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Commutators of Littlewood-Paley Operators on Herz Spaces with Variable Exponent

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Abstract. Let $\Omega\in L^2(\mathrm{S}^{n-1})$ be homogeneous function of degree zero and b be BMO functions. In this paper, we obtain some boundedness of the Littlewood-Paley Operators and their higher-order commutators on Herz spaces with variable exponent.

Key Words: Herz space, variable exponent, commutator, area integral, Littlewood-Paley g_λ^* function.

AMS Subject Classifications: 42B25, 42B35, 46E30

1 Introduction

The theory of function spaces with variable exponent has extensively studied by researchers since the work of Kováčik and Rákosník [7] appeared in 1991. In [9] and [10], the authors proved the boundedness of some Littlewood-Paley operators on variable L^p spaces, respectively.

Given an open set $E\subset\mathbb{R}^n$, and a measurable function $p(\cdot):E\longrightarrow[1,\infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda>0$,

$$\int_E\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx<\infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)}=\inf\Big\{\lambda>0:\int_E\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx\leq 1\Big\}.$$

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These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x)=p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\rm loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\rm loc}(\Omega):=\{f: f\in L^{p(\cdot)}(E) \text{ for all compact subsets } E\subset\Omega\}.$$

Define $\mathcal{P}^0(E)$ to be set of $p(\cdot):E\longrightarrow(0,\infty)$ such that

$$p^-=\operatorname{essinf}\{p(x):x\in E\}>0,\qquad p^+=\operatorname{esssup}\{p(x):x\in E\}<\infty.$$

Define $\mathcal{P}(E)$ to be set of $p(\cdot):E\longrightarrow[1,\infty)$ such that

$$p^-=\operatorname{essinf}\{p(x):x\in E\}>1,\qquad p^+=\operatorname{esssup}\{p(x):x\in E\}<\infty.$$

Denote $p'(x)=p(x)/(p(x)-1)$.

Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|f(y)|dy,$$

where $B_r(x)=\{y\in\mathbb{R}^n:|x-y|<r\}$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A\subset\mathbb{R}^n$ by $|A|$ and χ_A respectively. The notation $f\approx g$ means that there exist constants $C_1,C_2>0$ such that $C_1g\le f\le C_2g$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. *If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and satisfies*

$$|p(x)-p(y)|\le\frac{C}{-\log(|x-y|)},\quad |x-y|\le 1/2, \tag{1.1}$$

and

$$|p(x)-p(y)|\le\frac{C}{\log(|x|+e)},\quad |y|\ge|x|, \tag{1.2}$$

then $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, that is the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 1.2 (see [7])**.** *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\le r_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where

$$r_p=1+1/p^--1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.3 (see [5]). *Let $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S\subset B$,*

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\le C\frac{|B|}{|S|},\quad \frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\le C\Big(\frac{|S|}{|B|}\Big)^{\delta_1}\quad \text{and}\quad \frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\le C\Big(\frac{|S|}{|B|}\Big)^{\delta_2},$$

where δ_1, δ_2 are constants with $0<\delta_1,\ \delta_2<1$.

Throughout this paper δ_1, δ_2 is the same as in Lemma 1.3.

Lemma 1.4 (see [5]). *Suppose $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C>0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\le C.$$

Next we recall the definition of the Herz-type spaces with variable exponent. Let $B_k=\{x\in\mathbb{R}^n:|x|\le 2^k\}$ and $A_k=B_k\setminus B_{k-1}$ for $k\in\mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k=\chi_{A_k}$ for $k\in\mathbb{Z}$, $\tilde{\chi}_k=\chi_k$ if $k\in\mathbb{Z}_+$ and $\tilde{\chi}_0=\chi_{B_0}$.

Definition 1.1 (see [5]). Let $\alpha\in\mathbb{R}$, $0<p\le\infty$ and $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\{f\in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n\setminus\{0\}):\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}=\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}.$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\{f\in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n):\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}=\Big\{\sum_{k=0}^{\infty}2^{k\alpha p}\|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}.$$

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n\,(n\ge2)$ equipped with normalized Lebesgue measure. Let $\Omega\in L^1(\mathbb{R}^n)$, be homogeneous function of degree zero and

$$\int_{\mathrm{S}^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.3}$$

where $x'=x/|x|$ for any $x\neq 0$. The Littlewood-Paley area integral $\mu_{\Omega,S}$ and g_λ^* function μ_λ^* are defined by

$$\mu_{\Omega,S}(f)(x)=\Big(\iint_{\Gamma(x)}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)dz\Big|^2\frac{dydt}{t^{n+3}}\Big)^{1/2}$$

and

$$\mu^*_{\Omega,\lambda}(f)(x)=\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)dz\Big|^2\frac{dydt}{t^{n+3}}\Big)^{1/2},$$

where $\Gamma(x)=\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<t\}$ and $\lambda>1$.

