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Boundedness for the Singular Integral with Variable Kernel and Fractional Differentiation on Weighted Morrey Spaces

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Abstract. Let T be the singular integral operator with variable kernel, T^* be the adjoint of T and $T^\sharp$ be the pseudo-adjoint of T. Let T_1T_2 be the product of T_1 and T_2, $T_1\circ T_2$ be the pseudo product of T_1 and T_2. In this paper, we establish the boundedness for commutators of these operators and the fractional differentiation operator D^γ on the weighted Morrey spaces.

Key Words: Singular integral, variable kernel, fractional differentiation, BMO Sobolev space, weighted Morrey spaces.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ $(n\geq 2)$ with normalized Lebesgue measure $d\sigma$. The singular integral operator with variable kernel is defined by

$$Tf(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega(x,x-y)}{|x-y|^n}f(y)dy, \tag{1.1}$$

where $\Omega(x,z)$ satisfies the following conditions:

$$\Omega(x,\lambda z)=\Omega(x,z) \quad \text{for any } \ x,z\in\mathbb{R}^n \quad \text{and} \quad \lambda>0, \tag{1.2a}$$

$$\int_{S^{n-1}}\Omega(x,z')d\sigma(z')=0 \quad \text{for any } \ x\in\mathbb{R}^n. \tag{1.2b}$$

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Let $m\in\mathbb{N}$, denote by $\mathcal{H}_m$ the space of surface spherical harmonics of degree m on S^{n-1} with its dimension d_m. $\{Y_{m,j}\}_{j=1}^{d_m}$ denotes the normalized complete system in $\mathcal{H}_m$. We can write (see [1,3,9])

$$\Omega(x,z')=\sum_{m\geq 0}\sum_{j=1}^{d_m}a_{m,j}(x)Y_{m,j}(z'), \tag{1.3}$$

where

$$a_{m,j}(x)=\int_{S^{n-1}}\Omega(x,z')\overline{Y_{m,j}(z')}d\sigma(z'). \tag{1.4}$$

Let

$$T_{m,j}f(x)=\frac{Y_{m,j}}{|\cdot|^n}*f(x).$$

Then we can write

$$Tf(x)=\sum_{m\geq 0}\sum_{j=1}^{d_m}a_{m,j}(x)T_{m,j}f(x). \tag{1.5}$$

Let T^* and $T^\sharp$ denote the adjoint of T and the pseudo-adjoint of T respectively, which are defined by

$$T^*f(x)=\sum_{m=0}^{\infty}\sum_{j=1}^{d_m}(-1)^m T_{m,j}(\overline{a}_{m,j}f)(x) \tag{1.6}$$

and

$$T^\sharp f(x)=\sum_{m=0}^{\infty}\sum_{j=1}^{d_m}(-1)^m \overline{a}_{m,j}(x)T_{m,j}f(x). \tag{1.7}$$

Let T_1T_2 denote the product of T_1 and T_2, $T_1\circ T_2$ denote the pseudo product of T_1 and T_2 (see [1] for the definitions).

In 1955, Calderón and Zygmund [2] investigated the L^2 boundedness of the operator T. Let D be the square root of the Laplacian operator which is defined by $\widehat{Df}(\xi)=|\xi|\widehat{f}(\xi)$. Let

$$T_1f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega_1(x,x-y)}{|x-y|^n}f(y)dy \tag{1.8}$$

and

$$T_2f(x)=\text{p.v.}\int_{\mathbb{R}^n}\frac{\Omega_2(x,x-y)}{|x-y|^n}f(y)dy. \tag{1.9}$$

In [1], Calderón and Zygmund proved L^p $(1<p<\infty)$ boundedness of T_1^*, $T_1^\sharp$, T_1T_2, $T_1\circ T_2$ and D. In [3], Chen and Zhu proved the boundedness for commutator of these singular integral operators and the fractional differentiation operator D^γ on $L^p(\omega)$, where D^γ is defined by $\widehat{D^\gamma f}(\xi)=|\xi|^\gamma\widehat{f}(\xi)$. The Sobolev Space $I_\gamma(BMO)$ is the image of $BMO(\mathbb{R}^n)$ under I_γ (Riesz potential operator of order γ). A locally integrable function b is in $I_\gamma(BMO)$ if and only if $D^\gamma b\in BMO(\mathbb{R}^n)$. A weight function ω is called an A_p weight (or $\omega\in A_p$) $(1<p<\infty)$ if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q\omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q\omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1}<\infty,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes.

