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A Characterization of MRA Based Wavelet Frames Generated by the Walsh Polynomials

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Abstract. Extension Principles play a significant role in the construction of MRA based wavelet frames and have attracted much attention for their potential applications in various scientific fields. A novel and simple procedure for the construction of tight wavelet frames generated by the Walsh polynomials using Extension Principles was recently considered by Shah in [Tight wavelet frames generated by the Walsh polynomials, Int. J. Wavelets, Multiresolut. Inf. Process., 11(6) (2013), 1350042]. In this paper, we establish a complete characterization of tight wavelet frames generated by the Walsh polynomials in terms of the polyphase matrices formed by the polyphase components of the Walsh polynomials.

Key Words: Frame, wavelet frame, polyphase matrix, extension principles, Walsh polynomial, Walsh-Fourier transform.

AMS Subject Classifications: 42C15, 42C40, 42A38, 41A17, 22B99

1 Introduction

The most common method to construct tight wavelet frames relies on the so-called Unitary Extension Principles (UEP) introduced by Ron and Shen [11] and were subsequently extended by Daubechies et al. [2] in the form of the Oblique Extension Principle (OEP). They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function $\phi(x)$, which generates a multiresolution analysis. The resulting wavelet frames are based on multiresolution analysis, and the generators are often called *framelets*. These methods of construction of wavelet frames are generalized from one-dimension to higher-dimension, tight frames to dual frames, from single scaling function to a scaling function vector. Moreover, these principles are important because they can be

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used to construct wavelets from refinable functions which may not be scaling functions with desirable properties such as symmetry and antisymmetry, smoothness or compact support. To mention only a few references on tight wavelet frames, the reader is referred to [4, 6, 9].

The past decade has also witnessed a tremendous interest in the problem of constructing compactly supported orthonormal scaling functions and wavelets with an arbitrary dilation factor $p \geq 2$, $p \in \mathbb{N}$ (see Debnath and Shah [3]). The motivation comes partly from signal processing and numerical applications, where such wavelets are useful in image compression and feature extraction because of their small support and multifractal structure. Lang [10] constructed several examples of compactly supported wavelets for the Cantor dyadic group by following the procedure of Daubechies [1] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Kozyrev [8] found a compactly supported p-adic wavelet basis for $L^2(\mathbb{Q}_p)$ which is an analog of the Haar basis. The concept of multiresolution analysis on a positive half-line $\mathbb{R}^+$ was recently introduced by Farkov [5]. He pointed out a method for constructing compactly supported orthogonal p-wavelets related to the Walsh functions, and proved necessary and sufficient conditions for scaling filters with p^n many terms $(p,n\geq 2)$ to generate a p-MRA in $L^2(\mathbb{R}^+)$. Subsequently, dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ were constructed by Shah and Debnath in [17] using the machinery of Walsh-Fourier transforms. They have established a necessary and sufficient conditions for the system $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^+\}$ to be a frame for $L^2(\mathbb{R}^+)$. Wavelet packets and wavelet frame packets related to the Walsh polynomials were deeply investigated in a series of papers by the author in [13, 14, 18]. Recent results in this direction can also be found in [6, 16] and the references therein.

The second author of this article wrote an article [15] that focuses on the construction of tight wavelet frames generated by the Walsh polynomials on a half-line $\mathbb{R}^+$ based on the ideas from unitary extension principles. More precisely, the author provide a sufficient condition for finite number of functions $\{\psi_1, \psi_2, \cdots, \psi_L\}$ to form a tight wavelet frame for $L^2(\mathbb{R}^+)$ and established a characterization of tight wavelet frames on a positive half-line $\mathbb{R}^+$ by virtue of the modulation matrix $\mathcal{M}(\xi) = \{h_\ell(\xi + k/p)\}_{\ell,k=0}^{L,p-1}$ formed by the Walsh polynomials $h_\ell(\xi)$, $\ell = 0,1,\cdots,L$ associated with the scaling function $\phi(x)$ and basic wavelets $\psi_\ell(x)$, $\ell = 1,\cdots,L$.

