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Closed Range Composition Operators on a General Family of Function Spaces

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Abstract. In this paper, necessary and sufficient conditions for a closed range composition operator C_ϕ on the general family of holomorphic function spaces $F(p,q,s)$ and more generally on α-Besov type spaces $F(p,\alpha p-2,s)$ are given. We give a Carleson measure characterization on $F(p,\alpha p-2,s)$ spaces, then we indicate how Carleson measures can be used to characterize boundedness and compactness of C_ϕ on $F(p,q,s)$ and $F(p,\alpha p-2,s)$ spaces.

Key Words: Composition operators, $F(p,q,s)$ spaces, closed range, Carleson measure, Bloch space, Bergman type space.

AMS Subject Classifications: 47B33, 47B38, 30H25, 30H30

1 Introduction

Let ϕ be a holomorphic self-map of the unit disk $\mathbb{D}$. Associate to ϕ the composition operator C_ϕ is defined by $C_\phi = f\circ\phi$, for any function f that is holomorphic on $\mathbb{D}$.

This is the first setting that composition operators were studied boundedness, compactness, closed range have been studied in this setting. It is natural to study these properties on other function spaces.

In the early 70s, Cima, Thomson and Wogen in [7] were the first to study closed range composition operators, in the context of H^2 on $\mathbb{D}$. Their results are in terms of the boundary behaviour of the symbol ϕ. They asked the question of studying closed range composition operators in terms of properties of ϕ on $\mathbb{D}$ rather than on $\partial\mathbb{D}$. Zorboska in [23] answered the call and studied the problem in H^2 and also in weighted Bergman spaces. Jovovic and MacCluer in [10] studied the problem in weighted Dirichlet spaces; the closed range composition operators on Dirichlet-type spaces $D(\mu)$ (with μ is a positive Borel measure defined on the boundary of the unit disc) were introduced by Chacón in [6]. Also Ghatage, Zheng and Zorboska studied the problem in the Bloch

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space (see [9]), then Akeroyd and Ghatage revisited the problem in the context of the Bergman space (see [3]). Tjani in [16] studied the closed range composition operators on Besov spaces B_p and more generally on α-Besov spaces $B_{p,\alpha}$.

In this paper we study closed range composition operators on α-Besov type spaces $F(p,\alpha p-2,s)$ for $p>2$, and $\alpha,s>0$ with $\alpha p+s>1$. We will define and discuss properties of these spaces in Section 2, also we indicate how Carleson measures can be used to characterize boundedness of C_ϕ on $F(p,\alpha p-2,s)$, and we give reverse Carleson type conditions for a composition operator to be closed range on $F(p,\alpha p-2,s)$.

In Section 3, we concetrate on the spaces $F(p,q,s)$ for $p>2$, $q>-2$ and $s>0$ with $q+s>-1$. Let ϕ be a boundedly valent holomorphic self map of $\mathbb{D}$; we give reverse Carleson type conditions for a composition operator to be closed range on $F(p,q,s)$. We will show that, assuming that C_ϕ is bounded on $F(p,q,s)$, if C_ϕ is closed range on the Bloch space, then it is also closed range on $F(p,q,s)$. Moreover we will show that if ϕ is a boundedly valent holomorphic self map of $\mathbb{D}$ then C_ϕ is closed range on $F(p,q,s)$ if and only if it is closed range on Bloch space.

In Section 4, it has results on $F(p,\alpha p-2,s)$ spaces. Let ϕ be a boundedly valent holomorphic self map of $\mathbb{D}$; again we give reverse Carleson type conditions for a composition operator to be closed range on $F(p,\alpha p-2,s)$. Moreover assuming that C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$, if $\phi(\mathbb{D})$ or certain subsets of it contain an outer annulus then C_ϕ is closed range on $F(p,\alpha p-2,s)$.

Let C and K denote a positive and finite constants which may change from one occurrence to the next but will not depend on the functions involved.

Two quantities A_f and B_f, both depending on $f\in\mathcal{H}(\mathbb{D})$, are said to be equivalent, written as $A_f\approx B_f$, if there exists $C>0$ such that $\frac{1}{C}B_f\le A_f\le CB_f$, for every function $f\in\mathcal{H}(\mathbb{D})$. If the quantities A_f and B_f, are equivalent, then in particular we have $A_f<\infty$ if and only if $B_f<\infty$.

2 Prerequisites

This introductory section is dedicated to setting up the notation and introducing the main concepts along with a collection of some fundamental facts required for what is to follow.

- The symbol $\mathbb{D}=\{z:|z|<1\}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ the unit circle.
- The symbol $\mathcal{H}(\mathbb{D})$ denote the family of functions holomorphic on $\mathbb{D}$.
- H^2 the Hilbert space of $\mathcal{H}(\mathbb{D})$ with square summable power series coefficients.
- The symbol A denote two-dimensional Lebesgue measure on $\mathbb{D}$, so that $A(\mathbb{D})\equiv 1$.
- For $a\in\mathbb{D}$ the Möbius transformations $\varphi_a(z)$ is defined by

$$\varphi_a(z)=\frac{a-z}{1-\bar{a}z}\quad \text{for } z\in\mathbb{D}.$$

- For $a\in\mathbb{D}$ and $r\in(0,1)$, $\mathbb{D}(a,r)$ the pseudo-hyperbolic disc is defined by

$$\mathbb{D}(a,r)=\{z\in\mathbb{D}:|\varphi_a(z)|<r\}.$$

  Let $A(\mathbb{D}(a,r))$ denote its Lebesgue area measure. Then, there exist constants C_r depending only on r such that

$$\frac{(1-|a|)^2}{C_r}\le A(\mathbb{D}(a,r))\le C_r(1-|a|)^2. \tag{2.1}$$

- The Green's function for $\mathbb{D}$ with logarithmic singularity at $a\in\mathbb{D}$ is defined by

$$g(z,a)=\log\frac{1}{|\varphi_a(z)|}.$$

The following identity is easily verified:

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}=(1-|z|^2)|\varphi_a'(z)|. \tag{2.2}$$

Note that the Möbius transformation of $\mathbb{D}$, satisfying $\varphi_a(0)=a$, $\varphi_a(a)=0$, $\varphi_a(\varphi_a(z))=z$ and thus $\varphi_a^{-1}(z)=\varphi_a(z)$.