For an integer $m\geq 1$, let b be a locally integrable function on $\mathbb{R}^n$, the higher-order commutators $[b^m,\mu_{\Omega,S}]$ and $[b^m,\mu_\lambda^*]$ are defined by

$$[b^m,\mu_{\Omega,S}](f)(x)=\Big(\iint_{\Gamma(x)}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}[b(x)-b(z)]^m f(z)dz\Big|^2\frac{dydt}{t^{n+3}}\Big)^{1/2}$$

and

$$\begin{aligned}&[b^m,\mu^*_{\Omega,\lambda}](f)(x)\\ =&\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}[b(x)-b(z)]^m f(z)dz\Big|^2\frac{dydt}{t^{n+3}}\Big)^{1/2}.\end{aligned}$$

Motivated by [8,9], we will study the boundedness for the Littlewood-Paley operators and their commutators on the Herz space with variable exponent, where $\Omega\in L^2(\mathrm{S}^{n-1})$.

2 Estimate for the Littlewood-Paley operators

In this section we will prove the boundedness of the Littlewood-Paley area integral $\mu_{\Omega,S}$ and g_λ^* function $\mu^*_{\Omega,\lambda}$ on Herz spaces with variable exponent.

A nonnegative locally integrable function ω on $\mathbb{R}^n$ is said to belong to A_p $(1<p<\infty)$, if there is a constant $C>0$ such that

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q\omega(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q\omega(x)^{1-p'}dx\Big)^{p-1}<\infty,$$

where $p'=p/(p-1)$, Q denotes a cube in $\mathbb{R}^n$ with its sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

The weighted (L^p,L^p) boundedness of $\mu_{\Omega,S}$ and $\mu^*_{\Omega,\lambda}$ have been proved by Ding, Fan and Pan [3].

Lemma 2.1 (see [3])**.** *Suppose that* $\Omega\in L^s(\mathrm{S}^{n-1})$ $(s>1)$ *satisfying* (1.3). *If* $\omega\in A_{p/\beta}$, $\max\{s',2\}=\beta<p<\infty$, *then there is a constant* C, *independent of* f, *such that*

$$\int_{\mathbb{R}^n}|\mu_{\Omega,S}(f)(x)|^p\omega(x)dx\leq C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx$$

and

$$\int_{\mathbb{R}^n}|\mu^*_{\Omega,\lambda}(f)(x)|^p\omega(x)dx\le C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.$$

Lemma 2.2 (see [2]). *Given a family $\mathcal{F}$ and an open set $E\subset\mathbb{R}^n$, assume that for some p_0, $0<p_0<\infty$ and for every $\omega\in A_\infty$,*

$$\int_E f(x)^{p_0}\omega(x)dx\le C_0\int_E g(x)^{p_0}\omega(x)dx,\quad (f,g)\in\mathcal{F}.$$

Given $p(\cdot)\in\mathcal{P}^0(E)$ such that $p(\cdot)$ satisfies (1.1) *and* (1.2) *in Lemma 1.1. Then for all $(f,g)\in\mathcal{F}$ such that $f\in L^{p(\cdot)}(E)$,*

$$\|f\|_{L^{p(\cdot)}(E)}\le C\|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{p/\beta}\subset A_\infty$, by Lemma 2.1 and Lemma 2.2, it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n),L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the Littlewood-Paley area integral $\mu_{\Omega,S}$ and g^*_λ function $\mu^*_{\Omega,\lambda}$.

Now we give the main theorem in this section.

