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Commutators of Lipschitz Functions and Singular Integrals with Non-Smooth Kernels on Euclidean Spaces

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Abstract. In this article, we obtain the L^p-boundedness of commutators of Lipschitz functions and singular integrals with non-smooth kernels on Euclidean spaces.

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AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction

Consider the singular integral operator T defined by

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy, \tag{1.1}$$

where f is a continuous function with compact support, $x\notin \operatorname{supp} f$; and the kernel $K(x,y)$ is a measurable function defined on $(\mathbb{R}^n\times\mathbb{R}^n)\setminus\Delta$ with $\Delta=\{(x,x):x\in\mathbb{R}^n\}$. If $b\in \mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[b,T]$ of a BMO function b and the singular integral operator T is defined by

$$T_bf:=[b,T](f):=T(bf)-bT(f).$$

The L^p-boundedness $(1<p<\infty)$ of T and T_b are well known in the Euclidean setting, provided that the kernel $K(x,y)$ of the operator T satisfies Hörmander's conditions (see [1, 15–17] among many other good references). In 1999, Duong and McIntosh [3] obtained the L^p-boundedness of T, under the assumption that the kernel $K(x,y)$ satisfies some conditions which are weaker than Hörmander's integral conditions. The boundedness of the operator T with non-smooth kernel on $L^p(w)$ $(w\in\mathcal{A}_p(\mathbb{R}^n)$, $1<p<\infty)$ was proved by Martell [12]. Moreover, Duong and Yan [4] obtained the L^p-boundedness of

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the commutator T_b under some conditions which are weaker than Hörmander's pointwise conditions. Lin and Jiang [11] also obtained the L^p-boundedness of T_b, but with $b\in \mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)$. See also [8,9,13,18] for additional results on these topics.

The purpose of this paper is to extend the results in [11]. That is, we would like to obtain the L^p-boundedness $(1<p<\infty)$ of the operator $T_{\vec{b}}$, where

$$T_{\vec{b}}f(x)=\int_{\mathbb{R}^n}\Big\{\prod_{i=1}^{k}(b_i(x)-b_i(y))\Big\}K(x,y)f(y)dy, \tag{1.2}$$

$b_i\in \mathrm{Lip}_{\alpha_i,w}(\mathbb{R}^n)$ for $1\le i\le k$, and the weight w belongs to a subclass of $\mathcal{A}_1$.

2 Background

2.1 $\mathcal{A}_p$ weights

For a ball B in $\mathbb{R}^n$, let $|B|$ denote the measure of the ball B. A weight w is said to belong to the Muckenhoupt class $\mathcal{A}_p(\mathbb{R}^n)$, $1<p<\infty$, if there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w^{-p'/p}(x)dx\right)^{p/p'}\le C<\infty,$$

for all balls B in $\mathbb{R}^n$. The smallest constant C for which the above inequality holds is the $\mathcal{A}_p$ bound of w. The class $\mathcal{A}_1(\mathbb{R}^n)$ consists of non-negative functions w such that

$$\frac{w(B)}{|B|}:=\frac{1}{|B|}\int_B w(x)dx\le C\ \text{ess}\inf_{x\in B}w(x)$$

for all balls B in $\mathbb{R}^n$. It is well-known that (see [7,17] for instance) if $w\in\mathcal{A}_p(\mathbb{R}^n)$ for some $p\in[1,\infty)$, then for any measurable subset $E\subset B$, there exist positive constants γ and C such that

$$\frac{w(E)}{w(B)}\le C\left(\frac{|E|}{|B|}\right)^{\gamma}. \tag{2.1}$$

Inequality (2.1) indeed holds with $\gamma\in(0,1)$. This will be used in the estimate of (3.3) below. Furthermore, if $w\in\mathcal{A}_p(\mathbb{R}^n)$ $(1\le p\le\infty)$, then it satisfies the reverse Hölder inequality. That is, there exist $s'>1$ and $c>0$ (both depending on w) so that

$$\left(\frac{1}{|B|}\int_B w(x)^{s'}dx\right)^{1/s'}\le\frac{c}{|B|}\int_B w(x)dx\quad\text{for all balls }\ B\subset\mathbb{R}^n. \tag{2.2}$$

A weight w is said to belong to the class $\mathcal{A}_{p,q}(\mathbb{R}^n)$, $1<p,q<\infty$, if there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B w^q(x)dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w^{-p'}(x)dx\right)^{1/p'}\le C<\infty,$$

for all balls $B\subset\mathbb{R}^n$. Observe that

$$w\in\mathcal{A}_{p,q}(\mathbb{R}^n)\Leftrightarrow w^q\in\mathcal{A}_{1+q/p'}(\mathbb{R}^n).$$

When $p=1$ and $q>1$, we say that $w\in\mathcal{A}_{1,q}(\mathbb{R}^n)$ if there exists a positive constant C such that for all balls $B\subset\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B w^q(x)dx\right)^{1/q}\le C\ \mathrm{ess\ inf}_{x\in B}w(x).$$

It follows from Hölder's inequality that for $1<q_1<q_2<\infty$,

$$\mathcal{A}_{1,q_2}(\mathbb{R}^n)\subset\mathcal{A}_{1,q_1}(\mathbb{R}^n)\subset\mathcal{A}_1(\mathbb{R}^n).$$

Also, it is clear from the definition of $\mathcal{A}_{1,q}(\mathbb{R}^n)$ that $w\in\mathcal{A}_{1,q}(\mathbb{R}^n)$ implies $w^q\in A_1(\mathbb{R}^n)$.