For $0<\alpha<\infty$, the α-Bloch space $\mathcal{B}^\alpha$ is the space of functions $f\in\mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha}=\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|^2)^\alpha<\infty.$$

The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$ (see [22]).

Let $1<p<\infty$, $-1<\alpha<\infty$, the α-Besov space $B_{p,\alpha}$ is the space of functions $f\in\mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{B_{p,\alpha}}=\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^\alpha dA(z)<\infty.$$

If $\alpha=p-2$ then $B_{p,p-2}$ is the Besov space B_p (see [21]).

For $0<p,s<\infty$, $-2<q<\infty$, the general family function spaces $F(p,q,s)$ is defined as the set of all functions $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{F(p,q,s)}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)<\infty.$$

The family of spaces $F(p,q,s)$ was introduced by Zhao in [19] and Rättyä in [14]. For that $p\ge 1$, the spaces $F(p,q,s)$ are Banach spaces under the norm

$$\|f\|_{b_{p,q,s}}=\|f\|_{F(p,q,s)}+|f(0)|.$$

Moreover, the spaces $F(p,q,s)$ are $\frac{q+2}{p}$−Möbius invariant spaces (see [19, Proposition 4.3]). It contains, as special cases, many classical function spaces, such as the analytic Besov spaces B_p, weighted Bergman spaces, weighted Dirichlet spaces D_q, the α-Bloch spaces, $BMOA$ (the space of analytic functions of bounded mean oscillation) and the recently introduced Q_s spaces, the Q_s spaces, introduced in [5].

For convenience, we will write $q=\alpha p-2$, $F(p,q,s)$ is the α-Besove type space $F(p,\alpha p-2,s)$, where $\alpha>0$. It is known that, $F(p,q,s)$ contains only constant functions if $q+s\le -1$ or $\alpha p+s\le 1$ when $q=\alpha p-2$ (see [19, Proposition 2.12]). Therefore, later on we will assume that $\alpha p+s>1$, which guarantees that $F(p,\alpha p-2,s)$ is nontrivial. Among these $F(p,\alpha p-2,s)$ spaces, the case $\alpha=1$ is particularly interesting, since the spaces $F(p,p-2,s)$ are Möbius invariant, in the sense that for any function $f\in F(p,p-2,s)$ and any $a\in\mathbb{D}$ one has

$$\|f\circ\varphi_a\|_{F(p,p-2,s)}=\|f\|_{F(p,p-2,s)}.$$

It is known from [19] and [14], that the spaces $F(p,\alpha p-2,s)$ satisfy the following:

(1) $F(p,\alpha p-2,s)\subset\mathcal{B}^\alpha$ and $F(p,\alpha p-2,s)=\mathcal{B}^\alpha$, for $s>1$.
(2) $F(p,\alpha p-2,0)=B_{p,\alpha}$, for $1<p<\infty$.
(3) $F(2,0,s)=Q_s$ and $F(2,0,1)=BMOA$.
(4) $F(2,1,0)=H^2$ and $F(2,q,0)=D_q$.

The weighted Bergman space A^p_q consisting of those $f\in\mathcal{H}(\mathbb{D})$ for which

$$\|f\|^p_{A^p_q}=\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^q<\infty.$$

A function $f\in\mathcal{H}(\mathbb{D})$ is said to belong to $A^p_{q,s}$ (see [23]) if

$$\|f\|^p_{A^p_{q,s}}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)<\infty.$$

We note that by the derivative operator $f\mapsto f'$, $F(p,q,s)$ spaces are closely related to $A^p_{q,s}$ and $F(p,\alpha p-2,s)$ related to $A^p_{\alpha p-2,s}$.

Since $F(p,\alpha p-2,s)\subset\mathcal{B}^\alpha$, for $f\in F(p,\alpha p-2,s)$ one has (see [17, Lemma 2])

$$|f'(z)|\le C\frac{\|f\|^p_{F(p,\alpha p-2,s)}}{(1-|z|^2)^\alpha}. \tag{2.3}$$

Zhu in [20, Theorem 4.14] and [21, Theorem 9] gives the growth of any function in a Bergman space and the Besov space B_p respectively. Moreover, Tjani in [16] gives the growth of a function on any Besov type space $B_{p,\alpha}$. Below we describe in a similar fashion the growth of a function on $F(p,\alpha p-2,s)$ spaces. By Proposition 4.27 in [20] and Lemma 2 in [17], for any $f\in\mathcal{H}(\mathbb{D})$, we have

Lemma 2.1. *Therefor if p' is so that $\frac{1}{p}+\frac{1}{p'}=1$, for all $0<p,\alpha<\infty$ and $0<s<\infty$, with $\alpha p+s>1$. If $f\in F(p,\alpha p-2,s)$, then*

$$|f(z)-f(0)|\le C\begin{cases}\|f\|_{F(p,\alpha p-2,s)}, & \text{if } 0<\alpha<1,\\ \|f\|_{F(p,\alpha p-2,s)}\left(\log\frac{1}{1-|z|^2}\right)^{\frac{1}{p'}}, & \text{if } \alpha=0,\\ \|f\|_{F(p,\alpha p-2,s)}\frac{1}{(1-|z|^2)^{\alpha}}, & \text{if } \alpha>1.\end{cases}$$

The following lemma proved by Zhao (see [18, Theorem 1]):