The classical Morrey spaces $L^{p,\lambda}$ were introduced by Morrey in [7]. In [6], Komori and Shirai defined the weighted Morrey spaces $L^{p,\kappa}(\omega)$ and studied the boundedness of some classical operators on these weighted spaces. For a given weight function ω, we denote by $|Q|$ the Lebesgue measure of Q and denote by $\omega(Q)=\int_Q\omega(Q)dx$ the weighted measure of Q.

Definition 1.1 (see [6])**.** Let $1\le p<\infty$, $0<\kappa<1$ and ω be a weight function. Then the weighted Morrey space is defined by

$$L^{p,\kappa}(\omega)=\{f\in L^p_{loc}(\omega):\|f\|_{L^{p,\kappa}(\omega)}<\infty\},$$

where

$$\|f\|_{L^{p,\kappa}(\omega)}=\sup_Q\left(\frac{1}{\omega(Q)^\kappa}\int_Q|f(x)|^p\omega(x)dx\right)^{1/p}$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

The purpose of this paper is to establish the boundedness for T_1^*, $T_1^\sharp$, T_1T_2, $T_1\circ T_2$ and the fractional differentiation operator D^γ on the weighted Morrey spaces. Our results are stated as follows.

Theorem 1.1. *Let $0<\gamma<1$, $1<p<\infty$, $0<\kappa<1$ and $w\in A_p$. Suppose that $\Omega(x,y)$ satisfies* (1.2a), (1.2b) *and*

$$\max_{|j|\le 2n}\|D_x^\gamma(\partial^j/\partial y^j)\Omega(x,y)\|_{L^\infty(\mathbb{R}^n\times S^{n-1})}<\infty. \tag{1.10}$$

Then there is a constant $C>0$, such that

(i) $\|(TD^\gamma-D^\gamma T)f\|_{L^{p,\kappa}(\omega)}\le C\|f\|_{L^{p,\kappa}(\omega)}$;

(ii) $\|(T^*-T^\sharp)D^\gamma f\|_{L^{p,\kappa}(\omega)}\le C\|f\|_{L^{p,\kappa}(\omega)}$.

Theorem 1.2. *Let $0<\gamma<1$, $1<p<\infty$, $0<\kappa<1$ and $w\in A_p$. Suppose that $\Omega_1(x,y)$ and $\Omega_2(x,y)$ satisfy* (1.2a) *and* (1.2b). *If $\Omega_2(x,y)$ satisfies* (1.10) *and*

$$\max_{|j|\le 2n}\|(\partial^j/\partial y^j)\Omega_1(x,y)\|_{L^\infty(\mathbb{R}^n\times S^{n-1})}<\infty. \tag{1.11}$$

Then there is a constant $C>0$, such that

$$\|(T_1\circ T_2-T_1T_2)D^\gamma f\|_{L^{p,\kappa}(\omega)}\le C\|f\|_{L^{p,\kappa}(\omega)}.$$

2 Lemmas

We begin with some Lemmas.

Lemma 2.1 (see [6])**.** *If $1<p<\infty$, $0<\kappa<1$, $\omega\in A_p$ and T is a Calderón-Zygmund singular integral operator, then T is bounded on $L^{p,\kappa}(\omega)$.*

Given a weight ω, we say that ω satisfies the doubling condition if there exists a constant $D>0$ such that for any cube Q, we have $\omega(2Q)\le D\omega(Q)$. When ω satisfies this condition, we denote $\omega\in\Delta_2$. If $\omega\in A_p$, we know $\omega\in\Delta_2$ (see [5]).