Since the modulation matrix involved in the unitary extension principle has a particular structure and all the information is contained in the first column; the other columns can be derived from the first column by shifting the arguments. On contrary to this, a polyphase matrix is un-structured and this gives an opportunity to create new wavelets from existing ones by multiplying the polyphase matrix by some other appropriate matrix factor. In this paper, we take this opportunity and establish a complete characterization of tight wavelet frames generated by the Walsh polynomials in terms of the polyphase matrix $\Gamma(\xi) = \{\mu_{\ell,r}(\xi)\}_{\ell,r=0}^{p-1}$ formed by the polyphase components $\mu_{\ell,r}(\xi)$, $r = 0,1,\cdots,p-1$ of the Walsh polynomials $h_\ell(\xi)$.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and preliminaries related to the operations on positive half-line $\mathbb{R}^+$ including the definitions of Walsh-Fourier transform and MRA based wavelet frames related to the Walsh polynomials. In Section 3, we prove the main result of this article, shows that a unitary polyphase matrix leads to a tight wavelet frame generated by the Walsh polynomials.

2 Walsh-Fourier analysis and MRA based wavelet frames

We start this section with certain results on Walsh-Fourier analysis. We present a brief review of generalized Walsh functions, Walsh-Fourier transforms and its various properties.

As usual, let $\mathbb{R}^+=[0,+\infty)$, $\mathbb{Z}^+=\{0,1,2,\cdots\}$ and $\mathbb{N}=\mathbb{Z}^+-\{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x\in\mathbb{R}^+$ and any positive integer j, we set

$$x_j=[p^j x](\text{mod } p), \qquad x_{-j}=[p^{1-j}x](\text{mod } p), \tag{2.1}$$

where $x_j,x_{-j}\in\{0,1,\cdots,p-1\}$. It is clear that for each $x\in\mathbb{R}^+$, there exist $k=k(x)$ in $\mathbb{N}$ such that $x_{-j}=0$, $\forall j>k$.

Consider on $\mathbb{R}^+$ the addition defined as follows:

$$x\oplus y=\sum_{j<0}\zeta_j p^{-j-1}+\sum_{j>0}\zeta_j p^{-j},$$

with $\zeta_j=x_j+y_j(\text{mod } p)$, $j\in\mathbb{Z}\setminus\{0\}$, where $\zeta_j\in\{0,1,\cdots,p-1\}$ and x_j, y_j are calculated by (2.1). As usual, we write $z=x\ominus y$ if $z\oplus y=x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x\in[0,1)$, let $r_0(x)$ is given by

$$r_0(x)=\begin{cases} 1, & \text{if } x\in[0,1/p),\\ \varepsilon_p^{\ell}, & \text{if } x\in[\ell p^{-1},(\ell+1)p^{-1}), \quad \ell=1,2,\cdots,p-1,\end{cases}$$

where $\varepsilon_p=\exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1)=r_0(x)$, $x\in\mathbb{R}^+$. Then, the *generalized Walsh functions* $\{w_m(x):m\in\mathbb{Z}^+\}$ are defined by

$$w_0(x)\equiv 1 \quad \text{and} \quad w_m(x)=\prod_{j=0}^{k}\left(r_0(p^j x)\right)^{\mu_j},$$

where

$$m=\sum_{j=0}^{k}\mu_j p^j, \quad \mu_j\in\{0,1,\cdots,p-1\}, \quad \mu_k\neq 0.$$

They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions.

For $x,y\in\mathbb{R}^+$, let

$$\chi(x,y)=\exp\Big(\frac{2\pi i}{p}\sum_{j=1}^{\infty}(x_jy_{-j}+x_{-j}y_j)\Big), \tag{2.2}$$

where x_j, y_j are given by (2.1).

We observe that

$$\chi\Big(x,\frac{m}{p^n}\Big)=\chi\Big(\frac{x}{p^n},m\Big)=w_m\Big(\frac{x}{p^n}\Big),\qquad \forall x\in[0,p^n),\quad m,n\in\mathbb{Z}^+,$$
$$\chi(x\oplus y,z)=\chi(x,z)\chi(y,z),\quad \chi(x\ominus y,z)=\chi(x,z)\overline{\chi(y,z)},$$

where $x,y,z\in\mathbb{R}^+$ and $x\oplus y$ is p-adic irrational. It is well known that systems $\{\chi(\alpha,\cdot)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot,\alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in L^2 in $[0,1]$ (see Golubov et al. [7]).