Lemma 2.2. *Let $0<\alpha<\infty$, $0<r<1$, $0<p<\infty$ and $1<s<\infty$. Then, for an analytic function f in $\mathbb{D}$, the following quantities are equivalent:*

$$(A)\ \ \|f\|_{\mathcal{B}^\alpha},$$
$$(B)\ \ \sup_{a\in\mathbb{D}}\int_{\mathbb{D}(a,r)}|f'(z)|^p(1-|z|^2)^{p\alpha-2}dA(z),$$
$$(C)\ \ \sup_{a\in\mathbb{D}}\int_{\mathbb{D}(a,r)}|f'(z)|^p(1-|z|^2)^{p\alpha-2}(1-|\varphi_a(z)|^2)^s dA(z).$$

3 Carleson measures and composition operators on $F(p,\alpha p-2,s)$

In this section we consider the α-Besov type spaces $F(p,\alpha p-2,s)$ and we give conditions that guarantee that C_ϕ is a bounded operator on some spaces $F(p,\alpha p-2,s)$. But composition operators are not always bounded on all spaces $F(p,\alpha p-2,s)$. And we indicate how Carleson measures can be used to characterize boundedness and compactness of C_ϕ on $F(p,\alpha p-2,s)$.

Now, let $0<h<1$ and $0\le\theta<2\pi$. The Carleson type set $S(h,\theta)$ is:

$$S(h,\theta)=\{z\in\mathbb{D}:|z-e^{i\theta}|<h\}.$$

A positive measure μ on $\mathbb{D}$ is a Carleson measure if there is a constant C with

$$\mu(S(h,\theta))\le Ch,\quad \text{where}\ \ 0<h<1,\quad \text{and}\quad 0\le\theta<2\pi.$$

For $1<p<\infty$, μ is called a vanishing p-Carleson measure if

$$\lim_{h\to 0}\sup_{\theta\in[0,2\pi)}\frac{\mu(S(h,\theta))}{h^p}=0.$$

Definition 3.1. Let μ be a positive measure on $\mathbb{D}$, and let $0<p,\alpha<\infty$ and $0<s<\infty$. Then μ is an $(F(p,\alpha p-2,s);p)$-Carleson measure if there is a constant $K>0$, so that

$$\int_{\mathbb{D}}|f'(w)|^p d\mu(w)\le K\|f\|^p_{F(p,\alpha p-2,s)},$$

for all $f\in F(p,\alpha p-2,s)$.

Akeroyd and Fulmer in [1], Akeroyd and Ghatage in [2] were able to obtain a characterization of closed range composition operators on the Bergman space A^2 without the use of Nevanlinna type counting functions. But as Zorboska showed in [23] a full characterization of closed range of composition operators on the Hardy space requires the Nevanlinna counting function; similarly Tjani showed in [16] that Nevanlinna counting functions are needed for the characterization of closed range composition operators on Besov type spaces.

In order for us to obtain a full characterization of when C_ϕ is closed range on $F(p,\alpha p-2,s)$ we will have to use Nevanlinna type counting functions:

Definition 3.2. The counting function for the $F(p,\alpha p-2,s)$ spaces is

$$N_{p,\alpha,s}(\phi,w)=\sum_{\phi(z)=w}\{|\phi'(z)|^{p-2}(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s\}, \tag{3.1}$$

for $w\in\phi(\mathbb{D})$, $2\le p<\infty$, $0<\alpha<\infty$ and $0<s<\infty$.

The above counting functions come up in the change of variables formula in the respective spaces as follows:

For $f\in F(p,\alpha p-2,s)$, $2\le p<\infty$, $0<\alpha<\infty$, $0<s<\infty$ and $\alpha p+s>1$,

$$\begin{aligned}\|C_\phi f\|^p_{F(p,\alpha p-2,s)}&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z).\end{aligned} \tag{3.2}$$

By making a non-univalent change of variables as done in [8,15] we see that

$$\|C_\phi f\|^p_{F(p,\alpha p-2,s)}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^p N_{p,\alpha,s}(\phi,w)dA(w). \tag{3.3}$$

Let $a\in\mathbb{D}$, $f\in F(p,\alpha p-2,s)$. We know that for every $z\in\mathbb{D}$, $1-|\varphi_a(z)|^2\le 2g^s(z,a)$ (see [5]). Then

$$\begin{aligned}\|C_{\varphi_a} f\|^p_{F(p,\alpha p-2,s)}&=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(\varphi_a(z))|^p|\varphi_a'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\&\le 2^s\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(\varphi_a(z))|^p|\varphi_a'(z)|^p(1-|z|^2)^{\alpha p-2}g^s(z,a)dA(z)\\&\le 2^s\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^{\alpha p-2}|\varphi_a'(w)|^{\alpha p}\left(\log\frac{1}{|w|}\right)^s dA(w)\\&\le 2^s\|f\|^p_{\mathcal{B}^\alpha}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|\varphi_a'(w)|^{\alpha p}(1-|w|^2)^{-2}\left(\log\frac{1}{|w|}\right)^s dA(w)\\&\le 2^s\|f\|^p_{F(p,\alpha p-2,s)}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|\varphi_a'(w)|^{\alpha p}(1-|w|^2)^{-2}\left(\log\frac{1}{|w|}\right)^s dA(w).\end{aligned}$$

Recall that

$$\int_{\mathbb{D}}(1-|w|^2)^{-2}\left(\log\frac{1}{|w|}\right)^s dA(w)=C\int_0^1(1-r^2)^{-2}\left(\log\frac{1}{r}\right)^s dr\int_{\partial\mathbb{D}}dA(\zeta)\le C.$$

Thus by Lemma 2.1:

- If $0<\alpha<1$ then

$$\|C_{\varphi_a}f\|_{b_{p,q,s}}\le|f(a)|+\left(\frac{1-|a|}{1+|a|}\right)C\|f\|_{b_{p,q,s}}-\left(\frac{1-|a|}{1+|a|}\right)|f(0)|\le C\|f\|_{b_{p,q,s}}.$$