Theorem 2.1. *Suppose that $\lambda>2$, $0<p\le\infty$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfies conditions* (1.1) *and* (1.2) *in Lemma 1.1, $\Omega\in L^2(\mathrm{S}^{n-1})$ and $-n\delta_1<\alpha<n\delta_2$. Then the Littlewood-Paley g^*_λ function $\mu^*_{\Omega,\lambda}$ is bounded on $\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

Proof. We only prove homogeneous case. The non-homogeneous case can be proved in the same way. We suppose $0<p<\infty$, since the proof of the case $p=\infty$ is easier. Let $f\in\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Denote $f_j=f\chi_j$ for each $j\in\mathbb{Z}$, then we have $f(x)=\sum_{j=-\infty}^\infty f_j(x)$. Then we have

$$\begin{aligned}\|\mu^*_{\Omega,\lambda}(f)\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}&=\left\{\sum_{k=-\infty}^\infty 2^{k\alpha p}\|\mu^*_{\Omega,\lambda}(f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}\\&\le C\left\{\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\|\mu^*_{\Omega,\lambda}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\&\quad+C\left\{\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\|\mu^*_{\Omega,\lambda}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\&\quad+C\left\{\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\|\mu^*_{\Omega,\lambda}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\&=:CI_1+CI_2+CI_3.\end{aligned}\tag{2.1}$$

We first estimate I_2, by the $(L^{q(\cdot)}(\mathbb{R}^n),L^{q(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $\mu^*_{\Omega,\lambda}$ we have

$$I_2\le C\left\{\sum_{k=-\infty}^\infty 2^{k\alpha p}\|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}=C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\tag{2.2}$$

Now we estimate I_1. By the Minkowski inequality we have

$$
\begin{aligned}
|\mu^*_{\Omega,\lambda}(f_j)(x)| &= \left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}f_j(z)dz\right|^2\frac{dydt}{t^{n+3}}\right)^{1/2}\\
&= \left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}f_j(z)dz\right|^2\frac{dydt}{t^{n+3}}\right)^{1/2}\\
&\le \int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\right)^{1/2}dz\\
&\le \int_{\mathbb{R}^n}|f_j(z)|\left(\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\right)^{1/2}dz\\
&\quad+\int_{\mathbb{R}^n}|f_j(z)|\left(\int_{|x-z|}^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\right)^{1/2}dz. \qquad (2.3)
\end{aligned}
$$

Note that $z\in A_j$ and $|y-z|<t$. So we know that $|y-z|\sim|y|$, then for $\Omega\in L^2(\mathrm{S}^{n-1})$ we have

$$
\begin{aligned}
&\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}dy\le\int_{|y|<t}\frac{|\Omega(y)|^2}{|y|^{2n-2}}dy\\
\le&\int_0^t r^{1-n}dr\int_{\mathrm{S}^{n-1}}|\Omega(y')|^2d\sigma(y')\le t^{2-n}\|\Omega\|^2_{L^2(\mathrm{S}^{n-1})}. \qquad (2.4)
\end{aligned}
$$

For $\lambda>2$, we take $0<\theta<(\lambda-2)n$. Since $|x-z|\le|x-y|+|y-z|\le|x-y|+t$, by (2.4) we have

$$
\begin{aligned}
&\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\\
\le&\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n-2n-\theta}\frac{1}{|x-z|^{2n+\theta}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{3-n-\theta}}\\
\le&\frac{1}{|x-z|^{2n+\theta}}\int_0^{|x-z|}\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{3-n-\theta}}\\
\le&\frac{\|\Omega\|^2_{L^2(\mathrm{S}^{n-1})}}{|x-z|^{2n+\theta}}\int_0^{|x-z|}t^{\theta-1}dt\\
\le&C|x-z|^{-2n}. \qquad (2.5)
\end{aligned}
$$

Similarly, noting that $|y-z|\sim|y|$, by (2.4) we have

$$
\begin{aligned}
&\int_{|x-z|}^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\\
\le&\int_{|x-z|}^\infty\int_{|y-z|<t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{3+n}}
\end{aligned}
$$

$$
\begin{aligned}
&\le \|\Omega\|^2_{L^2(\mathrm{S}^{n-1})}\int_{|x-z|}^{\infty} t^{-2n-1}dt\\
&\le C|x-z|^{-2n}.
\end{aligned}\tag{2.6}
$$

Note that $x\in A_k$, $z\in A_j$ and $j\le k-2$. By (2.5), (2.6) and the generalized Hölder inequality we have

$$
|\mu^*_{\Omega,\lambda}(f_j)(x)|\le C\int_{\mathbb{R}^n}\frac{|f_j(z)|}{|x-z|^n}dz\le C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}. \tag{2.7}
$$