The *Walsh-Fourier transform* of a function $f\in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi)=\int_{\mathbb{R}^+}f(x)\overline{\chi(x,\xi)}dx, \tag{2.3}$$

where $\chi(x,\xi)$ is given by (2.2). The Walsh-Fourier operator $\mathcal{F}:L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)$, $\mathcal{F}f=\hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [7,12]). In particular, if $f\in L^2(\mathbb{R}^+)$, then $\hat{f}\in L^2(\mathbb{R}^+)$ and

$$\|\hat{f}\|_{L^2(\mathbb{R}^+)}=\|f\|_{L^2(\mathbb{R}^+)}.$$

For given $\Psi:=\{\psi_1,\cdots,\psi_L\}\subset L^2(\mathbb{R}^+)$, define the wavelet system

$$\mathcal{F}_\Psi:=\big\{\psi_{j,k}^{\ell}:1\le\ell\le L,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^+\big\}, \tag{2.4}$$

where $\psi_{j,k}^{\ell}=p^{j/2}\psi^{\ell}\big(p^j\cdot\ominus k\big)$. The wavelet system $\mathcal{F}_\Psi$ is called a *framelet system*, if there exist positive numbers $0<A\le B<\infty$ such that for all $f\in L^2(\mathbb{R}^+)$

$$A\|f\|_2^2\le\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\big|\langle f,\psi_{j,k}^{\ell}\rangle\big|^2\le B\|f\|_2^2. \tag{2.5}$$

The largest A and the smallest B for which (2.5) holds are called *wavelet frame bounds*. A wavelet frame is a *tight wavelet frame* if A and B are chosen such that $A=B$ and then generators $\psi_1,\psi_2,\cdots,\psi_L$ are often referred as *framelets*. If only the right-hand inequality in (2.5) holds, then $\mathcal{F}_\Psi$ is called a *Bessel sequence*.

Next, we give a brief account of the MRA based wavelet frames generated by the Walsh polynomials on a positive half-line $\mathbb{R}^+$. Following the unitary extension principle, one often starts with a refinable function or even with a refinement mask to construct desired wavelet frames. A compactly supported function $\phi \in L^2(\mathbb{R}^+)$ is called a *refinable function*, if it satisfies an equation of the type

$$\phi(x) = p\sum_{k=0}^{p^n-1} c_k \phi(px \ominus k), \quad x \in \mathbb{R}^+, \tag{2.6}$$

where c_k are complex coefficients. Applying the Walsh-Fourier transform, we can write this equation as

$$\hat{\phi}(\xi) = h_0(p^{-1}\xi)\hat{\phi}(p^{-1}\xi), \tag{2.7}$$

where

$$h_0(\xi) = \sum_{k=0}^{p^n-1} c_k \overline{w_k(\xi)} \tag{2.8}$$

is a *generalized Walsh polynomial*, which is called the *mask* or *symbol* of the refinable function ϕ and is of course a p-adic step function. Observe that

$$w_k(0) = \hat{\phi}(0) = 1.$$

Hence, letting $\xi = 0$ in (2.7) and (2.8), we obtain

$$\sum_{k=0}^{p^n-1} c_k = 1.$$

Since ϕ is compactly supported and in fact $\operatorname{supp}\phi \subset [0, p^{n-1})$, therefore $\hat{\phi} \in \mathcal{E}_{n-1}(\mathbb{R}^+)$ and hence as a result $\hat{\phi}(\xi) = 1$ for all $\xi \in [0, p^{1-n})$ as $\hat{\phi}(0) = 1$.

For a compactly supported refinable function $\phi \in L^2(\mathbb{R}^+)$, let V_0 be the closed shift invariant space generated by $\{\phi(x \ominus k) : k \in \mathbb{Z}^+\}$ and $V_j = \{\phi(p^j x) : \phi \in V_0\}$, $j \in \mathbb{Z}$. Then, it is proved in [5] that the closed subspaces $\{V_j : j \in \mathbb{Z}\}$ forms a p-multiresolution analysis (p-MRA) for $L^2(\mathbb{R}^+)$. Recall that a p-MRA is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^+)$ that satisfies: (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$, (ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and (iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. We further assume that

$$|\hat{\varphi}(0)|^2 = 1 \quad \text{and} \quad \lim_{j\to\infty} |\hat{\varphi}(p^j\xi)| = 0 \quad \text{for } a.e.\ \xi \in \mathbb{R}^+. \tag{2.9}$$

Given an p-MRA generated by a compactly supported refinable function $\phi(x)$, one can construct a set of basic tight framelets $\Psi = \{\psi_1, \cdots, \psi_L\} \subset V_1$ satisfying

$$\hat{\psi}^\ell(\xi) = h_\ell(p^{-1}\xi)\hat{\phi}(p^{-1}\xi), \tag{2.10}$$

where

$$h_\ell(\xi)=\sum_{k=0}^{p^n-1} d_k^\ell \overline{w_k(\xi)}, \quad \ell=1,\cdots,L, \tag{2.11}$$

are the generalized Walsh polynomials in $L^2[0,1]$ and are called the *framelet symbols* or *wavelet masks*.

