0$ such that,*

$$\|T^b(f)\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\le C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^{p,\kappa}(\omega)}.$$

Theorem 1.2. *Suppose that T^b is a Toeplitz type operator associated to singular integral operator with variable kernel, $\omega\in A_1$, and $b\in BMO(\omega)$. Let $1<p<\infty$, and $0<\kappa<1$. If $T^1(f)=0$ for any $f\in L^{p,\kappa}(\omega)$, then there exists a constant $C>0$ such that,*

$$\|T^b(f)\|_{L^{p,\kappa}(\omega^{1-p},\omega)}\le C\|b\|_{*,\omega}\|b\|_{L^{p,\kappa}(\omega)}.$$

2 Some preliminaries

Let $\alpha_1,\cdots,\alpha_n$ be real numbers, $\alpha_i\geq 1$ and define $\alpha=\sum_{i=1}^n\alpha_i$. Following Fabes and Riviére [6], the function $F(x,\rho)=\sum_{i=1}^n x_i^2\rho^{-2\alpha_i}$, for any fixed x, is a decreasing one with respect to $\rho>0$ and the equation $F(x,\rho)=1$ is uniquely solvable in $\rho(x)$. It is easy to check that $\rho(x-y)$ defines a distance between any two points $x,y\in\mathbb{R}^n$. Thus $\mathbb{R}^n$ endowed with the metric ρ results a homogeneous metric space [13,15]. The balls with respect to $\rho(x)$ centered at the origin and of radius r are the ellipsoids

$$\mathcal{E}_r(0)=\Big\{x\in\mathbb{R}^n:\ \frac{x_1^2}{\rho^{2\alpha_1}}+\cdots+\frac{x_n^2}{\rho^{2\alpha_n}}<1\Big\}$$

with Lebesgue measure $|\mathcal{E}_r|=C(n)r^\alpha$. It is easy to see that $\mathcal{E}_1(0)$ coincides with the unit sphere $\mathbb{S}^{n-1}$ with respect to the Euclidean metric.

Definition 2.1. The function $K(x,\xi):\ \mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ is called a variable kernel with mixed homogeneity if:

(i) for every fixed x, the function $K(x,\cdot)$ is a constant kernel satisfying

(1) $K(x,\cdot)\in C^\infty(\mathbb{R}^n\setminus\{0\})$,

(2) for any $\mu>0$, $\alpha_i\geq 1$, $\alpha=\sum_{i=1}^n\alpha_i$

$$K(x,\mu^{\alpha_1}\xi_1,\cdots,\mu^{\alpha_n}\xi_n)=\mu^{-\alpha}K(x,\xi),$$

(3) $\int_{\mathbb{S}^{n-1}}K(x,\xi)d\xi=0$ and $\int_{\mathbb{S}^{n-1}}|K(x,\xi)|d\xi<\infty$,

(ii) for every multiindex β, $\sup_{\xi\in\mathbb{S}^{n-1}}|D_\xi^\beta K(x,\xi)|\leq C(\beta)$ independent of x.

Note that in the special case $\alpha_i=1$, $1\leq i\leq n$, Definition 2.1 gives rise to the classical Calderón-Zygmund kernels. When $\alpha_i=1$, $1\leq i\leq n-1$, and $\alpha_n\geq 1$, we obtain the kernel studied by Jones in [14] and discussed in [13].

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $\mathcal{E}=\mathcal{E}_r(x_0)$ denote the ellipsoid with the center x_0 and radius r. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E)=\int_E\omega(x)dx$. For any given weight function ω on $\mathbb{R}^n$, $0<p<\infty$, denote by $L^p(\omega)$ the space of all function f satisfying

$$\|f\|_{L^p(\omega)}=\Big(\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx\Big)^{1/p}<\infty.$$

A weight ω is said to belong to the Muckenhoupt class A_p for $1<p<\infty$, if there exists a constant C such that

$$\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\omega(x)dx\Big)\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\omega(x)^{-\frac{1}{p-1}}dx\Big)^{p-1}\leq C$$

for every ellipsoid $\mathcal{E}$. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\omega(y)dy\le C\cdot\operatorname*{ess\,inf}_{x\in\mathcal{E}} w(x)$$

for every ball $\mathcal{E}$.