A locally integrable function f is said to belong to the spaces $\mathrm{Lip}^p_{\alpha,w}(\mathbb{R}^n)$ for $1\le p<\infty$, $0<\alpha<1$, and $w\in\mathcal{A}_\infty$ if

$$\|f\|_{\mathrm{Lip}^p_{\alpha,w}(\mathbb{R}^n)}:=\sup_B\left\{\frac{1}{w(B)^{\alpha/n}}\left(\frac{1}{w(B)}\int_B|f(x)-f_B|^p w^{1-p}(x)dx\right)^{1/p}\right\}<\infty,$$

where the supremum is taken over all balls $B\subset\mathbb{R}^n$, and $f_B:=\frac{1}{|B|}\int_B f(x)dx$.

When $p=1$, we simply denote $\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n):=\mathrm{Lip}^1_{\alpha,w}(\mathbb{R}^n)$. Note that if we set $w=1$, then the spaces $\mathrm{Lip}^p_{\alpha,w}(\mathbb{R}^n)$ are just the classical spaces $\mathrm{Lip}^p_\alpha(\mathbb{R}^n)$. Besides, when $w\in\mathcal{A}_1(\mathbb{R}^n)$, García-Cuerva [6] proved that the spaces $\mathrm{Lip}^p_{\alpha,w}(\mathbb{R}^n)$ are equivalent (with respect to the norms) for all $p\in[1,\infty)$. The interested reader may view [6,7,17] for more information on this subject. For $0<\alpha<1$, $1<r<p<n/\alpha$, $w\in\mathcal{A}_1(\mathbb{R}^n)$, and $f\in L^p(\mathbb{R}^n)$, define the maximal functions M_rf, $M_{\alpha,r}f$ and $M_{\alpha,r,w}f$ as follows:

$$M_rf(x):=\sup_{B\ni x}\left\{\frac{1}{|B|}\int_B|f(y)|^r dy\right\}^{1/r},$$
$$M_{\alpha,r}f(x):=\sup_{B\ni x}\left\{\frac{1}{|B|^{1-\alpha r/n}}\int_B|f(y)|^r dy\right\}^{1/r},$$

and

$$M_{\alpha,r,w}f(x):=\sup_{B\ni x}\left\{\frac{1}{w(B)^{1-\alpha r/n}}\int_B|f(y)|^r w(y)dy\right\}^{1/r}.$$

The following lemma is necessary for the proof of our theorem.

Lemma 2.1 (see [2,14]). *Let $0<\alpha<n$, $1\le r<p<n/\alpha$, $1/q=1/p-\alpha/n$. If $w^r\in\mathcal{A}_{p/r,q/r}(\mathbb{R}^n)$, then there exists a positive constant C independent of f such that*

$$\left(\int_{\mathbb{R}^n}|M_{\alpha,r}f(x)w(x)|^q dx\right)^{1/q}\le C\left(\int_{\mathbb{R}^n}|f(x)w(x)|^p dx\right)^{1/p}.$$

2.2 Approximation of the identity

We assume that there exists a class of operators A_t $(t>0)$ which can be represented by the kernels $a_t(x,y)$ in the sense that

$$A_t u(x)=\int_{\mathbb{R}^n} a_t(x,y)u(y)dy \quad \text{for every function } \ u\in L^1(\mathbb{R}^n)\cap L^r(\mathbb{R}^n), \quad (r>1).$$

Moreover, the kernels $a_t(x,y)$ satisfy the following conditions

$$|a_t(x,y)|\le h_t(x,y) \quad \text{for all } \ x,y\in\mathbb{R}^n, \tag{2.3}$$

where

$$h_t(x,y)=|B(x;t^{1/m})|^{-1}s(|x-y|^m t^{-1}) \quad \text{for some positive constant } \ m. \tag{2.4}$$

Here s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n(k+1)+\epsilon}s(r^m)=0 \quad \text{for some } \ \epsilon>0, \tag{2.5}$$

where k appears in (1.2).

Remark 2.1. The functions h_t above satisfy the following properties (see [4,5]):

i) There exist positive constants C_1 and C_2 such that

$$C_1\le\int_{\mathbb{R}^n} h_t(x,y)dx\le C_2 \quad \text{uniformly in } \ t \ \text{ and } \ y.$$

ii) There exists a positive constant C such that

$$\int_{\mathbb{R}^n} h_t(x,y)|f(x)|dx\le C\mathcal{M}f(y) \quad \text{and} \quad \int_{\mathbb{R}^n} h_t(x,y)|f(y)|dy\le C\mathcal{M}f(x),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator.

The class of operators A_t plays the role of approximation to the identity. The existence of such a class of operators A_t was verified in [3].

Now consider the operators T and $T_{\vec{b}}$ given in (1.1) and (1.2) respectively. Let A_t and B_t $(t>0)$ be two classes of operators which satisfy (2.3), (2.4) and (2.5). Denote by $K(x,y)-K_t(x,y)$ the kernels of the operators $(T-TB_t)$, and $K(x,y)-K^t(x,y)$ as the kernels of $(T-A_tT)$. We state below some assumptions which are necessary for our theorem.

(a) T is a bounded linear operator from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ for some $r\in(1,\infty)$;

(b) There exist positive constants c_1 and C_A such that

$$\int_{|x-y|\ge c_1 t^{1/m}} |K(x,y)-K_t(x,y)|dx\le C_A \quad \text{for all } \ y\in\mathbb{R}^n;$$

(c) There exist positive constants c_2, c_3 and $\beta>nk$ (k appears in (1.2)) such that

$$|K(x,y)-K^t(x,y)|\le\frac{c_3}{|B(x;|x-y|)|}\frac{t^{\beta/m}}{|x-y|^\beta} \quad \text{whenever } \ |x-y|\ge c_2 t^{1/m}.$$

In the sequel, the letter C will denote a constant, which may vary at different occurrences. However, it is independent of any essential variable.