- If $\alpha=1$ for all p', $\frac{1}{p}+\frac{1}{p'}=1$, then

$$\|C_{\varphi_a}f\|_{b_{p,q,s}}\le|f(a)|+C\|f\|_{b_{p,q,s}}-|f(0)|\le C\left[\frac{1}{C}+\left(\log\frac{1}{1-|a|^2}\right)^{\frac{1}{p'}}\right]\|f\|_{b_{p,q,s}}.$$

- If $\alpha>1$ then

$$\begin{aligned}\|C_{\varphi_a}f\|_{b_{p,q,s}}\le&|f(a)|+\left(\frac{1-|a|}{1+|a|}\right)(C\|f\|_{b_{p,q,s}})-\left(\frac{1-|a|}{1+|a|}\right)|f(0)|\\ \le&\left[1+\frac{C}{(1-|a|^2)^\alpha}+\left(\frac{1-|a|}{1+|a|}\right)\right]\|f\|_{b_{p,q,s}}.\end{aligned}$$

Remark 3.1. For each $a\in\mathbb{D}$, C_{φ_a} is a bounded linear onto operator on $F(p,\alpha p-2,s)$. Therefore $C_{\varphi_a\circ\phi}=C_\phi C_{\varphi_a}$ is closed range if and only if C_ϕ is closed range. Let $b_{p,\alpha,s}:=\{f\in F(p,\alpha p-2,s):f(0)=0\}$. If $\phi(0)=0$ then $b_{p,\alpha,s}$ is an invariant subspace of $F(p,\alpha p-2,s)$. Moreover since $F(p,\alpha p-2,s)$ is the direct sum of $b_{p,\alpha,s}$ and a one dimensional subspace, C_ϕ is closed range on $b_{p,\alpha,s}$ if and only if it is closed range on $F(p,\alpha p-2,s)$. If $\phi(0)\neq0$, let $\psi=\varphi_{\phi(0)}\circ\phi$. Then $\psi(0)=0$ and $C_\psi=C_\phi C_{\varphi_{\phi(0)}}$ is closed range on $F(p,\alpha p-2,s)$ if and only if C_ϕ is closed range on $b_{p,\alpha,s}$. It follows that in order to prove that C_ϕ is closed range on $F(p,\alpha p-2,s)$, for any $p>1$ and $0<\alpha<\infty$, we may assume that $\phi(0)=0$ and show that C_ϕ is closed range on $b_{p,\alpha,s}$. Moreover if C_ϕ is closed range on $F(p,\alpha p-2,s)$ then, since we may assume that $\phi(0)=0$, C_ϕ is closed range on $b_{p,\alpha,s}$.

Now we consider the restriction of C_ϕ to $F(p,\alpha p-2,s)$. Then C_ϕ is a bounded operator if and only if there is a positive constant K such that

$$\|C_\phi f\|^p_{F(p,\alpha p-2,s)}\le K\|f\|^p_{F(p,\alpha p-2,s)} \tag{3.4}$$

for all $f\in F(p,\alpha p-2,s)$ or, equivalently by (3.3),

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^pN_{p,\alpha,s}(\phi,w)dA(w)\le K\|f\|^p_{F(p,\alpha p-2,s)}$$

for all $f\in F(p,\alpha p-2,s)$.

In view of (3.4) above we see that C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$ if and only if the measure $N_{p,\alpha,s}(\phi,w)dA(w)$ is a $(F(p,\alpha p-2,s),p)$-Carleson measure. Now we give a characterization of the compact composition operator on $F(p,\alpha p-2,s)$ spaces in terms of p-Carleson measures.

Theorem 3.1. *Let $0<p,\alpha<\infty$ and $1<s<\infty$. The following are equivalent:*

(1) μ is a $(F(p,\alpha p-2,s),p)-$Carleson measure,

(2) There is a constant $K>0$ such that $\mu(S(h,\theta))\le Kh^{\alpha p}$ for all $h\in(0,1)$, and all $\theta\in[0,2\pi)$,

(3) There is a constant $C>0$, for all $a\in\mathbb{D}$, such that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|\varphi_a'(z)|^p d\mu(z)\le C.$$

The implication (2) $\Leftrightarrow$ (3) is proved by Arazy, Fisher and Peetre (see [4, Theorem 13]). The equivalence of (1) and (2) was given by El-Sayed Ahmed and Bakhit (see [8, Theorem 2.1]).

4 Reverse Carleson inequality for $F(p,q,s)$

Other than Cima, Thomson and Wogen's paper [7], the very first paper on closed range composition operators, all subsequent papers on the subject use a reverse type Carleson condition due to Luecking, see [11] and [12]. It is a condition on a subset of the unit disk that controls how large and how small is the intersection of the set with pseudohyperbolic disks or with Carleson type sets.

Definition 4.1. A set $H\subset\mathbb{D}$ satisfies the reverse Carleson condition if there is a constant $K>0$ so that $A(H\cap S(h,\theta))\ge Kh^2$ for all $0\le\theta<2\pi$ and $0<h<1$.

Luecking in [12] showed that $H\subset\mathbb{D}$ satisfies the reverse Carleson condition if and only if

$$\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\alpha dA(z)\le C\int_H|f(z)|^p(1-|z|^2)^\alpha dA(z),$$

for all $f\in A^p_\alpha$, $\alpha>-1$. Moreover he showed that pseudohyperbolic disks can be used in place of the Carleson type sets $S(h,\theta)$.

The main theorem on [12] can be extended to some (β,s)-weighted Bergman type spaces without much difficulty. In particular we have the following theorem.