Lemma 2.2 (see [6])**.** *If $\omega\in\Delta_2$, then there exists a constant $D_1>1$ such that*

$$\omega(2Q)\ge D_1\omega(Q).$$

Lemma 2.3. *If $1<p<\infty$, $0<\kappa<1$ and $\omega\in A_p$. Let $T_{m,j}$ be the convolution operator with kernel $\frac{Y_{m,j}}{|\cdot|^n}$, that is,*

$$T_{m,j}f(x)=\frac{Y_{m,j}}{|\cdot|^n}*f(x),$$

then

$$\|T_{m,j}f\|_{L^{p,\kappa}(\omega)}\le Cm^{n/2}\|f\|_{L^{p,\kappa}(\omega)}.$$

Proof. It is sufficient to prove that there exists $C>0$ such that

$$\frac{1}{\omega(B)^\kappa}\int_B|T_{m,j}f(x)|^p\omega(x)dx\le Cm^{np/2}\|f\|^p_{L^{p,\kappa}(\omega)}.$$

Fix a ball $B=B(x_0,r)$, where $B(x_0,r)$ denotes the ball with center x_0 and radius r. Decompose $f=f_1+f_2$ with $f_1=f_{\chi_{2B}}$. Since $T_{m,j}$ is linear, we can get

$$\begin{aligned}&\frac{1}{\omega(B)^\kappa}\int_B|T_{m,j}f(x)|^p\omega(x)dx\\ \le& C\left\{\frac{1}{\omega(B)^\kappa}\int_B|T_{m,j}f_1(x)|^p\omega(x)dx+\frac{1}{\omega(B)^\kappa}\int_B|T_{m,j}f_2(x)|^p\omega(x)dx\right\}\\ =&C\{I_1+I_2\}.\end{aligned}$$

For the term I_1, using the fact that if $\omega\in A_p$ then $\|T_{m,j}f\|_{L^p(\omega)}\le Cm^{n/2}\|f\|_{L^p(\omega)}$ (see [3]), we can get

$$\begin{aligned}\int_B|T_{m,j}f_1(x)|^p\omega(x)dx\le&\int_{\mathbb{R}^n}|T_{m,j}f_1(x)|^p\omega(x)dx\\ \le&Cm^{np/2}\int_{2B}|f(x)|^p\omega(x)dx\\ \le&Cm^{np/2}\|f\|^p_{L^{p,\kappa}(\omega)}\omega(B)^\kappa.\end{aligned}$$

Hence we have

$$\|T_{m,j}f_1\|_{L^{p,\kappa}(\omega)}\le Cm^{n/2}\|f\|_{L^{p,\kappa}(\omega)}.$$

For the term I_2, for $x\in B$ and $y\in(2B)^c$ we have $|x_0-y|<C|x-y|$. By the fact that $|Y_{m,j}|\le Cm^{(n-2)/2}$ (see [1]), we get

$$|T_{m,j}f_2(x)|\le Cm^{n/2}\int_{\mathbb{R}^n}\frac{|f_2(y)|}{|x-y|^n}dy\le Cm^{n/2}\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}dy.$$

Therefore we obtain

$$\frac{1}{\omega(B)^\kappa}\int_B|T_{m,j}f_2(x)|^p\omega(x)dx\le Cm^{np/2}\Big(\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}dy\Big)^p\omega(B)^{1-\kappa}.$$

Then, by Hölder's inequality and $\omega\in A_p$, we have

$$\begin{aligned}\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}dy=&\sum_{j=1}^\infty\int_{2^jr<|x_0-y|\le2^{j+1}r}\frac{|f(y)|}{|x_0-y|^n}dy\\ \le&\sum_{j=1}^\infty\frac{1}{|2^jB|}\int_{2^{j+1}B}|f(y)|dy\\ \le&\sum_{j=1}^\infty\frac{1}{|2^jB|}\Big(\int_{2^{j+1}B}|f(y)|^p\omega(y)dy\Big)^{1/p}\Big(\int_{2^{j+1}B}\omega(y)^{1-p'}dy\Big)^{(p-1)/p}\\ \le&C\|f\|_{L^{p,\kappa}(\omega)}\sum_{j=1}^\infty\frac{1}{\omega(2^{j+1}B)^{(1-\kappa)/p}}.\end{aligned}$$

By Lemma 2.2, we get

$$\begin{aligned}\Big(\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}dy\Big)^p\omega(B)^{1-\kappa}\le&C\|f\|^p_{L^{p,\kappa}(\omega)}\Big(\sum_{j=1}^\infty\frac{\omega(B)^{(1-\kappa)/p}}{\omega(2^{j+1}B)^{(1-\kappa)/p}}\Big)^p\\ \le&C\|f\|^p_{L^{p,\kappa}(\omega)}.\end{aligned}$$