3 Main theorem

Theorem 3.1. *Let $1<q_0<n/\alpha$, where $\alpha=\sum_{i=1}^k\alpha_i$, and $0<\alpha_1\le\alpha_2\le\cdots\le\alpha_k<1$. Fix an $s>1$ such that $1<s<q_0$ if $k=1$; otherwise, $1<s<\sqrt{q_0}$ if $k>1$. Let s' denote the conjugate of s. Set*

$$\tau_1=\max\Big\{s',\Big(1-\frac{\alpha q_0}{n}\Big)^{-1}\Big\},$$
$$\tau_2=1+(k-1)\Big(s'+\frac{1}{s'}-2\Big)q_0,$$
$$\tau_3=\begin{cases}0, & \text{if } k=1,\\ 1+\dfrac{n}{\alpha_1}, & \text{if } k>1.\end{cases}$$

Let $\tau=\max\{\tau_1,\tau_2,\tau_3\}$. Assume that $w\in\mathcal{A}_{1,\tau}(\mathbb{R}^n)$, and $b_i\in Lip_{\alpha_i,w}(\mathbb{R}^n)$ for $1\le i\le k$.

Let T, given by (1.1)*, satisfy assumptions (a), (b) and (c). Then there exists a constant $C>0$, independent of f, such that*

$$\|T_{\vec{b}}f\|_{L^{q_k}(w^{1-kq_k})}\le C\|\vec{b}\|_{Lip_{\alpha,w}(\mathbb{R}^n)}\|f\|_{L^{q_0}(w)},$$

where

$$\frac{1}{q_k}=\frac{1}{q_0}-\frac{\alpha}{n}\quad\text{and}\quad\|\vec{b}\|_{Lip_{\alpha,w}(\mathbb{R}^n)}=\prod_{i=1}^k\|b_i\|_{Lip_{\alpha_i,w}(\mathbb{R}^n)}.$$

Remark 3.1. Observe that for the case $k=1$, w is only required to be in $\mathcal{A}_{1,\tau_1}(\mathbb{R}^n)$.

Proof. First, we show that there exist $r_1,r_2,r_3>1$ such that $1<rs<q_0$, where $r:=r_1r_2r_3$. For the case $k=1$, since $1<s<q_0$, there exists an $r>1$ such that $1<rs<q_0$. We then choose some numbers $r_1,r_2,r_3>1$ such that $r=r_1r_2r_3$. Now suppose $k>1$. Since $s<\sqrt{q_0}$, there exists an r_3 such that $1<s<r_3<\sqrt{q_0}$. Then $sr_3<q_0$. Pick a number $t_1\in(sr_3,q_0)$, and let $t=t_1/sr_3>1$. We choose a number $r_2\in(1,t)$ and let $r_1=t/r_2$, $r:=r_1r_2r_3$. Then we have $r_1,r_2,r_3>1$, and $1<rs<q_0$. For the rest of the proof, we denote $t=r_1r_2$ and $r=r_1r_2r_3=tr_3$.

Let $\vec{\lambda}=(\lambda_1,\cdots,\lambda_k)=((b_1)_B,\cdots,(b_k)_B)$, where

$$(b_i)_B=\frac{1}{|B|}\int_B b_i(x)d\mu(x),\quad 1\le i\le k.$$

Following the notations in [16], let C_j^k $(1\le j\le k)$ stand for the family of all finite subsets $\sigma=\{\sigma(1),\cdots,\sigma(j)\}$ of j different elements of $\{1,\cdots,k\}$. For any $\sigma\in C_j^k$, we denote the complement sequence of σ by $\sigma'=\{1,\cdots,k\}\backslash\sigma$. Let $\vec{b}_\sigma=(b_{\sigma(1)},\cdots,b_{\sigma(j)})$ and let the product $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(j)}$. Similarly, denote $(\vec{b}-\vec{\lambda})_\sigma=(b_{\sigma(1)}-\lambda_{\sigma(1)},\cdots,b_{\sigma(j)}-\lambda_{\sigma(j)})$ and $(b-\lambda)_\sigma=(b_{\sigma(1)}-\lambda_{\sigma(1)})\cdots(b_{\sigma(j)}-\lambda_{\sigma(j)})$. For $\sigma=\{\sigma(1),\cdots,\sigma(l)\}$ $(1\le l\le k)$, let $\nu_l=\sum_{i=1}^l\alpha_{\sigma(i)}$, and let $\nu_0=0$. Note that $\nu_k=\alpha=\sum_{i=1}^k\alpha_i$. We define q_j $(0\le j\le k)$ by

$$\frac{1}{q_j}=\frac{1}{q_k}+\frac{\nu_{k-j}}{n}=\frac{1}{q_k}+\sum_{l=1}^{k-j}\frac{\alpha_{\sigma(l)}}{n}.\tag{3.1}$$

Then, the above equation implies that

$$\frac{1}{q_k}=\frac{1}{q_0}-\frac{\alpha}{n}, \tag{3.2a}$$

$$\frac{1}{q_j}=\frac{1}{q_0}-\frac{1}{n}\sum_{l=k-j+1}^{k}\alpha_{\sigma'(l)}, \quad 1\le j\le k. \tag{3.2b}$$

Note that Eqs. (3.1) and (3.2b) imply that for $1\le j\le k-1$,

$$1<rs<q_0<q_j<\frac{n}{\nu_{k-j}}.$$

We have the following lemmas.