Theorem 4.1. *Let $0<p,s<\infty$, $\beta>-1$ and $H\subset\mathbb{D}$. Then the following are equivalent.*

(1) For $f\in A^p_{\beta,s}$, there is a constant $C>0$ such that

$$\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\beta g^s(z,a)dA(z)\le C\int_H|f(z)|^p(1-|z|^2)^\beta g^s(z,a)dA(z).$$

(2) For all $\mathbb{D}(a,r)$ and $a\in\mathbb{D}$, there is a constant $K>0$ such that

$$A(H\cap\mathbb{D}(a,r))>KA(H\cap\mathbb{D}).$$

Proof. For the implication (1) ⇒ (2). Let proved by Luecking in [12].

For $z=re^{i\theta}$, let $E_1(w)=\{z:|z-w|<1-|w|\}$. Let $\chi_{E_1(w)}$ be the characteristic function of $E_1(w)$ evaluated at z, we can write

$$\chi_{E_1(w)}(z)\le\chi_{\mathbb{D}(a,r)}(z).$$

The crucial properties of the weight function $\Psi(z)=(1-|z|^2)^\beta(-\log|z|)^s$ are the following:

$$\Psi(a)\le C\inf\{\Psi(z):z\in\mathbb{D}(a,r)\}$$

and

$$\begin{aligned}\Psi(\varphi_a(z))&=(1-|\varphi_a(z)|^2)^\beta(-\log|\varphi_a(z)|)^s\\&=\left(\frac{(1-|a|^2)}{|1-\bar{a}z|^2}\right)^\beta(1-|z|^2)^\beta g^s(z,a).\end{aligned}$$

It implies that

$$\begin{aligned}&\int_G|f(w)|^p\Psi(w)dA(w)\\\le&C\int_{\mathbb{D}}|f(z)|^p\Psi(z)\left(\int_G\frac{1}{A(\mathbb{D}(a,r))}\chi_{\mathbb{D}(a,r)}(z)dA(w)\right)dA(z),\end{aligned}$$

where G is the set defined as:

$$G=\left\{w\in\mathbb{D}:|f(w)|<\frac{\varepsilon}{A(E_1(w))}\int_{E_1(w)}|f|dA\right\},\quad\varepsilon>0.$$

As in [12], we have that

$$\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\beta g^s(z,a)dA(z)\le C\int_H|f(z)|^p(1-|z|^2)^\beta g^s(z,a)dA(z).$$

This completes the proof of (2) ⇒ (1). □

If we let $\beta=\alpha p-2$ in Theorem 4.1, we have that $H\subset\mathbb{D}$ satisfies the reverse Carleson condition if and only if

$$\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^{\alpha p-2}g^s(z,a)dA(z)\le C\int_H|f(z)|^p(1-|z|^2)^{\alpha p-2}g^s(z,a)dA(z),$$

for all $f\in A^p_{\alpha p-2,s}$, $\alpha p+s>1$.

Moreover if $\beta=q$, Theorem 4.1 show that $H\subset\mathbb{D}$ satisfies the reverse Carleson condition if and only if

$$\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\le C\int_H|f(z)|^p(1-|z|^2)^q g^s(z,a)dA(z),$$

for all $f\in A^p_{q,s}$, $q+s>-1$.

Let $2<p<\infty$, $-2<q<\infty$ and $0<s<\infty$. Since $\{f':f\in b_{p,q,s}\}=A^p_{q,s}$, a set $H\subset\mathbb{D}$ satisfies the reverse Carleson condition if and only if there exists a positive constant C so that for all $f\in F(p,q,s)$

$$\int_H|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\ge C\|f\|^p_{F(p,q,s)}.$$

Let $2<p<\infty$, $-2<q<\infty$ and $0<s<\infty$. For any holomorphic self-map ϕ of the unit disk $\mathbb{D}$ define $\tau_{p,q,s}(\phi,z,a)$ on $\mathbb{D}$ by

$$\tau_{p,q,s}(\phi,z,a)=\frac{|\phi'(z)|(1-|z|^2)^{\frac{q}{p-2}}(1-|\varphi_a(z)|^2)^{\frac{s}{p-2}}}{(1-|\phi(z)|^2)^{\frac{q}{p-2}}(1-|\varphi_a(\phi(z))|^2)^{\frac{s}{p-2}}}. \tag{4.1}$$

For $\epsilon>0$, we write $G_\epsilon(\phi)=\phi(D_\epsilon)$, where $D_\epsilon=\{z\in\mathbb{D}:|\tau_{p,q,s}(\phi,z,a)|>\epsilon\}$. If $\varphi_a(0)=a$ and $\psi=\varphi_a\circ\phi$ then

$$\tau_{p,q,s}(\psi,z,a)=\tau_{p,q,s}(\phi,z,a),$$

and $G_\epsilon(\phi)$ satisfies the reverse Carleson condition if and only if $G_{\epsilon^*}(\psi)$ does, for some $\epsilon^*>0$.

Proposition 4.1. Let $2<p<\infty$, $-2<q<\infty$, $0<s<\infty$ and let ϕ be any holomorphic self-map of $\mathbb{D}$. Assume that $C_\phi:F(p,q,s)\to F(p,q,s)$ is a bounded operator. If there exists an $\epsilon>0$ so that $G_\epsilon(\phi)$ satisfies the reverse Carleson condition then $C_\phi:F(p,q,s)\to F(p,q,s)$ is closed range.