By Lemma 1.3 and Lemma 1.4, we have

$$
\begin{aligned}
&\|\mu^*_{\Omega,\lambda}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
\le& C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
\le& C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\|\chi_{B_j}\Big\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
\le& C\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
\le& C2^{(j-k)n\delta_2}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
I_1\le& C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_2}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
=& C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}.
\end{aligned}
$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_2-\alpha>0$, by the Hölder inequality we have

$$
\begin{aligned}
I_1\le& C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\Big)\\
&\times\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\alpha)p'/2}\Big)^{p/p'}\Big\}^{1/p}\\
\le& C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(j-k)(n\delta_2-\alpha)p/2}\Big)\Big\}^{1/p}\\
=& C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p/2}\Big)\Big\}^{1/p}\\
\le& C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}\\
=& C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{2.8}
$$

If $0<p\leq 1$, then we have

$$\begin{aligned}I_1\leq& C\Big\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha p}2^{(j-k)(n\delta_2-\alpha)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}\\ =&C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p}\Big)\Big\}^{1/p}\\ \leq& C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}\tag{2.9}$$

Let us now estimate I_3. Note that $x\in A_k, y\in A_j$ and $j\geq k+2$, so we have $|y-z|\sim|y|$. By (2.3)-(2.6) and the generalized Hölder inequality we have

$$|\mu^*_{\Omega,\lambda}(f_j)(x)|\leq C\int_{\mathbb{R}^n}\frac{|f_j(z)|}{|x-z|^n}dz\leq C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\tag{2.10}$$

By Lemma 1.3 and Lemma 1.4, we have

$$\begin{aligned}&\|\mu^*_{\Omega,\lambda}(f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \leq& C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \leq& C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \leq& C\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\ \leq& C2^{(k-j)n\delta_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}I_3\leq& C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\ =&C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}.\end{aligned}$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_1+\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}I_3\leq& C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\Big)\\ &\times\Big(\sum_{j=-\infty}^{k-2}2^{(k-j)(n\delta_1+\alpha)p'/2}\Big)^{p/p'}\Big\}^{1/p}\\ \leq& C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\right)\right\}^{1/p}\end{aligned}$$

$$\begin{aligned}&=C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)p/2}\Big)\Big\}^{1/p}\\&\le C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}\\&=C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.11}$$

If $0<p\le 1$, then we have

$$\begin{aligned}I_3&\le C\Big\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty}2^{j\alpha p}2^{(k-j)(n\delta_1+\alpha)p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}\\&=C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)p}\Big)\Big\}^{1/p}\\&\le C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}\tag{2.12}$$

Therefore, by (2.1), (2.2), (2.8), (2.9), (2.11) and (2.12), we complete the proof of Theorem 2.1. □

Since $\mu_{\Omega,S}(f)(x)\le C_\lambda\mu^*_{\Omega,\lambda}(f)(x)$, we easily obtain the following theorem.

Theorem 2.2. *Suppose that* $0<p\le\infty$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.1) *and* (1.2) *in Lemma 1.1,* $\Omega\in L^2(\mathrm{S}^{n-1})$ *and* $-n\delta_1<\alpha<n\delta_2$*. Then the Littlewood-Paley area integral* $\mu_{\Omega,S}$ *is bounded on* $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

3 BMO estimate for the commutators of Littlewood-Paley operators

Let us first recall that the space BMO$(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_*=\sup_Q\frac{1}{|Q|}\int_Q|f(x)-f_Q|dx<\infty,$$

where $f_Q=|Q|^{-1}\int_Q f(y)dy$, the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

Let $b\in$ BMO$(\mathbb{R}^n)$. The weighted (L^p,L^p) boundedness of $[b,\mu_\Omega]$ have been proved by Ding, Lu and Yabuta [4].

Lemma 3.1 (see [4]). *Suppose that* $\Omega\in L^s(\mathrm{S}^{n-1})(s>1)$ *satisfying* (1.3)*. For an integer* $m\ge 1$*, if* $b\in$ BMO$(\mathbb{R}^n)$ *and* $\omega\in A_{p/\beta}$, $\max\{s',2\}=\beta<p<\infty$*, then there is a constant* C*, independent of* f*, such that*

$$\int_{\mathbb{R}^n}|[b^m,\mu_{\Omega,S}](f)(x)|^p\omega(x)dx\le C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx$$

and

$$\int_{\mathbb{R}^n}|[b^m,\mu^*_{\Omega,\lambda}](f)(x)|^p\omega(x)dx\leq C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.$$

By Lemma 3.1 and Lemma 2.2, it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n),L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the commutators $[b^m,\mu_{\Omega,S}]$ and $[b^m,\mu^*_{\Omega,\lambda}]$.

Next, we will give the corresponding result about the commutator $[b,\mu_\Omega]$ on Herz-type Hardy spaces with variable exponent.