Lemma 3.1. *Assume that $x\in 2^{j+1}B$, where $j\in\mathbb{N}\cup\{0\}$. Consider $\sigma=\{\sigma(1),\cdots,\sigma(l)\}\in C_l^k$ with $1\le l\le k$. Denote $r=r_1r_2r_3$, where $r_1,r_2,r_3>1$. Let $\nu_l=\sum_{i=1}^{l}\alpha_{\sigma(i)}$. Then*

$$\begin{aligned}I_2(x):=&w(x)^{1/r_1'}\left\{\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|(b(y)-\lambda)_\sigma f(y)|^{r_1}w(y)^{1-r_1}dy\right\}^{\frac{1}{r_1}}\\ \le& C2^{(j+1)nl}\Big(\prod_{i=1}^{l}\|b_{\sigma(i)}\|_{Lip_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}\Big)w(x)^l M_{\nu_l,r,w}f(x).\end{aligned}$$

Proof. Observe that

$$\begin{aligned}|(b_{\sigma(i)})_B-(b_{\sigma(i)})_{2^{j+1}B}|\le& C(j+1)2^{(j+1)n(1-\gamma)}w(2^{j+1}B)^{\alpha_{\sigma(i)}/n}\frac{w(2^{j+1}B)}{|2^{j+1}B|}\|b_{\sigma(i)}\|_{\mathrm{Lip}_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}\\ \le& C2^{(j+1)n}w(2^{j+1}B)^{\alpha_{\sigma(i)}/n}\frac{w(2^{j+1}B)}{|2^{j+1}B|}\|b_{\sigma(i)}\|_{\mathrm{Lip}_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}.\end{aligned} \tag{3.3}$$

By Hölder's inequality, we have

$$\begin{aligned}&\left\{\int_{2^{j+1}B}|(b(y)-\lambda)_\sigma f(y)|^{r_1}w(y)^{1-r_1}dy\right\}^{\frac{1}{r_1}}\\ \le&\left\{\int_{2^{j+1}B}|(b(y)-\lambda)_\sigma|^{r_1r_2'}w(y)^{1-lr_1r_2'}dy\right\}^{\frac{1}{r_1r_2'}}\left\{\int_{2^{j+1}B}|f(y)|^t w(y)^{1+t(l-1)}dy\right\}^{1/t}\\ \equiv& J_1(x)J_2(x),\end{aligned} \tag{3.4}$$

where $t=r_1r_2$. Note that if $k=1$, then necessarily $l=1$, and thus $w^{1+(l-1)r_1r_2r_3'}=w\in\mathcal{A}_1(\mathbb{R}^n)$. On the other hand, if $k>1$, then since $r_3>s$, $r_3'<s'$. Moreover,

$$r_1r_2=t=\frac{t_1}{sr_3}<\frac{q_0}{s^2}.$$

Thus,

$$1+(l-1)r_1r_2r_3'<1+(k-1)\frac{q_0}{s^2}s'=1+(k-1)q_0\Big(s'+\frac{1}{s'}-2\Big)=\tau_2.$$

This shows that $w^{1+(l-1)r_1r_2r_3'}\in\mathcal{A}_1(\mathbb{R}^n)$. Therefore by Hölder's inequality,

$$J_2(x)\le Cw(2^{j+1}B)^{\frac{1}{r}-\frac{\nu_l}{n}}|2^{j+1}B|^{\frac{1}{r_1r_2r_3'}}w(x)^{\frac{1}{r_1r_2r_3'}+l-1}M_{\nu_l,r,w}f(x). \tag{3.5}$$

Now let $\gamma_1,\cdots,\gamma_l$ be such that $1<r_1r_2'\le\gamma_1,\cdots,\gamma_l$ and $\sum_{i=1}^l\gamma_i^{-1}=1/r_1r_2'$.

Let

$$g_i(y)=(b_{\sigma(i)}(y)-\lambda_{\sigma(i)})w(y)^{\frac{1}{\gamma_i}-1},$$

and recall that $\lambda_{\sigma(i)}=(b_{\sigma(i)})_B$. An application of the generalized Hölder inequality gives

$$\begin{aligned}J_1(x)=&\Big\{\int_{2^{j+1}B}|\prod_{i=1}^l g_i(y)|^{r_1r_2'}dy\Big\}^{\frac{1}{r_1r_2'}}\le\prod_{i=1}^l\Big\{\int_{2^{j+1}B}|g_i(y)|^{\gamma_i}dy\Big\}^{\frac{1}{\gamma_i}}\\ \le&\prod_{i=1}^l\Big[\Big\{\int_{2^{j+1}B}|b_{\sigma(i)}(y)-(b_{\sigma(i)})_{2^{j+1}B}|^{\gamma_i}w(y)^{1-\gamma_i}dy\Big\}^{\frac{1}{\gamma_i}}\\ &+\Big\{\int_{2^{j+1}B}|(b_{\sigma(i)})_B-(b_{\sigma(i)})_{2^{j+1}B}|^{\gamma_i}w(y)^{1-\gamma_i}dy\Big\}^{\frac{1}{\gamma_i}}\Big]\\ \le&C2^{(j+1)nl}w(2^{j+1}B)^{\frac{\nu_l}{n}+\frac{1}{r_1r_2'}}\prod_{i=1}^l\|b_{\sigma(i)}\|_{\mathrm{Lip}_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)},\end{aligned} \tag{3.6}$$

where the last inequality follows from (3.3). Combining (3.5) and (3.6) yields

$$I_2(x)\le\frac{w(x)^{1/r_1'}}{|2^{j+1}B|^{1/r_1}}J_1(x)J_2(x)\le C2^{(j+1)nl}w(x)^lM_{\nu_l,r,w}f(x)\prod_{i=1}^l\|b_{\sigma(i)}\|_{\mathrm{Lip}_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}.$$