Proof. First suppose that $\phi(0)=0$. Let $f\in F(p,q,s)$ and $f(0)=0$. Then

$$\begin{aligned}\|f\circ\phi\|^p_{F(p,q,s)}\ge&\int_{D_\epsilon}|f'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\ \ge&\epsilon^{p-2}\int_{D_\epsilon}|f'(\phi(z))|^p(1-|\phi(z)|^2)^q(1-|\varphi_a(\phi(z))|^2)^s|\phi'(z)|^2dA(z).\end{aligned}$$

By making a non-univalent change of variables as done in [15, page 186], we have

$$\|f\circ\phi\|^p_{F(p,q,s)}\geq \epsilon^{p-2}\int_{G_\epsilon(\phi)}|f'(w)|^p(1-|w|^2)^q(1-|\varphi_a(w)|^2)^s N_{2,0,0}(\phi,w)dA(w).$$

Since $G_\epsilon(\phi)$ satisfies the reverse Carleson condition and using the obvious fact that for $w\in G_\epsilon(\phi)$, $N_{2,0,0}(\phi,w)\geq 1$, we see that

$$\begin{aligned}\|f\circ\phi\|^p_{F(p,q,s)}\geq&\epsilon^{p-2}\int_{G_\epsilon(\phi)}|f'(w)|^p(1-|w|^2)^q(1-|\varphi_a(w)|^2)^s dA(w)\\ \geq& C\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^q(1-|\varphi_a(w)|^2)^s dA(w). \end{aligned}\tag{4.2}$$

Therefore C_ϕ is closed range on $b_{p,q,s}$ and by Remark 3.1, C_ϕ is also closed range on $F(p,q,s)$. □

Theorem 4.2. *Let $2<p<\infty$, $-2<q<\infty$, $0<s<\infty$ and let ϕ be any holomorphic self-map of $\mathbb{D}$. Then $C_\phi:F(p,q,s)\to F(p,q,s)$ is closed range if and only if there exists an $\epsilon>0$ so that $G_\epsilon(\phi)$ satisfies the reverse Carleson condition.*

Proof. First we may assume that $\phi(0)=0$, and $C_\phi:F(p,q,s)\to F(p,q,s)$ is closed range. If there does not exist an $\epsilon>0$ such that $G_\epsilon(\phi)$ satisfies the reverse Carleson condition then we can find a sequence (f_k) in $F(p,q,s)$ with $f_k(0)=0$ and $\|f_k\|_{F(p,q,s)}=1$ for all k, and

$$\lim_{k\to\infty}\int_{G_{\frac{1}{k}}(\phi)}|f_k'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)=0. \tag{4.3}$$

By (3.2) we have

$$\begin{aligned}\|f_k\circ\phi\|^p_{F(p,q,s)}=&\sup_{a\in\mathbb{D}}\int_{D_{\frac{1}{k}}}|f_k'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\ &+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}-D_{\frac{1}{k}}}|f_k'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\ =&I_1+I_2.\end{aligned}$$

Thus by making a non-univalent change of variables as done in [15] and the Schwarz-Pick Lemma, since ϕ is boundedly valent, we obtain that

$$\begin{aligned}I_1=&\sup_{a\in\mathbb{D}}\int_{D_{\frac{1}{k}}}|f_k'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\ \leq&\sup_{a\in\mathbb{D}}\int_{D_{\frac{1}{k}}}|f'(\phi(z))|^p(1-|\phi(z)|^2)^q(1-|\varphi_a(\phi(z))|^2)^s|\phi'(z)|^2 dA(z)\\ =&\sup_{a\in\mathbb{D}}\int_{G_{\frac{1}{k}}(\phi)}|f'(w)|^p(1-|w|^2)^q(1-|\varphi_a(w)|^2)^s N_{2,0,0}(\phi,w)dA(w)\\ \leq& C\sup_{a\in\mathbb{D}}\int_{G_{\frac{1}{k}}(\phi)}|f'(w)|^p(1-|w|^2)^q(1-|\varphi_a(w)|^2)^s dA(w).\end{aligned}$$

Thus by (4.3) we get

$$\lim_{k\to\infty} I_1 = 0. \tag{4.4}$$

We also observe that

$$\begin{aligned}I_2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}-D_{\frac{1}{k}}} |f_k'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z)\\ &\le \frac{1}{k^{p-2}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k'(\phi(z))|^2 |\phi'(z)|^p (1-|\phi(z)|^2)^q (1-|\varphi_a(\phi(z))|^2)^s dA(z).\end{aligned}$$

Thus another appeal to the non-univalent change of variables as done in [15] and since ϕ is boundedly valent

$$\begin{aligned}I_2 &\le C\frac{1}{k^{p-2}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^q (1-|\varphi_a(w)|^2)^s dA(w)\\ &\le C\frac{1}{k^{p-2}},\end{aligned}$$

and

$$\lim_{k\to\infty} I_2 = 0. \tag{4.5}$$

Therefore by (4.4) and (4.5) we see that $\|f_k\circ\phi\|^p_{F(p,q,s)} \to 0$, as $k\to\infty$, for some sequence $(f_k)\in F(p,q,s)$ with $f_k(0)=0$ and $\|f_k\|_{F(p,q,s)}=1$.

This contradicts that C_ϕ is closed range on $F(p,q,s)$ and one direction follows. The converse follows by Proposition 4.1. □

In [3] Akeroyd, Ghatage and Tjani proved that C_ϕ is closed range on Bloch space if and only if there exists an $\epsilon>0$ so that $G_\epsilon(\phi)$ satisfies the reverse Carleson condition. In [16] Tjani proved that C_ϕ is closed range on Bloch space $\mathcal{B}$ then it is also closed range in Besov space $B_p, p>2$. Thus by Proposition 4.1 if C_ϕ is closed range on Bloch space $\mathcal{B}$ then it is also closed range on $F(p,q,s)$, $p>2$. Jovovic and MacCluer proved in [10] that if C_ϕ is bounded on Dirichlet space D and $\phi(\mathbb{D})$ satisfies the reverse Carleson condition then C_ϕ is closed range on D. It is clear that if for some $\epsilon>0$, $G_\epsilon(\phi)$ satisfies the reverse Carleson condition then so does $\phi(\mathbb{D})$. Thus by Proposition 4.1, we obtain the following corollary:

Corollary 4.1. For $2<p<\infty$, $-2<q<\infty$, $0<s<\infty$, suppose that C_ϕ is a bounded operator on $F(p,q,s)$. If C_ϕ is closed range on Bloch space $\mathcal{B}$ then it is also closed range on $F(p,q,s)$.