So we have

$$\|T_{m,j}f_2\|_{L^{p,\kappa}(\omega)}\le Cm^{n/2}\|f\|_{L^{p,\kappa}(\omega)}.$$

This completes the proof. □

3 Proof of Theorem 1.1: $L^{p,\kappa}(\omega)$ norm of $TD^\gamma - D^\gamma T$

Proof of Theorem 1.1: Let

$$Tf(x) = \int_{\mathbb{R}^n} \frac{\Omega(x,x-y)}{|x-y|^n} f(y)dy.$$

Write

$$\Omega(x,y) = \sum_{m=0}^{\infty} \sum_{j=1}^{d_m} a_{m,j}(x) Y_{m,j}(y),$$

where

$$a_{m,j}(x) = \int_{S^{n-1}} \Omega(x,z') \overline{Y_{m,j}(z')} d\sigma(z').$$

For $\Omega(x,y')$ satisfies (1.10), we have (see [3])

$$(TD^\gamma - D^\gamma T)f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} [a_{m,j}, D^\gamma] T_{m,j} f,$$

and

$$\|D^\gamma a_{m,j}\|_{L^\infty} \leq C m^{-2n}. \tag{3.1}$$

In fact, $[b,D^\gamma]$ (see [8]) is a generalized Calderón-Zygmund operator, then by Lemma 2.1, we can get that $[b,D^\gamma]$ is bounded on $L^{p,\kappa}(\omega)$ for $1<p<\infty$, $0<\kappa<1$ and $w \in A_p$, namely

$$\|[b,D^\gamma]f\|_{L^{p,\kappa}(\omega)} \leq C\|D^\gamma b\|_{BMO} \|f\|_{L^{p,\kappa}(\omega)}. \tag{3.2}$$

Then by $d_m \simeq m^{n-2}$ (see [4]), (3.1), (3.2) and Lemma 2.3, we get

$$\begin{aligned}
\|(TD^\gamma - D^\gamma T)f\|_{L^{p,\kappa}(\omega)} &\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \|[a_{m,j}, D^\gamma] T_{m,j} f\|_{L^{p,\kappa}(\omega)} \\
&\leq C \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \|D^\gamma a_{m,j}\|_{BMO} \|T_{m,j} f\|_{L^{p,\kappa}(\omega)} \\
&\leq C \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} m^{n/2} \|D^\gamma a_{m,j}\|_{L^\infty} \|f\|_{L^{p,\kappa}(\omega)} \\
&\leq C \sum_{m=1}^{\infty} m^{n-2} m^{n/2} m^{-2n} \|f\|_{L^{p,\kappa}(\omega)} \\
&\leq C\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}$$

Thus, we complete the proof. □

4 Proof of Theorem 1.1: $L^{p,\kappa}(\omega)$ norm of $(T^\sharp - T^*)D^\gamma$

Proof of Theorem 1.1: By (1.6) and (1.7), we can write

$$(T^\sharp - T^*)D^\gamma f = \sum_{m=1}^{\infty}\sum_{j=1}^{d_m}(-1)^m[\overline{a}_{m,j},T_{m,j}]D^\gamma f. \tag{4.1}$$

We first estimate the $L^{p,\kappa}(\omega)$ norm of $[b,T_{m,j}]D^\gamma$ for any fixed $b\in I_\gamma(BMO)$. We get

$$[b,T_{m,j}]D^\gamma f = [b,D^\gamma T_{m,j}]f - T_{m,j}[b,D^\gamma]f.$$

By (3.2) and Lemma 2.3, we get

$$\|T_{m,j}[b,D^\gamma]f\|_{L^{p,\kappa}(\omega)} \le Cm^{n/2}\|D^\gamma b\|_{BMO}\|f\|_{L^{p,\kappa}(\omega)}. \tag{4.2}$$