So, we complete the proof of the lemma. ☐

Lemma 3.2. *Consider $\sigma=\{\sigma(1),\cdots,\sigma(l)\}\in C_l^k$ with $1\le l\le k$. We use the convention that $T_{\vec{b}_{\sigma'}}f=Tf$ when $\sigma=\{\sigma(1),\cdots,\sigma(k)\}\in C_k^k$, i.e., $\sigma'=\emptyset$. Let $t_B=r_B^m$, where r_B is the radius of the ball B, and m appears in* (2.4)-(2.5). *Then*

$$\begin{aligned}&\sup_{B\ni x}\Big\{\frac{1}{|B|}\int_B|A_{t_B}((b-\lambda)_\sigma T_{\vec{b}_{\sigma'}}f)(y)|dy\Big\}\\ \le&C\Big(\prod_{i=1}^l\|b_{\sigma(i)}\|_{Lip_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}\Big)w(x)^lM_{\nu_l,r,w}(T_{\vec{b}_{\sigma'}}f)(x).\end{aligned}$$

Proof. Take a ball B which contains x. We have

$$\begin{aligned}
&\frac{1}{|B|}\int_B |A_{t_B}((b-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f)(y)|dy\\
\le&\frac{1}{|B|}\int_B\int_{\mathbb{R}^n} h_{t_B}(y,z)|(b(z)-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f(z)|dzdy\\
\le&\frac{1}{|B|}\int_B\int_{2B} h_{t_B}(y,z)|(b(z)-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f(z)|dzdy\\
&\quad+\sum_{j=1}^{\infty}\frac{1}{|B|}\int_B\int_{2^{j+1}B\backslash 2^jB} h_{t_B}(y,z)|(b(z)-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f(z)|dzdy\\
\equiv&J_3(x)+J_4(x).
\end{aligned}$$

Recall that $t_B^{1/m}=r_B$. For $y\in B$ and $z\in 2B$, we have

$$h_{t_B}(y,z)=\frac{s(|y-z|^m t_B^{-1})}{|B(y;t_B^{1/m})|}\le\frac{s(0)}{|B(y;r_B)|}\le\frac{C}{|B|}\le\frac{C}{|2B|}.$$

Thus, by Hölder's inequality and Lemma 3.1, we see that

$$\begin{aligned}
J_3(x)\le&\frac{C}{|2B|}\int_{2B}|(b(z)-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f(z)|dz\\
\le&Cw(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2B|}\int_{2B}|(b(z)-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f(z)|^{r_1}w(z)^{1-r_1}dz\right\}^{\frac{1}{r_1}}\\
\le&C\Big(\prod_{i=1}^{l}\|b_{\sigma(i)}\|_{\mathrm{Lip}_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}\Big)w(x)^l M_{\nu_l,r,w}(T_{\vec{b}_{\sigma'}} f)(x).
\end{aligned}$$

On the other hand, $y\in B$ and $z\in 2^{j+1}B\backslash 2^jB$ imply that $|y-z|\ge 2^{j-1}r_B$. So,

$$h_{t_B}(y,z)=\frac{s(|y-z|^m t_B^{-1})}{|B(y;t_B^{1/m})|}\le C\frac{s(2^{(j-1)m})}{|B|}\le C\frac{2^{(j+1)n}s(2^{(j-1)m})}{|2^{j+1}B|}.$$

Hence, an application of Lemma 3.1 yields

$$\begin{aligned}
J_4(x)\le&C\sum_{j=1}^{\infty}2^{(j+1)n}s(2^{(j-1)m})\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|(b(z)-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f(z)|dz\\
\le&C\sum_{j=1}^{\infty}2^{(j+1)n}s(2^{(j-1)m})w(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|(b(z)-\lambda)_\sigma T_{\vec{b}_{\sigma'}} f(z)|^{r_1}w(z)^{1-r_1}dz\right\}^{\frac{1}{r_1}}\\
\le&C\sum_{j=1}^{\infty}2^{(j+1)n[k+1]}s(2^{(j-1)m})\Big(\prod_{i=1}^{l}\|b_{\sigma(i)}\|_{\mathrm{Lip}_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}\Big)w(x)^l M_{\nu_l,r,w}(T_{\vec{b}_{\sigma'}} f)(x)\\
\le&C\Big(\prod_{i=1}^{l}\|b_{\sigma(i)}\|_{\mathrm{Lip}_{\alpha_{\sigma(i)},w}(\mathbb{R}^n)}\Big)w(x)^l M_{\nu_l,r,w}(T_{\vec{b}_{\sigma'}} f)(x),
\end{aligned}$$

provided that

$$\lim_{r\to\infty} r^{n(k+1)+\epsilon} s(r^m)=0$$

for some $\epsilon>0$. Combining the estimates of $J_3(x)$ and $J_4(x)$ and taking the supremum over all balls B containing x yields the conclusion. □

Lemma 3.3. *It holds*

$$\begin{aligned}M^{\sharp}_A(T_{\vec{b}}f)(x):=&\sup_{B\ni x}\Big\{\frac{1}{|B|}\int_B |T_{\vec{b}}f(y)-A_{t_B}(T_{\vec{b}}f)(y)|dy\Big\}\\ \le& C\|\vec{b}\|_{Lip_{\alpha,w}(\mathbb{R}^n)}w(x)^k\{M_{\alpha,r,w}f(x)+M_{\alpha,r,w}(Tf)(x)\}\\ &+C\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\Big[\prod_{l=1}^{i}\|b_{\sigma(l)}\|_{Lip_{\alpha_{\sigma(l)},w}(\mathbb{R}^n)}\Big]w(x)^iM_{\nu_i,r,w}(T_{\vec{b}_{\sigma'}}f)(x),\end{aligned}$$

where $c_{k,i}$ are constants depending on k and i.