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p-boundedness of Hardy-Littlewood maximal function in [17].

Lemma 2.1. *Suppose $\omega\in A_1$. Then*
(i) there exists a $\epsilon>0$ such that

$$\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\omega(x)^{1+\epsilon}dx\Big)^{1/(1+\epsilon)}\le\frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\omega(x)dx, \tag{2.1}$$

(ii) there exist two constant C_1 and C_2, such that

$$C_1\omega(\mathcal{E})\le|\mathcal{E}|\,\inf_{x\in\mathcal{E}}\omega(x)\le C_2\omega(\mathcal{E}). \tag{2.2}$$

Let us recall the definition of weighted Lipschitz function space and weighted BMO function space.

Definition 2.2. For $1\le p<\infty$, $0<\beta<1$, and $\omega\in A_\infty$. A locally integrable function b is said to be in the weighted Lipschitz function space if

$$\sup_{\mathcal{E}}\frac{1}{\omega(\mathcal{E})^{\beta/n}}\Big[\frac{1}{\omega(\mathcal{E})}\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^p\omega(x)^{1-p}dx\Big]^{1/p}\le C<\infty,$$

where $b_{\mathcal{E}}=|\mathcal{E}|^{-1}\int_{\mathcal{E}}b(y)dy$, and the supremum is taken over all ellipsoids $\mathcal{E}$.

The Banach space of such functions modulo constants is denoted by $Lip_{\beta,p}(\omega)$. The smallest bound C satisfying conditions above is then taken to be the norm of b denoted by $\|b\|_{Lip_{\beta,p}(\omega)}$. Put $Lip_{\beta,\omega}=Lip_{\beta,1}(\omega)$. Obviously, for the case $\omega=1$, the $Lip_{\beta,p}(\omega)$ space is the classical Lip_β space. Let $\omega\in A_1$. García-Cuerva in [18] proved that the spaces $Lip_{\beta,p}(\omega)$ coincide, and the norms $\|b\|_{Lip_{\beta,p}(\omega)}$ are equivalent with respect to different values of p provided that $1\le p<\infty$. Since we always discuss under the assumption $\omega\in A_1$ in the following, then we denote the norm of $Lip_{\beta,p}(\omega)$ by $\|\cdot\|_{Lip_{\beta,\omega}}$ for $1\le p<\infty$.

Definition 2.3 (see [6])**.** Let b be a locally integrable function and ω be a weight function. A locally integrable function b is said to be in the weighted BMO function space $BMO(\omega)$, if there exists a constant C such that

$$\|b\|_{*,\omega}=\sup_{\mathcal{E}}\frac{1}{\omega(\mathcal{E})}\int_{\mathcal{E}}|b(y)-b_{\mathcal{E}}|dy<\infty,$$

where $b_{\mathcal{E}}=|\mathcal{E}|^{-1}\int_{\mathcal{E}}b(y)dy$, and the supremum is taken over all ellipsoids $\mathcal{E}$.

If $\omega\in A_1$, Garcín-Cuera in [19] showed that

$$C_1\|b\|_{*,\omega}\le\sup_{\mathcal{E}}\Big(\frac{1}{\omega(\mathcal{E})}\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^p\omega(x)^{1-p}dy\Big)^{1/p}\le C_2\|b\|_{*,\omega}$$

for $1\le p<\infty$.

Now we shall introduce the Hardy-Littlewood maximal operator and several variants.