To estimate $L^{p,\kappa}(\omega)$ norm of $[b,D^\gamma T_{m,j}]f$, we know that (see [3]) $[b,D^\gamma T_{m,j}]f$ is a generalized Calderón-Zygmund operator with kernel

$$|k_m,j(x,y)| \le Cm^{n/2-1+\gamma}\|D^\gamma b\|_{BMO}\frac{1}{|x-y|^n},$$

then by Lemma 2.1, we get

$$\|[b,D^\gamma T_{m,j}]f\|_{L^{p,\kappa}(\omega)} \le Cm^{n/2+\gamma}\|D^\gamma b\|_{BMO}\|f\|_{L^{p,\kappa}(\omega)}, \tag{4.3}$$

where C is independent of m and j. Then by (4.2) and (4.3), we have

$$\begin{aligned}\|[b,T_{m,j}]D^\gamma f\|_{L^{p,\kappa}(\omega)} &\le C\|[b,D^\gamma T_{m,j}]f\|_{L^{p,\kappa}(\omega)} + C\|T_{m,j}[b,D^\gamma]f\|_{L^{p,\kappa}(\omega)}\\ &\le Cm^{n/2+\gamma}\|D^\gamma b\|_{BMO}\|f\|_{L^{p,\kappa}(\omega)} + Cm^{n/2}\|D^\gamma b\|_{BMO}\|f\|_{L^{p,\kappa}(\omega)}\\ &\le Cm^{n/2+\gamma}\|D^\gamma b\|_{BMO}\|f\|_{L^{p,\kappa}(\omega)}. \end{aligned}\tag{4.4}$$

By (4.1), (4.4) and (3.1), we get

$$\begin{aligned}\|(T^\sharp - T^*)D^\gamma f\|_{L^{p,\kappa}(\omega)} &\le \sum_{m=1}^{\infty}\sum_{j=1}^{d_m}\|[\overline{a}_{m,j},T_{m,j}]D^\gamma f\|_{L^{p,\kappa}(\omega)}\\ &\le C\sum_{m=1}^{\infty}\sum_{j=1}^{d_m} m^{n/2+\gamma}\|D^\gamma \overline{a}_{m,j}\|_{BMO}\|f\|_{L^{p,\kappa}(\omega)}\\ &\le C\sum_{m=1}^{\infty}\sum_{j=1}^{d_m} m^{n/2+\gamma}\|D^\gamma \overline{a}_{m,j}\|_{L^\infty}\|f\|_{L^{p,\kappa}(\omega)}\\ &\le C\sum_{m=1}^{\infty} m^{n-2}m^{n/2+\gamma}m^{-2n}\|f\|_{L^{p,\kappa}(\omega)}\\ &\le C\|f\|_{L^{p,\kappa}(\omega)}.\end{aligned}$$

Thus, we complete the proof. □

5 Proof of Theorem 1.2

Proof of Theorem 1.2: Let T_1, T_2 be like in (1.8) and (1.9). Write

$$\Omega_1(x,y)=\sum_{m=0}^{\infty}\sum_{j=1}^{d_m}a_{m,j}(x)Y_{m,j}(y),$$

and

$$\Omega_2(x,y)=\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda}b_{\lambda,\mu}(x)Y_{\lambda,\mu}(y),$$

where

$$a_{m,j}(x)=\int_{S^{n-1}}\Omega_1(x,z')\overline{Y_{m,j}(z')}d\sigma(z'),$$
$$b_{\lambda,\mu}(x)=\int_{S^{n-1}}\Omega_2(x,z')\overline{Y_{\lambda,\mu}(z')}d\sigma(z'),$$

and (see [3])

$$a_{m,j}(x)=(-1)^n m^{-n}(m+n-2)^{-n}\int_{S^{n-1}}L^n_{y'}(\Omega_1(x,y'))Y_{m,j}(y')d\sigma(y'), \qquad m\geq 1,$$
$$D^\gamma b_{\lambda,\mu}(x)=(-1)^l\lambda^{-l}(\lambda+n-2)^{-l}\int_{S^{n-1}}D^\gamma_x L^l_{y'}\Omega_2(x,y')Y_{mj}(y')d\sigma(y'), \qquad m\geq 1.$$