Proof. For an arbitrary fixed $x\in\mathbb{R}^n$, choose a ball B which contains x. Following [16], we split $f=f\chi_{2B}+f\chi_{(2B)^c}\equiv f_1+f_2$, and write

$$\begin{aligned}&\frac{1}{|B|}\int_B|T_{\vec{b}}f(y)-A_{t_B}(T_{\vec{b}}f)(y)|dy\\ \le&\frac{1}{|B|}\int_B\Big|\Big[\prod_{i=1}^k(b_i(y)-\lambda_i)\Big]Tf(y)\Big|dy+\frac{1}{|B|}\int_B\Big|A_{t_B}\Big(\Big[\prod_{i=1}^k(b_i-\lambda_i)\Big]Tf\Big)(y)\Big|dy\\ &+\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\frac{1}{|B|}\int_B|(b(y)-\lambda)_\sigma T_{\vec{b}_{\sigma'}}f(y)|dy\\ &+\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\frac{1}{|B|}\int_B|A_{t_B}((b-\lambda)_\sigma T_{\vec{b}_{\sigma'}}f)(y)|dy\\ &+\frac{1}{|B|}\int_B\Big|T\Big(\Big[\prod_{i=1}^k(b_i-\lambda_i)\Big]f_1\Big)(y)\Big|dy+\frac{1}{|B|}\int_B\Big|A_{t_B}T\Big(\Big[\prod_{i=1}^k(b_i-\lambda_i)\Big]f_1\Big)(y)\Big|dy\\ &+\frac{1}{|B|}\int_B\Big|(T-A_{t_B}T)\Big(\Big[\prod_{i=1}^k(b_i-\lambda_i)\Big]f_2\Big)(y)\Big|dy\equiv\sum_{i=1}^7K_i(x).\end{aligned}$$

By Hölder's inequality, Lemma 3.1 and Lemma 3.2 respectively, we have

$$K_1(x),K_2(x)\le C\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}w(x)^kM_{\alpha,r,w}(Tf)(x),$$

and

$$K_3(x),K_4(x)\le C\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\Big[\prod_{l=1}^i\|b_{\sigma(l)}\|_{\mathrm{Lip}_{\alpha_{\sigma(l)},w}(\mathbb{R}^n)}\Big]w(x)^iM_{\nu_i,r,w}(T_{\vec{b}_{\sigma'}}f)(x),$$

where

$$\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}=\prod_{i=1}^{k}\|b_i\|_{\mathrm{Lip}_{\alpha_i,w}(\mathbb{R}^n)}.$$

Note that

$$w\in\mathcal{A}_1(\mathbb{R}^n)\Rightarrow w^{1-r_1}\in\mathcal{A}_{r_1}(\mathbb{R}^n).$$

Therefore, by Theorem 5.3 [12] and Lemma 3.1,

$$\begin{aligned}K_5(x)&\leq w(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|B|}\int_B\Big|T\Big(\Big[\prod_{i=1}^{k}(b_i-\lambda_i)\Big]f_1\Big)(y)\Big|^{r_1}w(y)^{1-r_1}dy\right\}^{\frac{1}{r_1}}\\&\leq Cw(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2B|}\int_{2B}\Big|\Big[\prod_{i=1}^{k}(b_i-\lambda_i)\Big]f_1(y)\Big|^{r_1}w(y)^{1-r_1}dy\right\}^{\frac{1}{r_1}}\\&\leq C\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}w(x)^kM_{\alpha,r,w}f(x).\end{aligned}$$

Observe that

$$\begin{aligned}K_6(x)\leq&\frac{C}{|2B|}\int_{2B}\Big|T\Big(\Big[\prod_{i=1}^{k}(b_i-\lambda_i)\Big]f_1\Big)(z)\Big|dz\\&+C\sum_{j=1}^{\infty}2^{(j+1)n}s(2^{(j-1)m})\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\Big|T\Big(\Big[\prod_{i=1}^{k}(b_i-\lambda_i)\Big]f_1\Big)(z)\Big|dz\\\leq&Cw(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2B|}\int_{2B}\Big|T\Big(\Big[\prod_{i=1}^{k}(b_i-\lambda_i)\Big]f_1\Big)(z)\Big|^{r_1}w(z)^{1-r_1}dz\right\}^{\frac{1}{r_1}}\\&+C\sum_{j=1}^{\infty}2^{(j+1)n}s(2^{(j-1)m})w(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\Big|T\Big(\Big[\prod_{i=1}^{k}(b_i-\lambda_i)\Big]f_1\Big)(z)\Big|^{r_1}w(z)^{1-r_1}dz\right\}^{\frac{1}{r_1}}\\\leq&Cw(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2B|}\int_{2B}\Big|\Big[\prod_{i=1}^{k}(b_i(z)-\lambda_i)\Big]f_1(z)\Big|^{r_1}w(z)^{1-r_1}dz\right\}^{\frac{1}{r_1}}\\&+C\sum_{j=1}^{\infty}2^{\frac{(j+1)n}{r_1'}}s(2^{(j-1)m})w(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2B|}\int_{2B}\Big|\Big[\prod_{i=1}^{k}(b_i(z)-\lambda_i)\Big]f_1(z)\Big|^{r_1}w(z)^{1-r_1}dz\right\}^{\frac{1}{r_1}}\\\leq&C\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}w(x)^kM_{\alpha,r,w}f(x).\end{aligned}$$

The third and last inequalities are due to Theorem 5.3 [12] and Lemma 3.1 respectively.