Theorem 3.1. *Suppose that $b\in \mathrm{BMO}(\mathbb{R}^n)$, $\lambda>2$, $0<p\leq\infty$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfies conditions* (1.1) *and* (1.2) *in Lemma 1.1, $\Omega\in L^2(\mathrm{S}^{n-1})$ and $-n\delta_1<\alpha<n\delta_2$. Then $[b^m,\mu^*_{\Omega,\lambda}]$ is bounded on $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

In the proof of Theorem 3.1, we also need the following lemma.

Lemma 3.2 (see [6]). *Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, m be a positive integer and B be a ball in $\mathbb{R}^n$. Then we have that for all $b\in\mathrm{BMO}(\mathbb{R}^n)$ and all $j,i\in\mathbb{Z}$ with $j>i$,*

$$\frac{1}{C}\|b\|^m_*\leq\sup_B\frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)^m\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq C\|b\|^m_*,$$
$$\|(b-b_{B_i})^m\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\leq C(j-i)^m\|b\|^m_*\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i=\{x\in\mathbb{R}^n:|x|\leq 2^i\}$ and $B_j=\{x\in\mathbb{R}^n:|x|\leq 2^j\}$.

Proof of Theorem 3.1. Similar to Theorem 2.1, we only prove homogeneous case and still suppose $0<p<\infty$. Let $f\in\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, and we write

$$f(x)=\sum_{j=-\infty}^{\infty}f\chi_j(x)=\sum_{j=-\infty}^{\infty}f_j(x).$$

Then we have

$$\begin{aligned}
&\|[b^m,\mu^*_{\Omega,\lambda}](f)\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
=&\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|[b^m,\mu^*_{\Omega,\lambda}](f)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}\\
\leq& C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=-\infty}^{k-2}\|[b^m,\mu^*_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
&+C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=k-1}^{k+1}\|[b^m,\mu^*_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
&+C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty}\|[b^m,\mu^*_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\Big\}^{1/p}\\
=:&CJ_1+CJ_2+CJ_3. \qquad (3.1)
\end{aligned}$$

Noting $[b^m,\mu^*_{\Omega,\lambda}]$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, so we have

$$J_2\le C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}=C\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{3.2}$$

Now we estimate J_1. By the Minkowski inequality we have

$$\begin{aligned}
&|[b^m,\mu^*_{\Omega,\lambda}](f_j)(x)|\\
=&\bigg(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}[b(x)-b(z)]^m f_j(z)dz\Big|^2\frac{dydt}{t^{n+3}}\bigg)^{1/2}\\
=&\bigg(\int_0^\infty\int_{\mathbb{R}^n}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-1}}[b(x)-b(z)]^m f_j(z)dz\Big|^2\frac{dydt}{t^{n+3}}\bigg)^{1/2}\\
\le&\int_{\mathbb{R}^n}|b(x)-b(z)|^m|f_j(z)|\bigg(\int_0^\infty\int_{|y-z|<t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\bigg)^{1/2}dz\\
\le&\int_{\mathbb{R}^n}|b(x)-b(z)|^m|f_j(z)|\bigg(\int_0^{|x-z|}\int_{|y-z|<t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\bigg)^{1/2}dz\\
&+\int_{\mathbb{R}^n}|b(x)-b(z)|^m|f_j(z)|\bigg(\int_{|x-z|}^\infty\int_{|y-z|<t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2}}\frac{dydt}{t^{n+3}}\bigg)^{1/2}dz.
\end{aligned}\tag{3.3}$$

Note that $x\in A_k$, $z\in A_j$ and $j\le k-2$. By (2.5), (2.6) and the generalized Hölder inequality we have

$$\begin{aligned}
&|[b^m,\mu^*_{\Omega,\lambda}](f_j)(x)|\\
\le&C\int_{\mathbb{R}^n}\frac{|f_j(z)|}{|x-z|^n}|b(x)-b(z)|^m dz\\
\le&C\bigg(|b(x)-b_{B_j}|^m\int_{A_j}\frac{|f_j(z)|}{|x-z|^n}dz+\int_{A_j}\frac{|f_j(z)|}{|x-z|^n}|b_{B_j}-b(z)|^m dz\bigg)\\
\le&C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(|b(x)-b_{B_j}|^m\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_j}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big).
\end{aligned}\tag{3.4}$$