With $h_\ell(\xi)$, $\ell=0,1,\cdots,L$, $L\geq p-1$, as the Walsh polynomials (wavelet masks), we formulate the matrix $\mathcal{M}(\xi)$ as:

$$\mathcal{M}(\xi)=\begin{pmatrix} h_0(\xi) & h_0(\xi\oplus 1/p) & \cdots & h_0(\xi\oplus(p-1)/p) \\ h_1(\xi) & h_1(\xi\oplus 1/p) & \cdots & h_1(\xi\oplus(p-1)/p) \\ \vdots & \vdots & \ddots & \vdots \\ h_L(\xi) & h_L(\xi\oplus 1/p) & \cdots & h_L(\xi\oplus(p-1)/p) \end{pmatrix}. \tag{2.12}$$

The so-called unitary extension principle (UEP) provides a sufficient condition on $\Psi=\{\psi_1,\cdots,\psi_L\}$ such that the wavelet system $\mathcal{F}_\Psi$ given by (2.4) forms a tight frame of $L^2(\mathbb{R}^+)$. In this connection, Shah [15] gave an explicit construction scheme for the construction of tight wavelet frames generated by the Walsh polynomials using unitary extension principles in the following way.

Theorem 2.1. *Let $\phi(x)$ be a compactly supported refinable function and $\hat{\phi}(0)=1$. Then, the wavelet system $\mathcal{F}_\Psi$ given by* (2.4) *constitutes a normalized tight wavelet frame in $L^2(\mathbb{R}^+)$ provided the matrix $\mathcal{M}(\xi)$ as defined in* (2.12) *satisfies*

$$\mathcal{M}(\xi)\mathcal{M}^*(\xi)=I_p, \quad \textit{for a.e. } \xi\in\sigma(V_0), \tag{2.13}$$

where

$$\sigma(V_0):=\Big\{\xi\in[0,1]:\sum_{k\in\mathbb{Z}^+}|\hat{\phi}(\xi\oplus k)|^2\neq 0\Big\}.$$

3 Polyphase matrix characterization of tight wavelet frames

Motivated and inspired by the construction of tight wavelet frames generated by the Walsh polynomials [15] using the machinery of unitary extension principles. In this section, we shall first derive the polyphase representation of the Walsh polynomials (wavelet masks) and then establish a complete characterization of tight wavelet frames generated by the Walsh polynomials by means of their polyphase components.

The *polyphase representation* of the refinement mask $h_0(\xi)$ can be derived by using the properties of Walsh polynomials as

$$\begin{aligned} h_0(\xi)&=\sum_{k=0}^{p^n-1} c_k\overline{w_k(\xi)}\sum_{k=0}^{p^n-1}\sum_{m=0}^{p-1} c_{pk+m}\overline{w_{pk+m}(\xi)} \\ &=\sum_{m=0}^{p-1}\overline{w_m(\xi)}\sum_{k=0}^{p^n-1} c_{pk+m}\overline{w_k(p\xi)}=\frac{1}{\sqrt{p}}\sum_{m=0}^{p-1}\mu_{0,m}(p\xi)\overline{w_m(\xi)}, \end{aligned}$$

where

$$\mu_{0,m}(\xi)=\sqrt{p}\sum_{k=0}^{p^n-1}c_{pk+m}\overline{w_k(\xi)},\quad m=0,1,\cdots,p-1. \tag{3.1}$$

Similarly, the wavelet masks $h_\ell(\xi)$, $1\le\ell\le L$, as defined in (2.11) can be splitted into polyphase components as

$$h_\ell(\xi)=\frac{1}{\sqrt{p}}\sum_{m=0}^{p-1}\mu_{\ell,m}(p\xi)\overline{w_m(\xi)}, \tag{3.2}$$

where

$$\mu_{\ell,m}(\xi)=\sqrt{p}\sum_{k=0}^{p^{n-1}-1}d^{\ell}_{pk+m}\overline{w_k(\xi)},\quad m=0,1,\cdots,p-1. \tag{3.3}$$