It remains to estimate $K_7(x)$. By hypothesis and Lemma 3.1, we have

$$\begin{aligned}K_7(x)&\leq\frac{1}{|B|}\int_B\int_{(2B)^c}|K(y,z)-K^{t_B}(y,z)|\Big|\Big[\prod_{i=1}^{k}(b_i(z)-\lambda_i)\Big]f_2(z)\Big|dzdy\\&\leq C\sum_{j=1}^{\infty}\frac{1}{|B|}\int_B\int_{2^{j+1}B\backslash 2^jB}\frac{t_B^{\beta/m}\Big|\Big[\prod_{i=1}^{k}(b_i(z)-\lambda_i)\Big]f_2(z)\Big|}{|B(y;|y-z|)||y-z|^{\beta}}dzdy\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{\infty}2^{-(j-1)\beta}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\Big|\Big[\prod_{i=1}^{k}(b_i(z)-\lambda_i)\Big]f(z)\Big|dz\\
&\leq C\sum_{j=1}^{\infty}2^{-(j-1)\beta}w(x)^{\frac{1}{r_1'}}\left\{\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}\Big|\Big[\prod_{i=1}^{k}(b_i(z)-\lambda_i)\Big]f(z)\Big|^{r_1}w(z)^{1-r_1}dz\right\}^{\frac{1}{r_1}}\\
&\leq C\sum_{j=1}^{\infty}2^{(j+1)nk}2^{-(j-1)\beta}\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}w(x)^kM_{\alpha,r,w}f(x)\\
&\leq C\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}w(x)^kM_{\alpha,r,w}f(x),
\end{aligned}
$$

provided that $\beta>nk$. Finally, the result follows from combining all of the estimates above and taking the supremum over all balls B containing x. □

Lemma 3.4. *If $w\in\mathcal{A}_{1,s'}(\mathbb{R}^n)$ for some $s'>1$, then there exists a constant $C>0$ such that*

$$
M_{\alpha,r,w}f(x)\leq Cw(x)^{\alpha/n}M_{\alpha,rs}f(x),\quad \textit{where }\ \frac{1}{s}+\frac{1}{s'}=1.
$$

Proof. Let B be a ball which contains x. By Hölder's inequality,

$$
\begin{aligned}
&\left\{\frac{1}{w(B)^{1-\frac{\alpha r}{n}}}\int_B|f(y)|^rw(y)dy\right\}^{1/r}\\
\leq&\frac{|B|^{\frac{1}{rs'}}}{w(B)^{\frac{1}{r}-\frac{\alpha}{n}}}\left\{\int_B|f(y)|^{rs}dy\right\}^{\frac{1}{rs}}\left(\frac{1}{|B|}\int_Bw(y)^{s'}dy\right)^{\frac{1}{rs'}}\\
\leq&C\left(\frac{w(B)}{|B|}\right)^{\alpha/n}\left(\frac{w(B)}{|B|}\right)^{-1/r}\left\{\operatorname{ess\,inf}_{y\in B}w(y)\right\}^{1/r}\left\{\frac{1}{|B|^{1-\frac{\alpha rs}{n}}}\int_B|f(y)|^{rs}dy\right\}^{\frac{1}{rs}}\\
\leq&C\left(\frac{w(B)}{|B|}\right)^{\alpha/n}\left\{\frac{1}{|B|^{1-\frac{\alpha rs}{n}}}\int_B|f(y)|^{rs}dy\right\}^{\frac{1}{rs}}\\
\leq&Cw(x)^{\alpha/n}M_{\alpha,rs}f(x).
\end{aligned}
$$

Taking the supremum over all balls B which contain x yields the desired result. □

By Eqs. (3.2a) and (3.2b), we have that, for $1\leq j\leq k$,

$$
\frac{q_j}{q_0}\leq\frac{q_k}{q_0}=\Big(1-\frac{\alpha q_0}{n}\Big)^{-1}\leq\tau_1.
$$

Thus for $1\leq j\leq k$,

$$
w^{q_j/q_0}=w^{\tilde{q}_j/\tilde{q}_0}\in\mathcal{A}_1(\mathbb{R}^n)\subset\mathcal{A}_{1+\frac{\tilde{q}_j}{\tilde{q}_0'}}(\mathbb{R}^n)\Rightarrow w^{\frac{1}{\tilde{q}_0}}\in\mathcal{A}_{\tilde{q}_0,\tilde{q}_j}(\mathbb{R}^n),
$$

where

$$
\tilde{q}_0=\frac{q_0}{rs}\quad\text{ and }\quad\tilde{q}_j=\frac{q_j}{rs}.
$$

Therefore, we may apply Lemma 2.1, Lemma 3.4, Theorem 5.3 [12], together with equation (3.2a) to conclude that

$$\|w^k M_{\alpha,r,w} f\|_{L^{q_k}(w^{1-kq_k})} \le C\|f\|_{L^{q_0}(w)}, \tag{3.7}$$

and

$$\|w^k M_{\alpha,r,w}(Tf)\|_{L^{q_k}(w^{1-kq_k})} \le C\|Tf\|_{L^{q_0}(w)} \le C\|f\|_{L^{q_0}(w)}, \tag{3.8}$$

provided that $1<rs<q_0<n/\alpha$. Now for $1\le j\le k$, we denote $\beta_j=\sum_{l=k-j+1}^{k}\alpha_{\sigma'(l)}/n$. Note that there are j terms in the sum β_j. So $\beta_j\ge j\alpha_1/n$. Then by Eq. (3.2b),

$$\frac{jq_j-1}{q_j-1}=1+\frac{(j-1)}{\beta_j+1/q_0'}\le 1+\frac{n(j-1)}{j\alpha_1}\le 1+\frac{n}{\alpha_1}=\tau_3.$$

So $w^{\frac{jq_j-1}{q_j-1}}\in\mathcal{A}_1(\mathbb{R}^n)$, which implies that $w^{1-jq_j}\in\mathcal{A}_{q_j}(\mathbb{R}^n)$ for all $1\le j\le k$.