By Lemma 1.3, Lemma 1.4 and Lemma 3.2 we have

$$\begin{aligned}
&\|[b^m,\mu^*_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
\le&C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|(b(\cdot)-b_{B_j})^m\chi_k(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&+\|b\|^m_*\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\
\le&C2^{-kn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big((k-j)^m\|b\|^m_*\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&+\|b\|^m_*\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le C2^{-kn}(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\le C(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\\
&\le C2^{(j-k)n\delta_2}(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
J_1 &\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_2}(k-j)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&=C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\alpha)}(k-j)^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.
\end{aligned}
$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_2-\alpha>0$, by the Hölder inequality we have

$$
\begin{aligned}
J_1 &\le C\|b\|_*^m\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p2^{(j-k)(n\delta_2-\alpha)p/2}\Big)\\
&\qquad\times\Big(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\alpha)p'/2}(k-j)^{mp'}\Big)^{p/p'}\Big\}^{1/p}\\
&\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p}\\
&=C\|b\|_*^m\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p/2}\right)\right\}^{1/p}\\
&\le C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p}\\
&=C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.5}
$$

If $0<p\le 1$, then we have

$$
\begin{aligned}
J_1 &\le C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha p}2^{(j-k)(n\delta_2-\alpha)p}(k-j)^{mp}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p}\\
&=C\|b\|_*^m\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\left(\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\alpha)p}(k-j)^{mp}\right)\right\}^{1/p}\\
&\le C\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.6}
$$

Let us now estimate J_3. Note that $x\in A_k$, $y\in A_j$ and $j\ge k+2$, so we have $|y-z|\sim|y|$.

Similar to (3.4), we get

$$\begin{aligned}&|[b^m,\mu^*_{\Omega,\lambda}](f_j)(x)|\\ \le& C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(|b(x)-b_{B_k}|^m\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}+\|(b_{B_k}-b(\cdot))\chi_j(\cdot)\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\Big).\end{aligned}\tag{3.7}$$

By Lemma 1.3, Lemma 1.4 and Lemma 3.2, we have

$$\begin{aligned}&\|[b^m,\mu^*_{\Omega,\lambda}](f_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \le& C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\|b\|_*^m\|\chi_j\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &+\|(b_{B_k}-b(\cdot))^m\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\ \le& C2^{-jn}\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\|b\|_*^m\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &+(j-k)^m\|b\|_*^m\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\ \le& C2^{-jn}(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\|\chi_{B_j}\Big\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ \le& C(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\\ \le& C2^{(k-j)n\delta_1}(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}J_3\le& C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}(j-k)^m\|b\|_*^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\ =& C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}(j-k)^m\|f_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\right\}^{1/p}.\end{aligned}$$

If $1<p<\infty$, take $1/p+1/p'=1$. Since $n\delta_1+\alpha>0$, by the Hölder inequality we have

$$\begin{aligned}J_3\le& C\|b\|_*^m\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\Big)\\ &\times\Big(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1+\alpha)p'/2}(j-k)^{mp'}\Big)^{p/p'}\Big\}^{1/p}\\ \le& C\|b\|_*^m\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}2^{(k-j)(n\delta_1+\alpha)p/2}\right)\right\}^{1/p}\\ =& C\|b\|_*^m\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)p/2}\right)\right\}^{1/p}\end{aligned}$$

$$\begin{aligned}
&\le C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}\\
&=C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \qquad (3.8)
\end{aligned}$$

If $0<p\le 1$, then we have

$$\begin{aligned}
J_3\le& C\|b\|^m_*\Big\{\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty}2^{j\alpha p}2^{(k-j)(n\delta_1+\alpha)p}(j-k)^{mp}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big\}^{1/p}\\
=&C\|b\|^m_*\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha p}\|f_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)p}(j-k)^{mp}\Big)\Big\}^{1/p}\\
\le& C\|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \qquad (3.9)
\end{aligned}$$

Therefore, by (3.1), (3.2), (3.5), (3.6), (3.8), (3.9), we complete the proof of Theorem 3.1. □

Since $[b^m,\mu_{\Omega,S}](f)(x)\le C_\lambda[b^m,\mu^*_{\Omega,\lambda}](f)(x)$, we easily obtain the following theorem.

Theorem 3.2. *Suppose that* $b\in \mathrm{BMO}(\mathbb{R}^n)$, $0<p\le\infty$, $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (1.1) *and* (1.2) *in Lemma 1.1,* $\Omega\in L^2(\mathrm{S}^{n-1})$ *and* $-n\delta_1<\alpha<n\delta_2$. *Then* $[b^m,\mu_{\Omega,S}]$ *is bounded on* $\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

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