For a given measurable function $f\in L^1_{loc}(\mathbb{R}^n)$, define the Hardy-Littlewood maximal operator Mf and the sharp maximal operator $M^\sharp f$ as

$$M(f)(x)=\sup_{x\in\mathcal{E}}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|f(y)|dy,$$
$$M^\sharp(f)(x)=\sup_{x\in\mathcal{E}}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|f(y)-f_{\mathcal{E}}|dy\approx\sup_{x\in\mathcal{E}}\inf_{c}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|f(y)-c|dy.$$

For $1\le r<\infty$, the weighted maximal operator $M_{\omega,r}f$ is defined by

$$M_{\omega,r}(f)(x)=\sup_{x\in\mathcal{E}}\Big(\frac{1}{\omega(\mathcal{E})}\int_{\mathcal{E}}|f(y)|^r\omega(y)dy\Big)^{1/r}.$$

For $0<\beta<n$, and $1\le r<\infty$, we define the fractional weighted maximal operator $M_{\beta,\omega,r}f$ by

$$M_{\beta,\omega,r}(f)(x)=\sup_{x\in\mathcal{E}}\Big(\frac{1}{\omega(\mathcal{E})^{1-r\beta/n}}\int_{\mathcal{E}}|f(y)|^r\omega(y)dy\Big)^{1/r},$$

where the supremum is taken over all ellipsoids $\mathcal{E}$.

Definition 2.4. Let $1\le p<\infty$, $0\le\kappa<1$ and ω be a weight function. Then for two weights μ and ν, the weighted Morrey space is defined by

$$L^{p,\kappa}(\mu,\nu)=\big\{f\in L^p_{loc}(\mu):\|f\|_{L^{p,\kappa}(\mu,\nu)}<\infty\big\},$$

where

$$\|f\|_{L^{p,\kappa}(\mu,\nu)}=\sup_{\mathcal{E}}\Big(\frac{1}{\nu(\mathcal{E})^\kappa}\int_{\mathcal{E}}|f(x)|^p\mu(x)dx\Big)^{1/p},$$

and the supremum is taken over all ellipsoids $\mathcal{E}$.

If $\nu=\mu$, then we have the classical Morrey space $L^{p,\kappa}(\mu)$ with measure μ.

Lemma 2.2 (see [20]). *Suppose* $\omega\in\cup_{1\le t<\infty}A_t$.

(i) If $1\le r<p<\infty$, *and* $0<\kappa<1$, *then*

$$\|M_{\omega,r}f\|_{L^{p,\kappa}(\omega)}\le C\|f\|_{L^{p,\kappa}(\omega)}. \tag{2.3}$$

(ii) If $0<\beta<n$, $1\le r<p<n/\beta$, $1/q=1/p-\beta/n$ *and* $0<\kappa<p/q$, *then*

$$\|M_{\beta,\omega,r}f\|_{L^{q,\kappa q/p}(\omega)}\le C\|f\|_{L^{p,\kappa}(\omega)}. \tag{2.4}$$

Lemma 2.3 (see [16]). *Let T be a singular integral operator with variable kernel, $1<p<\infty$ and $1<\kappa<1$. If $\omega\in A_p$, then there exists a constant $C>0$ such that*

$$\|Tf\|_{L^{p,\kappa}(\omega)}\le C\|f\|_{L^{p,\kappa}(\omega)}. \tag{2.5}$$

In view of Proposition 3.1 in [20], we have

Lemma 2.4. *Let $0<\kappa<1$ and $1<p<\infty$. If $\mu,\nu\in A_\infty$, then for every $f\in L_{loc}$ with $M^\sharp f\in L^{p,\kappa}(\mu,\nu)$, there exists a constant C such that*

$$\|M(f)\|_{L^{p,\kappa}(\mu,\nu)}\le C\|M^\sharp f\|_{L^{p,\kappa}(\mu,\nu)}. \tag{2.6}$$

The following lemmas play a critical role in the proof of our theorems.

Lemma 2.5. *Suppose $\omega\in A_1$, and $b\in Lip_{\beta,\omega}$ $(0<\beta<1)$. Then there exist a sufficiently large number s and a constant $C>0$ such that, for every $f\in L^p(\omega)$ with $p>1$ and $1<r<p$, we have*

$$\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^{s'}|f(x)|^{s'}dx\Big)^{\frac{1}{s'}}\le C\|b\|_{Lip_{\beta,\omega}}\omega(x)M_{\beta,\omega,r}(f)(x), \tag{2.7}$$

where $1/s+1/s'=1$.