Since $1<rs<q_0$, it follows that for $1\le j\le k-1$,

$$\frac{(jq_j-1)}{\tilde{q}_j-1}=\frac{rs(jq_j-1)}{q_j-rs}\le q_0\frac{(jq_j-1)}{q_j-q_0}=\frac{(j+\beta_j)q_0-1}{q_0\beta_j}\le 1+\frac{j}{\beta_j}\le 1+\frac{n}{\alpha_1}=\tau_3.$$

Hence

$$w^{1+(jrs-1)\tilde{q}_j'}=w^{\frac{rs(jq_j-1)}{q_j-rs}}=w^{\frac{jq_j-1}{\tilde{q}_j-1}}\in\mathcal{A}_1(\mathbb{R}^n),$$

which implies that

$$w^{(\frac{1}{q_j}-j)q_k}\in\mathcal{A}_{1+\frac{\tilde{q}_k}{\tilde{q}_j'}}(\mathbb{R}^n),$$

or equivalently,

$$w^{(\frac{1}{q_j}-j)rs}\in\mathcal{A}_{\tilde{q}_j,\tilde{q}_k}(\mathbb{R}^n).$$

Again, by Lemma 2.1, Lemma 3.4 and Eq. (3.1), we infer that for $1\le i\le k-1$,

$$\|w^i M_{\nu_i,r,w}(T_{\vec{b}_{\sigma'}}f)\|_{L^{q_k}(w^{1-kq_k})}\le C\|T_{\vec{b}_{\sigma'}}f\|_{L^{q_{k-i}}(w^{1-(k-i)q_{k-i}})}, \tag{3.9}$$

provided that $1<rs<q_0<q_{k-i}<n/\nu_i$ for $1\le i\le k-1$. Consequently, by Theorems 4.2, 5.3 [12], Lemma 3.3, inequalities (3.7)-(3.9), and induction argument, we conclude that

$$\begin{aligned}\|T_{\vec{b}}f\|_{L^{q_k}(w^{1-kq_k})}&\le\|\mathcal{M}(T_{\vec{b}}f)\|_{L^{q_k}(w^{1-kq_k})}\le\|M_A^\sharp(T_{\vec{b}}f)\|_{L^{q_k}(w^{1-kq_k})}\\&\le C\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}\|f\|_{L^{q_0}(w)}\\&\quad+C\sum_{i=1}^{k-1}\sum_{\sigma\in C_i^k}c_{k,i}\Big[\prod_{l=1}^{i}\|b_{\sigma(l)}\|_{\mathrm{Lip}_{\alpha_{\sigma(l)},w}(\mathbb{R}^n)}\Big]\|T_{\vec{b}_{\sigma'}}f\|_{L^{q_{k-i}}(w^{1-(k-i)q_{k-i}})}\\&\le C\|\vec{b}\|_{\mathrm{Lip}_{\alpha,w}(\mathbb{R}^n)}\|f\|_{L^{q_0}(w)}.\end{aligned}$$

Thus, we complete the proof. □

Remark 3.2. We could apply the reverse Hölder inequality to prove Lemma 3.4, thereby eliminating the assumption that $w \in \mathcal{A}_{1,s'}(\mathbb{R}^n)$. However, the exponent s' appearing in the reverse Hölder inequality (see (2.2)) depends on w and may be very close to 1, which means that its conjugate exponent s could be very large (see [7,17]). This limits the value of q_0 for which the theorem holds, since q_0 is necessarily greater than s.

Remark 3.3. Let φ be a non-decreasing positive function on $\mathbb{R}^+$. Denote by $\Omega(f,B)$, the mean oscillation of a function f on a ball $B \subset \mathbb{R}^n$, as $|B|^{-1}\int_B |f(x)-f_B|dx$. Define BMO_φ as the space of all functions f satisfying $\Omega(f,B) \le C\varphi(r)$, whenever B is a ball with radius r (see [10]). Note that when $\varphi \equiv 1$, then $\mathrm{BMO}_\varphi = \mathrm{BMO}$, the space of all functions of bounded mean oscillation. Let Λ_α, $0<\alpha\le 1$, be the space of Lipschitz continuous functions, $\Lambda_\alpha = \{f : |f(x)-f(y)| \le C|x-y|^\alpha,\ \forall x,y \in \mathbb{R}^n\}$. A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a Young function if it is continuous, convex, increasing and satisfying $\psi(0)=0$ and $\lim_{t\to\infty}\psi(t)=\infty$. The Orlicz space L_ψ is defined as the space of all functions f such that $\int \psi(\lambda|f|) < \infty$, for some $\lambda > 0$.

Now consider the singular integral Tf and the commutator $T_b f$ (as defined in the introduction), but with the convolution kernel

$$K(x)=\frac{K(x/|x|)}{|x|^n}, \quad \int_{S^{n-1}} K(x')d\sigma(x')=0 \quad \text{and} \quad K\in C^\infty(S^{n-1}).$$

With this type of kernel, Janson [10] proved that b belongs to BMO_φ if and only if T_b maps L^p $(1<p<\infty)$ boundedly into L_ψ, where φ and ψ are related by the equation $\varphi(r)=r^{n/q}\psi^{-1}(r^{-n})$, or equivalently, $\psi^{-1}(t)=t^{1/p}\varphi(t^{-1/n})$. When $\varphi(t)=t^\alpha$ $(0<\alpha<1)$, $\psi(t)=t^q$ with $1/q=1/p-\alpha/n$, then it is evident that $\mathrm{Lip}_\alpha(\mathbb{R}^n)=\mathrm{BMO}_{t^\alpha}(\mathbb{R}^n)$ and $L_\psi(\mathbb{R}^n)=L^q(\mathbb{R}^n)$. In this particular case, Janson's Theorem says that b belongs to $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ if and only if T_b maps $L^p(\mathbb{R}^n)$ $(1<p<\infty)$ boundedly into $L^q(\mathbb{R}^n)$, where $1/q=1/p-\alpha/n$. It is interesting to note that the above necessary condition is the same as in Theorem 3.1, when $k=1$ and $w\equiv 1$, but with different kernel K.

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