With the polyphase components given by (3.1) and (3.3), we formulate the *polyphase matrix* $\Gamma(\xi)$ as:

$$\Gamma(\xi)=\begin{pmatrix}\mu_{0,0}(\xi) & \mu_{1,0}(\xi) & \cdots & \mu_{L,0}(\xi)\\ \mu_{0,1}(\xi) & \mu_{1,1}(\xi) & \cdots & \mu_{L,1}(\xi)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{0,p-1}(\xi) & \mu_{1,p-1}(\xi) & \cdots & \mu_{L,p-1}(\xi)\end{pmatrix}. \tag{3.4}$$

The polyphase matrix $\Gamma(\xi)$ is called a *unitary matrix* if

$$\Gamma(p\xi)\Gamma^*(p\xi)=I_p,\quad a.e.\ \ \xi\in[0,1], \tag{3.5}$$

which is equivalent to

$$\begin{aligned}&\sum_{\ell=0}^{L}\overline{\mu_{\ell,r}(p\xi)}\mu_{\ell,r'}(p\xi)=\delta_{r,r'}\\ \Leftrightarrow&\sum_{\ell=1}^{L}\overline{\mu_{\ell,r'}(p\xi)}\mu_{\ell,r}(p\xi)=\delta_{r,r'}-\overline{\mu_{0,r}(p\xi)}\mu_{0,r'}(p\xi),\quad 0\le r,r'\le p-1.\end{aligned} \tag{3.6}$$

The following theorem, the main result of this paper, shows that a unitary polyphase matrix leads to a tight wavelet frame generated by Walsh polynomial on a half-line $\mathbb{R}^+$.

Theorem 3.1. *Let $\phi\in L^2(\mathbb{R}^+)$ be a compactly supported refinable function and every element of the framelet symbols, $h_0(\xi)$, $h_\ell(\xi)$, $\ell=1,2,\cdots,L$, in* (2.8) *and* (2.11) *is a Walsh polynomial. Moreover, if the polyphase matrix $\Gamma(\xi)$ given by* (3.4) *satisfy UEP condition* (3.5)*, then the wavelet system $\mathcal{F}_\Psi$ given by* (2.4) *constitutes a tight frame for $L^2(\mathbb{R}^+)$.*

Proof. By Parseval's formula, we have

$$\begin{aligned}
&\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\langle f,\psi_{j,k}^{\ell}\rangle\right|^2\\
&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,p^{j/2}\psi^{\ell}(p^jx\ominus k)\right\rangle\right|^2\\
&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle \hat{f},p^{-j/2}\hat{\psi}^{\ell}(p^{-j}\xi)w(p^{-j}\xi)\right\rangle\right|^2\\
&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}p^j\sum_{k\in\mathbb{Z}^+}\left|\left\langle \hat{f}(p^j\xi)\overline{\hat{\psi}^{\ell}(\xi)},w(\xi)\right\rangle\right|^2\\
&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}p^j\int_{\mathbb{R}^+}\left|\hat{f}(p^j\xi)\right|^2\left|\hat{\psi}^{\ell}(\xi)\right|^2d\xi. \qquad (3.7)
\end{aligned}$$

Implementing the polyphase component formula (3.3) of wavelet masks $h_\ell(\xi)$, $\ell=1,\cdots,L$, we obtain

$$\begin{aligned}
&\sum_{\ell=1}^{L}\left|\hat{\psi}^{\ell}(\xi)\right|^2\\
&=\sum_{\ell=1}^{L}\left|h_\ell(p^{-1}\xi)\hat{\phi}(p^{-1}\xi)\right|^2\\
&=\sum_{\ell=1}^{L}\overline{h_\ell(p^{-1}\xi)\hat{\phi}(p^{-1}\xi)}h_\ell(p^{-1}\xi)\hat{\phi}(p^{-1}\xi)\\
&=\overline{\hat{\phi}(p^{-1}\xi)}\sum_{\ell=1}^{L}\overline{\left(\sum_{r=0}^{p-1}\overline{w_r(\xi)}\mu_{\ell,r}(\xi)\right)}\left(\sum_{r'=0}^{p-1}\overline{w_{r'}(\xi)}\mu_{\ell,r'}(\xi)\right)\hat{\phi}(p^{-1}\xi)\\
&=\overline{\hat{\phi}(p^{-1}\xi)}\sum_{r=0}^{p-1}\sum_{r'=0}^{p-1}w_{r-r'}(\xi)\left(\sum_{\ell=1}^{L}\overline{\mu_{\ell,r}(\xi)}\mu_{\ell,r'}(\xi)\right)\hat{\phi}(p^{-1}\xi).
\end{aligned}$$