Proof. Let $r_2=r/s'$, $r_3=\epsilon/(s'-1)$ and $1/r_1+1/r_2+1/r_3=1$, where ϵ is the constant in Lemma 2.1. Choosing a sufficiently large number s such that $1<s'<r(1+\epsilon)/(r+\epsilon)$, then $r_1,r_2,r_3>1$. By Hölder's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^{s'}|f(x)|^{s'}dx\Big)^{\frac{1}{s'}}\\
=&|\mathcal{E}|^{-\frac{1}{s'}}\Big(\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^{s'}\omega(x)^{\frac{1}{r_1}-s'}|f(x)|^{s'}\omega(x)^{\frac{1}{r_2}}\omega(x)^{s'-\frac{1}{r_1}-\frac{1}{r_2}}dx\Big)^{\frac{1}{s'}}\\
\le&C|\mathcal{E}|^{-\frac{1}{s'}}\Big(\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^{r_1s'}\omega(x)^{1-r_1s'}dx\Big)^{\frac{1}{r_1s'}}\Big(\int_B|f(x)|^{r_2s'}\omega(x)dx\Big)^{\frac{1}{r_2s'}}\\
&\times\Big(\int_{\mathcal{E}}\omega(x)^{1+r_3(s'-1)}dx\Big)^{\frac{1}{r_3s'}}.
\end{aligned}$$

Since $b\in Lip_{\beta,\omega}$, and $\omega\in A_1$, by (2.1), (2.2) we get

$$\begin{aligned}
&\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^{s'}|f(x)|^{s'}dx\Big)^{\frac{1}{s'}}\\
\le&C\|b\|_{Lip_{\beta,\omega}}|\mathcal{E}|^{-\frac{1}{s'}}\omega(\mathcal{E})^{\frac{\beta}{n}+\frac{1}{r_1s'}}\Big(\int_B|f(x)|^r\omega(x)dx\Big)^{\frac{1}{r}}\Big(\int_{\mathcal{E}}\omega(x)^{1+\epsilon}dx\Big)^{\frac{1}{r_3s'}}\\
\le&C\|b\|_{Lip_{\beta,\omega}}\frac{\omega(\mathcal{E})^{1+\frac{\beta}{n}}}{|\mathcal{E}|}\Big(\frac{1}{\omega(\mathcal{E})}\int_B|f(x)|^r\omega(x)dx\Big)^{\frac{1}{r}}\\
\le&C\|b\|_{Lip_{\beta,\omega}}\omega(x)M_{\beta,\omega,r}(f)(x).
\end{aligned}$$

∎

Similar to the proof of Lemma 2.5, we have

Lemma 2.6. *Suppose $\omega\in A_1$, and $b\in BMO(\omega)$. Then there exist sufficiently large number s and constant $C>0$ such that, for every $f\in L^p(\omega)$ with $p>1$ and $1<r<p$, we have*

$$\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|b(x)-b_{\mathcal{E}}|^{s'}|f(x)|^{s'}dx\Big)^{\frac{1}{s'}}\le C\omega(x)\|b\|_{*,\omega}M_{\omega,r}(f)(x), \tag{2.8}$$

where $1/s+1/s'=1$.

Finally, we need the spherical harmonics and their properties (see more detail in [13, 15]). Recall that any homogeneous polynomial $P:\mathbb{R}^n\to\mathbb{R}$ of degree m that satisfies $\Delta P=0$ is called an n-dimensional solid harmonic of degree m. Its restriction to the unit sphere $\mathbb{S}^{n-1}$ will be called an n-dimensional spherical harmonic of degree m Denote by $\mathbb{H}_m$ the space of all n-dimensional spherical harmonics of degree m. In general it results in a finite dimensional linear space with $g_m=dim\mathbb{H}_m$ such that $g_0=1$, $g_1=n$ and

$$g_m=C^{n-1}_{m+n-1}-C^{n-1}_{m+n-3}\le C(n)m^{n-2},\quad m\ge 2. \tag{2.9}$$

Furthermore, let $\{Y_{sm}\}^{g_m}_{s=1}$ be an orthonormal base of $\mathbb{H}_m$, then $\{Y_{sm}\}^{g_m\infty}_{s=1m=0}$ is a complete orthonormal system in $L^2(\mathbb{S}^{n-1})$ and

$$\sup_{x\in\mathbb{S}^{n-1}}|D^\beta_x Y_{sm}(x)|\le C(n)m^{|\beta|+(n-2)/2},\quad m=1,2,\cdots. \tag{2.10}$$