Moreover by Theorem 4.2 we obtain the following corollary:

Corollary 4.2. Let $2<p<\infty$, $-2<q<\infty$, $0<s<\infty$, and ϕ be a boundedly valen holomorphic self-map of $\mathbb{D}$. Then C_ϕ is closed range on Bloch space $\mathcal{B}$ if and only if it is also closed range on $F(p,q,s)$.

5 Reverse Carleson inequality for $F(p,\alpha p-2,s)$

Let $2<p<\infty$, $0<\alpha<\infty$ and $0<s<\infty$. Since $\{f': f\in b_{p,\alpha,s}\}=A^p_{\alpha p-2,s}$, a set $H\subset\mathbb{D}$ satisfies the reverse Carleson condition if and only if there exists a positive constant C so that for all $f\in F(p,\alpha p-2,s)$

$$\int_H |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\geq C\|f\|^p_{F(p,\alpha p-2,s)}.$$

For any holomorphic self-map ϕ of the unit disk $\mathbb{D}$ define $\tau_{p,\alpha,s}(\phi,z,a)$ on $\mathbb{D}$ by

$$\tau_{p,\alpha,s}(\phi,z,a)=\frac{|\phi'(z)|(1-|z|^2)^{\frac{\alpha p-2}{p-2}}(1-|\varphi_a(z)|^2)^{\frac{s}{p-2}}}{(1-|\phi(z)|^2)^{\frac{\alpha p-2}{p-2}}(1-|\varphi_a(\phi(z))|^2)^{\frac{s}{p-2}}}. \tag{5.1}$$

For $\epsilon>0$, we write $H_\epsilon(\phi)=\phi(\Delta_\epsilon)$, where $\Delta_\epsilon=\{z\in\mathbb{D}: |\tau_{p,\alpha,s}(\phi,z,a)|>\epsilon\}$. If $\varphi_a(0)=a$ and $\psi=\varphi_a\circ\phi$ then

$$\tau_{p,\alpha,s}(\psi,z,a)=\tau_{p,\alpha,s}(\phi,z,a),$$

and $H_\epsilon(\phi)$ satisfies the reverse Carleson condition if and only if $H_{\epsilon^*}(\psi)$ does, for $\epsilon^*>0$.

Proposition 5.1. Let $2<p<\infty$, $0<\alpha,s<\infty$ with $\alpha p+s>1$ and let ϕ be any holomorphic self-map of $\mathbb{D}$. Assume that $C_\phi: F(p,\alpha p-2,s)\to F(p,\alpha p-2,s)$ is a bounded operator. If there exists an $\epsilon>0$ so that $H_\epsilon(\phi)$ satisfies the reverse Carleson condition then $C_\phi: F(p,\alpha p-2,s)\to F(p,\alpha p-2,s)$ is closed range.

Proof. The proof follows the steps of the proof of Proposition 4.1: change the powers in the integrands from q to $\alpha p-2$, change $G_\epsilon(\phi)$ to $H_\epsilon(\phi)$; finally change D_ϵ with Δ_ϵ. □

Theorem 5.1. *Let $2<p<\infty$, $0<\alpha,s<\infty$ with $\alpha p+s>1$ and let ϕ be any holomorphic self-map of $\mathbb{D}$. Then $C_\phi: F(p,\alpha p-2,s)\to F(p,\alpha p-2,s)$ is closed range if and only if there exists an $\epsilon>0$ so that $H_\epsilon(\phi)$ satisfies the reverse Carleson condition.*

Proof. The proof follows the steps of the proof of Theorem 4.2: change the powers in the integrands from q to $\alpha p-2$, change $G_\epsilon(\phi)$ to $H_\epsilon(\phi)$; finally change D_ϵ with Δ_ϵ. □

Thus by Proposition 5.1 and Theorem 5.1 we obtain the following corollary:

Corollary 5.1. For $2<p<\infty$, $0<\alpha,s<\infty$ with $\alpha p+s>1$. Suppose that C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$. If C_ϕ is closed range on α-Bloch space $\mathcal{B}^\alpha$ then it is also closed range on $F(p,\alpha p-2,s)$.

Corollary 5.2. For $2<p<\infty$, $0<\alpha,s<\infty$ with $\alpha p+s>1$ and ϕ be a boundedly valen holomorphic self-map of $\mathbb{D}$. Then C_ϕ is closed range on α-Bloch space $\mathcal{B}^\alpha$ if and only if it is also closed range on $F(p,\alpha p-2,s)$.

As was mentioned in the introduction Nevanlinna type counting functions will have to be used for the case where ϕ is not boundedly valent. Let $2<p<\infty$, $0<\alpha<\infty$. Recall the counting function for $F(p,\alpha p-2,s)$:

$$N_{p,\alpha,s}(\phi,w)=\sum_{\phi(z)=w}\{|\phi'(z)|^{p-2}(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s\}.$$

Proposition 5.2. For $2<p<\infty$, $0\le s,\alpha<\infty$ with $\alpha p+s>1$. Let C_ϕ be a bounded operator on $F(p,\alpha p-2,s)$. Then C_ϕ is closed range on $F(p,\alpha p-2,s)$ if and only if there exists an $\epsilon>0$ so that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(w)|^pN_{p,\alpha,s}(\phi,w)dA(w)\ge\epsilon\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(w)|^2)^s dA(z) \quad (5.2)$$

for all $f\in A^p_{\alpha p-2,s}$.