Since $\Omega_1(x,y)$ satisfies (1.11), then we get

$$\|a_{m,j}\|_{L^\infty}\leq Cm^{-2n}, \tag{5.1}$$

where C is independent of m and j. Since $\Omega_2(x,y')$ satisfies (1.10), we get

$$\|D^\gamma b_{\lambda,\mu}\|_{L^\infty}\leq C\lambda^{-2n}. \tag{5.2}$$

Let

$$T_{m,j}f(x)=\frac{Y_{m,j}}{|\cdot|^n}*f(x)$$

and

$$T_{\lambda,\mu}f(x)=\frac{Y_{\lambda,\mu}}{|\cdot|^n}*f(x).$$

Since $\Omega_1(x,y)$ and $\Omega_2(x,y)$ satisfies (1.2b), then we get

$$T_1f(x)=\sum_{m=1}^{\infty}\sum_{j=1}^{d_m}a_{m,j}(x)T_{m,j}f(x)$$

and

$$T_2 f(x)=\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda} b_{\lambda,\mu}(x)T_{\lambda,\mu}f(x).$$

Write (see [3])

$$(T_1\circ T_2)f(x)=\sum_{m=1}^{\infty}\sum_{j=1}^{d_m}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda} a_{m,j}(x)b_{\lambda,\mu}(x)(T_{m,j}T_{\lambda,\mu}f)(x),$$

and

$$(T_1T_2)f(x)=\sum_{m=1}^{\infty}\sum_{j=1}^{d_m}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda} a_{m,j}(x)T_{m,j}(b_{\lambda,\mu}T_{\lambda,\mu}f)(x).$$

Then

$$(T_1\circ T_2-T_1T_2)D^\gamma f=\sum_{m=1}^{\infty}\sum_{j=1}^{d_m}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda} a_{m,j}[b_{\lambda,\mu},T_{m,j}]D^\gamma T_{\lambda,\mu}f.$$

So, by Lemma 2.3, (4.4), (5.1) and (5.2), we get

$$\begin{aligned}
&\|(T_1\circ T_2-T_1T_2)D^\gamma f\|_{L^{p,\kappa}(\omega)}\\
\leq& C\sum_{m=1}^{\infty}\sum_{j=1}^{d_m}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda}\|a_{m,j}\|_{L^\infty}\|[b_{\lambda,\mu},T_{m,j}]D^\gamma T_{\lambda,\mu}f\|_{L^{p,\kappa}(\omega)}\\
\leq& C\sum_{m=1}^{\infty}\sum_{j=1}^{d_m}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda}\|a_{m,j}\|_{L^\infty}\|D^\gamma b_{\lambda,\mu}\|_{BMO}m^{n/2+\gamma}\|T_{\lambda,\mu}f\|_{L^{p,\kappa}(\omega)}\\
\leq& C\sum_{m=1}^{\infty}\sum_{j=1}^{d_m}\sum_{\lambda=1}^{\infty}\sum_{\mu=1}^{d_\lambda}\|a_{m,j}\|_{L^\infty}\|D^\gamma b_{\lambda,\mu}\|_{L^\infty}m^{n/2+\gamma}\lambda^{n/2}\|f\|_{L^{p,\kappa}(\omega)}\\
\leq& C\sum_{m=1}^{\infty}m^{n-2}m^{-2n}m^{n/2+\gamma}\sum_{\lambda=1}^{\infty}\lambda^{n-2}\lambda^{-2n}\lambda^{n/2}\|f\|_{L^{p,\kappa}(\omega)}\\
\leq& C\|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned}$$

Thus, we complete the proof. □

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References

[1] A. Calderón and A. Zygmund, Singular integral operators and differential equations, Amer. J. Math., 79 (1957), 901–921.

[2] A. Calderón and A. Zygmund, On a problem of Mihlin, Trans. Amer. Math. Soc., 78 (1955), 209–224.

[3] Y. Chen and K. Zhu, Weighted norm inequality for the singular integral with variable kernel and fractional differentiation, J. Math. Anal. Appl., 423 (2015), 1610–1629.

[4] G. Di Fazio and M. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, J. Funct. Anal., 112 (1993), 241–256.

[5] L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education, Inc. Upper Saddle River, New Jersey, 2004.

[6] Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, Math. Nachr., 282 (2009), 219–231.

[7] C. B. Morrey, On the solution of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 43 (1938), 126–166.

[8] M. A. M. Murray, Commutators with fractional differentiation and BMO Sobolev spaces, Indiana Univ. Math. J., 34 (1985), 205–215.

[9] E. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, N. J., 1971.