If, for instance, $\phi\in C^\infty(\mathbb{S}^{n-1})$, then $\sum_{s,m}b_{sm}Y_{sm}$ is the Fourier series expansion of $\phi(x)$ with respect to $\{Y_{sm}\}_{sm}$ then

$$b_{sm}=\int_{\mathbb{S}^{n-1}}\phi(y)Y_{sm}(y)d\sigma,\quad |b_{sm}|\le C(n,l)m^{-2l}\sup_{|\beta|=2l}\sup_{y\in\mathbb{S}^{n-1}}|D^\beta_y\phi(y)|, \tag{2.11}$$

for any integer l. In particular, the expansion of ϕ into spherical harmonics converges uniformly to ϕ. For the proof of the above results see [21].

Let $x,y\in\mathbb{R}^n$, and

$$\overline{y}=\frac{y}{\rho(y)}=\Big(\frac{y_1}{\rho(y)^{\alpha_1}},\cdots,\frac{y_n}{\rho(y)^{\alpha_n}}\Big)\in\mathbb{S}^{n-1}.$$

In view of the properties of the kernel K with respect to the second variable and the complete of $\{Y_{sm}(x)\}$ in $L^2(\mathbb{S}^{n-1})$, we get

$$\begin{aligned}K(x,x-y)=&\rho(x-y)^{-\alpha}K(x,\overline{x-y})\\ =&\rho(x-y)^{-\alpha}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}b_{sm}(x)Y_{sm}(\overline{x-y}).\end{aligned}$$

Replacing the kernel with its series expansion, (1.2) can be written as

$$\begin{aligned}T(f)(x)=&\lim_{\epsilon\to 0}T_\epsilon(f)(x)\\=&\lim_{\epsilon\to 0}\int_{\rho(x-y)>\epsilon}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}b_{sm}(x)\rho(x-y)^{-\alpha}Y_{sm}(\overline{x-y})f(y)dy.\end{aligned}$$

From the properties of (2.9)-(2.11), the series expansion

$$\begin{aligned}&\Big|\sum_{m=1}^{N}\sum_{s=1}^{g_m}b_{sm}(x)\rho(x-y)^{-\alpha}Y_{sm}(\overline{x-y})f(y)f(y)\Big|\\\le& C(n,\alpha)\frac{|f(y)|}{\rho(x-y)^\alpha}\sum_{m=1}^{\infty}m^{3(n-2)/2-2l},\end{aligned}$$

where the integer l is preliminarily chosen greater than $(3n-2)/4$. Along with the $\rho(x-y)^{-\alpha}f(y)\in L^1(\mathbb{R}^n)$ for almost everywhere $x\in\mathbb{R}^n$, by the Fubini dominated convergence theorem, we have

$$\begin{aligned}T(f)(x)=&\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}b_{sm}(x)\lim_{\epsilon\to 0}\int_{\rho(x-y)>\epsilon}H_{sm}(x-y)f(y)dy\\=&\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}b_{sm}(x)T_{sm}f(x),\end{aligned}\tag{2.12}$$

where

$$H_{sm}(x-y)=\rho(x-y)^{-\alpha}Y_{sm}(\overline{x-y}),$$

and H_{sm} satisfies pointwise Hörmander condition as following

$$|H_{sm}(x-y)-H_{sm}(x_0-y)|\le C(n,\alpha)m^{n/2}\frac{\rho(x_0-x)}{\rho(x-y)^{\alpha+1}}\tag{2.13}$$

for each $x\in\mathcal{E}$ and $y\notin 2\mathcal{E}$ (see [15, Lemma 3.2]). Then

$$\begin{aligned}T_{sm}f(x)=&\lim_{\epsilon\to 0}\int_{\rho(x-y)>\epsilon}H_{sm}(x-y)f(y)dy\\=&p.v.\int_{\mathbb{R}^n}H_{sm}(x-y)f(y)dy\end{aligned}\tag{2.14}$$

is a classical Calderón-Zygmund operator with a constant kernel.