Proof. By Remark 3.1 without loss of generality we may assume that $\phi(0)=0$. Then C_ϕ is closed range on $F(p,\alpha p-2,s)$ if and only if C_ϕ is closed range on $b_{p,\alpha,s}$. By (3.3) this is equivalent to

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^pN_{p,\alpha,s}(\phi,w)dA(w)\ge\epsilon\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)$$

for all $f\in b_{p,q,s}$. Since $\{f':f\in b_{p,q,s}\}=A^p_{\alpha,s}$ the conclusion follows. □

Given $2<p<\infty$, $0\le s,\alpha<\infty$ with $\alpha p+s>1$. Let

$$D_\epsilon(p,\alpha,s;a)=\left\{w\in\mathbb{D}:\frac{N_{p,\alpha,s}(\phi,w)}{(1-|w|^2)^{\alpha p-2}(1-|\varphi_a(w)|^2)^s}>\epsilon\right\}.$$

Note that if ϕ is univalent then $D_\epsilon(p,\alpha,s;a)=G_\epsilon$.

Proposition 5.3. Let $2<p<\infty$, $0\le s,\alpha<\infty$ with $\alpha p+s>1$. For some $r<1$, $\epsilon>0$ and $D_\epsilon(p,\alpha,s;a)$ contains an annulus of the form $\{w:r<|w|<1\}$. If C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$, then C_ϕ is closed range on $F(p,\alpha p-2,s)$.

Proof. For each $f\in A^p_{\alpha,s}$ and $\mathbb{D}_r=\{w\in\mathbb{D}:|w|>r\}$,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(w)|^pN_{p,\alpha,s}(\phi,w)dA(w)\\ \ge&\sup_{a\in\mathbb{D}}\int_{D_\epsilon(p,\alpha,s;a)}|f(w)|^pN_{p,\alpha,s}(\phi,w)dA(w)\\ \ge&\epsilon\sup_{a\in\mathbb{D}}\int_{D_\epsilon(p,\alpha,s;a)}|f(w)|^p(1-|w|^2)^{\alpha p-2}(1-|\varphi_a(w)|^2)^s dA(w)\\ \ge&\epsilon\sup_{a\in\mathbb{D}}\int_{\mathbb{D}_r}|f(w)|^p(1-|w|^2)^{\alpha p-2}(1-|\varphi_a(w)|^2)^s dA(w)\\ \ge&\epsilon C\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(w)|^p(1-|w|^2)^{\alpha p-2}(1-|\varphi_a(w)|^2)^s dA(w).\end{aligned}$$

By Proposition 5.2 the result follows. □

In the case $0<p\leq 2$, $0<\alpha\leq 1$ one can obtain the same conclusion with a weaker assumption. Note in particular that if $0<p<2$ it is valid for a composition operator C_ϕ on $F(p,q,s)$.

Proposition 5.4. Let $2<p<\infty$, $0\leq s,\alpha<\infty$ with $\alpha p+s>1$. For some $r<1$ and $\phi(\mathbb{D})$ contains an annulus of the form $\{w:r<|w|<1\}$. If C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$, then C_ϕ is closed range on $F(p,\alpha p-2,s)$.

Proof. Suppose that C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$. By Remark 3.1 we may assume that $\phi(0)=0$. By the Schwarz-Pick lemma and since $p\leq 2$

$$|\phi'(z)|^{p-2}(1-|z|^2)^{\alpha p-2}\geq(1-|\phi(z)|^2)^{p-2}(1-|z|^2)^{\alpha p-p}.$$

For $w\in\phi(\mathbb{D})$ we have

$$N_{p,\alpha,s}(\phi,w)\geq(1-|w|^2)^{p-2}N_{2,\alpha-\frac{p-2}{p},s}(\phi,w).$$

By Schwarz's lemma and since $0<\alpha\leq 1$ we conclude that for all $w\in\phi(\mathbb{D})$

$$N_{p,\alpha,s}(\phi,w)\geq(1-|w|^2)^{\alpha p-2}N_{2,\frac{p}{2},s}(\phi,w).$$

Thus for each $f\in A^p_{\alpha,s}$

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(w)|^pN_{p,\alpha,s}(\phi,w)dA(w)\\
\geq&\sup_{a\in\mathbb{D}}\int_{\phi(\mathbb{D})}|f(w)|^pN_{p,\alpha,s}(\phi,w)dA(w)\\
\geq&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}_r}|f(w)|^p(1-|w|^2)^{\alpha p-2}(1-|\varphi_a(w)|^2)^s dA(w)\\
\geq&C\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f(w)|^p(1-|w|^2)^{\alpha p-2}(1-|\varphi_a(w)|^2)^s dA(w).
\end{aligned}$$

By Proposition 5.2 the result follows. □

Proposition 5.5. Let $2<p<\infty$, $0\leq s,\alpha<\infty$ with $\alpha p+s>1$ and assume that C_ϕ is a bounded operator on $F(p,\alpha p-2,s)$. If there exists an $\epsilon>0$ so that $D_\epsilon(p,\alpha,s;a)$ satisfies the reverse Carleson condition, then C_ϕ is closed range on $F(p,\alpha p-2,s)$.

Proof. Suppose that there exists an $\epsilon>0$ so that $D_\epsilon(p,\alpha,s;a)$ satisfies the reverse Carleson condition. Then by Theorem 4.1, we have

$$\begin{aligned}
&\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\
\leq&C\int_{D_\epsilon(p,\alpha,s;a)}|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z).
\end{aligned}\tag{5.3}$$

Let $f\in F(p,\alpha p-2,s)$. Then

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^p N_{p,\alpha,s}(\phi,w)dA(w)\\
\geq&\sup_{a\in\mathbb{D}}\int_{D_\epsilon(p,\alpha,s;a)}|f'(w)|^p N_{p,\alpha,s}(\phi,w)dA(w)\\
\geq&\epsilon\sup_{a\in\mathbb{D}}\int_{D_\epsilon(p,\alpha,s;a)}|f'(w)|^p(1-|w|^2)^q(1-|\varphi_a(w)|^2)^s dA(w)\\
\geq&C\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^q(1-|\varphi_a(w)|^2)^s dA(w).
\end{aligned}$$

Thus, by the proof of Proposition 5.2, C_ϕ is closed range on $F(p,\alpha p-2,s)$. □

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