Since the polyphase matrix $\Gamma(\xi)$ is unitary, which is equivalent to (3.6), the above expression reduces to

$$\begin{aligned}
\sum_{\ell=1}^{L}\left|\hat{\psi}^{\ell}(\xi)\right|^2&=\overline{\hat{\phi}(p^{-1}\xi)}\sum_{r=0}^{p-1}\sum_{r'=0}^{p-1}w_{r-r'}(\xi)\left(\delta_{r,r'}-\mu_{0,r}(\xi)\mu_{0,r'}(\xi)\right)\hat{\phi}(p^{-1}\xi)\\
&=\overline{\hat{\phi}(p^{-1}\xi)}\hat{\phi}(p^{-1}\xi)-\overline{\hat{\phi}(p^{-1}\xi)}\sum_{r=0}^{p-1}\sum_{r'=0}^{p-1}w_{r-r'}(\xi)\overline{\mu_{0,r}(\xi)}\mu_{0,r'}(\xi)\hat{\phi}(p^{-1}\xi)\\
&=\left|\hat{\phi}(p^{-1}\xi)\right|^2-\overline{\hat{\phi}(p^{-1}\xi)h_0(p^{-1}\xi)}h_0(p^{-1}\xi)\hat{\phi}(p^{-1}\xi)\\
&=\left|\hat{\phi}(p^{-1}\xi)\right|^2-\left|h_0(p^{-1}\xi)\hat{\phi}(p^{-1}\xi)\right|^2\\
&=\left|\hat{\phi}(p^{-1}\xi)\right|^2-\left|\hat{\phi}(\xi)\right|^2. \qquad (3.8)
\end{aligned}$$

By substituting Eq. (3.8) in (3.7), we obtain

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\left|\left\langle f,\psi_{j,k}^{\ell}\right\rangle\right|^2=\sum_{j\in\mathbb{Z}}p^j\int_{\mathbb{R}^+}\left|\hat{f}(p^j\xi)\right|^2\left(\left|\hat{\phi}(p^{-1}\xi)\right|^2-\left|\hat{\phi}(\xi)\right|^2\right)d\xi$$
$$=\int_{\mathbb{R}^+}\left|\hat{f}(\xi)\right|^2\sum_{j\in\mathbb{Z}}\left(\left|\hat{\phi}(p^{-j-1}\xi)\right|^2-\left|\hat{\phi}(p^{-j}\xi)\right|^2\right)d\xi. \tag{3.9}$$

Using the assumption (2.9), the summand in the above expression can be written as

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\left(\left|\hat{\phi}(p^{-j-1}\xi)\right|^2-\left|\hat{\phi}(p^{-j}\xi)\right|^2\right)d\xi\\&=\lim_{j\to\infty}\left|\hat{\phi}(p^{-j-1}\xi)\right|^2-\lim_{j\to-\infty}\left|\hat{\phi}(p^{-j}\xi)\right|^2\\&=\lim_{j\to\infty}\left|\hat{\phi}(p^{-j}\xi)\right|^2-\lim_{j\to\infty}\left|\hat{\phi}(p^{j}\xi)\right|^2\\&=\left|\hat{\phi}(0)\right|^2-\lim_{j\to\infty}\left|\hat{\phi}(p^{j}\xi)\right|^2\\&=1.\end{aligned}$$

By using the above estimate in Eq. (3.9), we obtain

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\sum_{\ell=1}^{L}\left|\left\langle f,\psi_{j,k}^{\ell}\right\rangle\right|^2=\int_{\mathbb{R}^+}\left|\hat{f}(\xi)\right|^2d\xi=\left\|\hat{f}\right\|_2^2=\|f\|_2^2.$$

This completes the proof of the theorem. □

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