3 Proof of theorems

Proof of Theorem 1.1. We only give the proof of Theorem 1.1, since the proof of Theorem 1.2 is similar to Theorem 1.1. Let

$$T^b(f)(x)=\sum_{k=1}^{Q} T^{k,1}M^bT^{k,2}(f)(x).$$

Without loss generality, we may assume $T^{k,1}$ $(k=1,\cdots,Q)$ are singular integral operators with variable kernel. By (2.12),

$$T^b(f)(x)=\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m} b_{sm}^{k,1}(x)T_{sm}^{k,1}M^bT^{k,2}(f)(x),$$

where

$$T_{sm}^{k,1}(f)(x)=\int_{\mathbb{R}^n} H_{sm}^{k,1}(x-y)f(y)dy$$

are classical Calderón-Zygmund operator with constant kernel as (2.14). Set $\mathcal{E}$ for the ellipsoid centered at x_0 and of radius r, and let $\mathcal{E}\ni x$. Since $T^1(g)=0$ for any $g\in L^{p,\kappa}(\omega)$, then

$$T^b(f)(x)=\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m} b_{sm}^{k,1}(x)T_{sm}^{k,1}M^{b-b_{2\mathcal{E}}}T^{k,2}(f)(x).$$

We first prove

$$\begin{aligned}&M^{\sharp}T_{sm}^{k,1}M^{b-b_{2\mathcal{E}}}T^{k,2}(f)(x)\\ \le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)\big(M_{\beta,\omega,r}(T^{k,2}(f))(x)+M_{\beta,\omega,1}(T^{k,2}(f))(x)\big)\end{aligned} \tag{3.1}$$

for arbitrary $x\in\mathcal{E}$. We write $T_{sm}^{k,1}M^{b-b_{2\mathcal{E}}}T^{k,2}(f)(x)$ as

$$\begin{aligned}&T_{sm}^{k,1}M^{b-b_{2\mathcal{E}}}T^{k,2}(f)(y)\\ =&T_{sm}^{k,1}M^{(b-b_{2\mathcal{E}})\chi_{2\mathcal{E}}}T^{k,2}(f)(y)+T_{sm}^{k,1}M^{(b-b_{2\mathcal{E}})\chi_{(2\mathcal{E})^c}}T^{k,2}(f)(y)\\ =&\mathrm{U}_1(y)+\mathrm{U}_2(y).\end{aligned}$$

Taking $c=\mathrm{U}_2(x_0)$, then

$$\begin{aligned}&\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|T_{sm}^{k,1}M^{b-b_{2\mathcal{E}}}T^{k,2}(f)(y)-c|dy\\ \le&\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|\mathrm{U}_1(y)|dy+\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|\mathrm{U}_2(y)-\mathrm{U}_2(x_0)|dy\\ =& M_1+M_2.\end{aligned}$$

Choosing a sufficiently large number s and by Hölder's inequality, the boundedness of $T_{sm}^{k,1}$ in $L^{s'}(\mathbb{R}^n)$ and Lemma 2.5, we have

$$\begin{aligned}M_1=&\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|T_{sm}^{k,1}M^{(b-b_{2\mathcal{E}})\chi_{2\mathcal{E}}}T^{k,2}(f)(y)|dy\\ \le&\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|T_{sm}^{k,1}M^{(b-b_{2\mathcal{E}})\chi_{2\mathcal{E}}}T^{k,2}(f)(y)|^{s'}dy\Big)^{1/s'}\\ \le&C\Big(\frac{1}{|\mathcal{E}|}\int_{\mathbb{R}^n}|M^{(b-b_{2\mathcal{E}})\chi_{2\mathcal{E}}}T^{k,2}(f)(y)|^{s'}dy\Big)^{1/s'}\\ \le&C\|b\|_{Lip_{\beta,\omega}}\omega(x)M_{\beta,\omega,r}(T^{k,2}(f))(x).\end{aligned}$$

For any $y\in\mathcal{E}$, and $z\in(2\mathcal{E})^c$, we have $\rho(y-z)\sim\rho(x_0-z)$. Then by (2.13) we get,

$$\begin{aligned}M_2\le&\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|T_{sm}^{k,1}M^{(b-b_{2\mathcal{E}})\chi_{(2\mathcal{E})^c}}T^{k,2}(f)(y)-T_{sm}^{k,1}M^{(b-b_{2\mathcal{E}})\chi_{(2\mathcal{E})^c}}T^{k,2}(f)(x_0)|dy\\ \le&C\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\int_{(2\mathcal{E})^c}|b(z)-b_{2\mathcal{E}}||H_{sm}^{k,1}(y-z)-H_{sm}^{k,1}(x_0-z)||T^{k,2}(f)(z)|dzdy\\ \le&Cm^{n/2}\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\int_{(2\mathcal{E})^c}|b(z)-b_{2\mathcal{E}}|\frac{\rho(x_0-y)}{\rho(y-z)^{\alpha+1}}|T^{k,2}(f)(z)|dzdy\\ \le&Cm^{n/2}\sum_{j=1}^{\infty}\int_{2^{j+1}\mathcal{E}\backslash 2^j\mathcal{E}}|b(z)-b_{2\mathcal{E}}|\frac{\rho(x_0-y)}{\rho(x_0-z)^{\alpha+1}}T^{k,2}(f)(z)|dz\\ \le&Cm^{n/2}\sum_{j=1}^{\infty}\frac{r}{(2^kr)^{\alpha+1}}\int_{2^{j+1}\mathcal{E}}|b(z)-b_{2\mathcal{E}}||T^{k,2}(f)(z)|dz\\ \le&Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}\mathcal{E}}-b_{2\mathcal{E}}|\frac{1}{|2^{j+1}\mathcal{E}|}\int_{2^{j+1}\mathcal{E}}|T^{k,2}(f)(z)|dz\\ &+Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}\mathcal{E}|}\int_{2^{j+1}\mathcal{E}}|b(z)-b_{2^{j+1}\mathcal{E}}||T^{k,2}(f)(z)|dz\\ =&M_{21}+M_{22}.\end{aligned}$$

Note that $\omega\in A_1$, and

$$\begin{aligned}|b_{2^{j+1}\mathcal{E}}-b_{2\mathcal{E}}|\le&\sum_{k=1}^{j}\frac{1}{|2^k\mathcal{E}|}\int_{2^{k+1}\mathcal{E}}|b(z)-b_{2^{k+1}\mathcal{E}}|dz\\ \le&C\|b\|_{Lip_{\beta,\omega}}\sum_{k=1}^{j}\frac{\omega(2^{k+1}\mathcal{E})^{1+\beta/n}}{|2^k\mathcal{E}|}\\ \le&C\|b\|_{Lip_{\beta,\omega}}\sum_{k=1}^{j}\inf_{x\in 2^{k+1}\mathcal{E}}\omega(x)\omega(2^{k+1}\mathcal{E})^{\frac{\beta}{n}}\\ \le&Cj\|b\|_{Lip_{\beta,\omega}}\omega(x)\omega(2^{j+1}\mathcal{E})^{\beta/n},\end{aligned}$$

then by (2.2), we get

$$
\begin{aligned}
M_{21} =& Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}\mathcal{E}}-b_{2\mathcal{E}}|\frac{1}{|2^{j+1}\mathcal{E}|}\int_{2^{j+1}\mathcal{E}}|T^{k,2}(f)(z)|dz\\
\le& Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}\mathcal{E}}-b_{2\mathcal{E}}|\frac{1}{\omega(2^{j+1}\mathcal{E})}\int_{2^{j+1}\mathcal{E}}|T^{k,2}(f)(z)|\omega(z)dz\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)\sum_{j=1}^{\infty}j2^{-j}\frac{1}{\omega(2^{j+1}\mathcal{E})^{1-\beta/n}}\int_{2^{j+1}\mathcal{E}}|T^{k,2}(f)(z)|\omega(z)dz\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)M_{\beta,\omega,1}(T^{k,2}(f))(x)\sum_{j=1}^{\infty}j2^{-j}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)M_{\beta,\omega,1}(T^{k,2}(f))(x).
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
M_{22} =& Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\frac{1}{|2^{j+1}\mathcal{E}|}\int_{2^{j+1}\mathcal{E}}|b(z)-b_{2^{j+1}\mathcal{E}}||T^{k,2}(f)(z)|dz\\
\le& Cm^{n/2}\sum_{j=1}^{\infty}2^{-j}\Big(\frac{1}{|2^{j+1}\mathcal{E}|}\int_{2^{j+1}\mathcal{E}}|b(z)-b_{2^{j+1}\mathcal{E}}|^{r'}\omega(z)^{1-r'}dz\Big)^{\frac{1}{r'}}\\
&\times\Big(\frac{1}{|2^{j+1}\mathcal{E}|}\int_{2^{j+1}\mathcal{E}}|T^{k,2}(f)(z)|^{r}\omega(z)dz\Big)^{\frac{1}{r}}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\sum_{j=1}^{\infty}2^{-j}\frac{\omega(2^{j+1}\mathcal{E})^{1+\beta/n}}{|2^{j+1}\mathcal{E}|}\\
&\times\Big(\frac{1}{\omega(2^{j+1}\mathcal{E})}\int_{2^{j+1}\mathcal{E}}|T^{k,2}(f)(z)|^{r}\omega(z)dz\Big)^{\frac{1}{r}}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)M_{\beta,\omega,r}(T^{k,2}(f))(x)\sum_{j=1}^{\infty}2^{-j}\\
\le& Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)M_{\beta,\omega,r}(T^{k,2}(f))(x).
\end{aligned}
$$

Hence

$$
M_2\le Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\omega(x)(M_{\beta,\omega,r}(T^{k,2}(f))(x)+M_{\beta,\omega,1}(T^{k,2}(f))(x)).
$$

Combining the estimates for M_1 and M_2, we finish the proof of (3.1).

Since $\omega\in A_1$ implies $\omega^{1-q}\in A_q$, by Lemma 2.4, (3.1), Lemma 2.2 and Lemma 2.3, we have

$$
\begin{aligned}
&\|T_{sm}^{k,1}M^{b-b_{2\varepsilon}}T^{k,2}(f)\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\\
\le&\|MT_{sm}^{k,1}M^{b-b_{2\varepsilon}}T^{k,2}(f)\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\\
\le&\|M^{\sharp}T_{sm}^{k,1}M^{b-b_{2\varepsilon}}T^{k,2}(f)\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\\
\le&Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\|\omega(\cdot)(M_{\beta,\omega,r}(T^{k,2}(f))+M_{\beta,\omega,1}(T^{k,2}(f)))\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\\
=&Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\|M_{\beta,\omega,r}(T^{k,2}(f))+M_{\beta,\omega,1}(T^{k,2}(f)\|_{L^{q,\kappa q/p}(\omega)}\\
\le&Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\|T^{k,2}f\|_{L^{p,\kappa}(\omega)}\\
\le&Cm^{n/2}\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}
$$

Choosing $l>(3n-2)/4$, then

$$
\begin{aligned}
&\|T^{b}(f)\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\\
\le&\Big\|\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}b_{sm}^{k,1}(x)T_{sm}^{k,1}M^{b-b_{2\varepsilon}}T^{k,2}(f)(x)\Big\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\\
\le&\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}\|b_{sm}^{k,1}(x)\|_{L^{\infty}}\|T_{sm}^{k,1}M^{b-b_{2\varepsilon}}T^{k,2}(f)\|_{L^{q,\kappa q/p}(\omega^{1-q},\omega)}\\
\le&C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^{p,\kappa}(\omega)}\sum_{k=1}^{Q}\sum_{m=1}^{\infty}\sum_{s=1}^{g_m}m^{-2l+n/2}\\
\le&C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^{p,\kappa}(\omega)}\sum_{m=1}^{\infty}m^{-2l+n/2+n-2}\\
\le&C\|b\|_{Lip_{\beta,\omega}}\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}
$$

This finishes the proof of Theorem 1.1. □

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