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Variable Hardy Spaces on the Heisenberg Group

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Abstract. We consider Hardy spaces with variable exponents defined by grand maximal function on the Heisenberg group. Then we introduce some equivalent characterizations of variable Hardy spaces. By using atomic decomposition and molecular decomposition we get the boundedness of singular integral operators on variable Hardy spaces. We investigate the Littlewood-Paley characterization by virtue of the boundedness of singular integral operators.

Key Words: Hardy spaces, variable exponents, Heisenberg group, atomic decomposition, Littlewood-Paley characterization.

AMS Subject Classifications: 42B20, 42B25, 42B30, 42B35, 43A80

1 Introduction

Hardy spaces have a number of applications in harmonic analysis, as well as in control theory and in scattering theory. The classical Hardy spaces H^p can be characterized by maximal functions, atomic decomposition, and Littlewood-Paley decomposition (see [8, 18, 22, 34, 36, 43], etc.). We can see from [5, 13, 14, 24, 31, 33, 39, 54], etc., that there are many further studies about different kinds of Hardy spaces.

Variable Hardy spaces have been well studied by [11, 37, 41, 48, 49, 52, 55, 56], etc.. We refer to [3, 10, 15–17, 46, 47, 50, 51], etc., for some other kinds of variable function spaces and their applications. We can see that increasing attention has been paid to the study of function spaces with variable exponent in harmonic analysis.

The Heisenberg group, denoted by $\mathbb{H}^n$, plays an important role in several branches of mathematics, such as representation theory, partial differential equations, several complex variables and number theory.

However, as far as we know there is no work investigating variable Hardy spaces on the Heisenberg group. Inspired by the studies of Hardy spaces on some kinds of abstract spaces e.g., [4, 6, 23, 25, 27, 30, 38, 45], we turn to consider characterizing variable Hardy

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spaces on the Heisenberg group. This is based on [37] but since the Heisenberg group possesses a special geometry structure, it is a kind of non-Abelian groups and the Fourier transform on it is operator-valued, we need more complicated calculations to extend the classical theories on it. Furthermore, because of the properties of the left invariant vector fields on the Heisenberg group, we have to use sub-Laplace operator $\mathcal{L}$ instead of the classical Laplace operator Δ on $\mathbb{R}^n$ and thus the heat kernel is quite different from the one on the Euclidean spaces.

Firstly, we need to use the structure of dyadic cubes on the doubling metric space given by Tuomas Hytönen and Anna Kairema (see [29]) to generalize some basic theory about variable Lebesgue spaces introduced by David V. Cruz-Uribe and Alberto Fiorenza (see [12]). Then we use a lot of analysis tools on stratified groups introduced by G. B. Folland and E. M. Stein (see [19]) and some basic properties of Fourier transform on the Heisenberg group (see the refrences [44] and [35]) to investigate the variable Hardy spaces on it.

This paper is organized as follows. In Section 2, we recall some basic properties of the Heisenberg group shown in [44] and [19] and then give the definition of variable Hardy spaces and atoms on $\mathbb{H}^n$. In Section 3, we first introduce the log-Hölder continuity and decay condition for the variable exponent $p(\cdot)$. Then under these conditons we prove the equivalence of Hardy norms given by maximal functions, i.e.,

$$\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}} \sim \|M^*_\varphi f\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \sim \|M_\varphi f\|_{L^{p(\cdot)}_{\mathbb{H}^n}},$$

in Theorem 3.2. Then we characterize variable Hardy spaces by heat kernel in Theorem 3.3. In Section 4, we give the equivalent characterization of variable Hardy spaces by atomic decomposition in Theorem 4.4 and Theorem 4.5 by virtue of the conclusions from Section 3 and then we give the boundedness of singular integral operators in Theorem 4.7 and Theorem 4.9 as an application of atomic decomposition. Finally, in Section 5, by the boundedness of singular integral operators we can give the Littlewood-Paley characterization in Theorem 5.2.

2 Preliminary

In this section we first introduce some basic properties of the Heisenberg group (see [44] and [19]) and then give the definition of variable Hardy spaces on the Heisenberg group.

We denote by $\mathbb{H}^n$ the Heisenberg group, which is a Lie group with the underlying manifold $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{C}^n \times \mathbb{R}$. The multiplication is given by

$$(x,y,t)(u,v,s) = \Big(x+u, y+v, t+s+\frac{1}{2}(u\cdot y - x\cdot v)\Big),$$

where $u\cdot y = \sum_{j=1}^n u_j y_j$. $\mathbb{H}^n$ is an unimodular group, whose Haar measure coincides with the Lebesgue measure of $\mathbb{R}^{2n+1}$. Different from Euclidean spaces, the dilations on $\mathbb{H}^n$ is defined by

$$\delta(x,y,t)=(\delta x,\delta y,\delta^2 t),\quad \delta>0,$$

and the homogeneous norm on the Heisenberg group is as follow:

$$|\mu|=|(x,y,t)|=((|x|^2+|y|^2)^2+16t^2)^{\frac{1}{4}},$$

which satisfies the trigonometric inequality $|\mu\nu|\le|\mu|+|\nu|$ and $|\mu^{-1}|=|\mu|$. $Q=2n+2$ is the homogeneous dimension of $\mathbb{H}^n$ and the ball $B_r(u)=\{v\in\mathbb{H}^n:|u^{-1}v|<r\}$ has volume Cr^Q. The distance of two sets A and B is defined by

$$dist(A,B)=\inf_{x\in A,\ y\in B}|x^{-1}y|.$$

Note that from this definition, the distance of two points x and y can be written as $dist(x,y)=|x^{-1}y|$ and the distance of a point x and a set A can be written as

$$dist(x,A)=\inf_{y\in A}|x^{-1}y|.$$

Denote by X_i the left invariant vector fields of $\mathbb{H}^n$, where $i=1,2,\cdots,2n+1$. By the formula

$$X_j f(y)=\frac{d}{dt}f(y\cdot e^{tX_j})\Big|_t=0$$

(see [19, pp. 20]), we can calculate that

$$X_j=\frac{\partial}{\partial x_j}+\frac{1}{2}y_j\frac{\partial}{\partial t},\quad X_{j+n}=\frac{\partial}{\partial y_j}-\frac{1}{2}x_j\frac{\partial}{\partial t},\quad X_{2n+1}=\frac{\partial}{\partial t},\quad \text{where } j=1,2,\cdots,n.$$

Let $X=(X_1,X_2,\cdots,X_{2n+1})$. The Schwartz class $\mathcal{S}$ on $\mathbb{H}^n$ is defined by

$$\mathcal{S}=\Big\{\phi\in C^\infty(\mathbb{H}^n):(1+|x|)^\alpha|X^\beta\phi|<\infty \text{ for all } \alpha\in\mathbb{N}\cup\{0\},\ \beta\in(\mathbb{N}\cup\{0\})^{2n+1},\ x\in\mathbb{H}^n\Big\}.$$

Now we topologize $\mathcal{S}(\mathbb{H}^n)$ by the semi–norms $\{p_N\}_{N\in\mathbb{N}}$ which is given by

$$p_N(\varphi)\equiv\sum_{d(\alpha)\le N}\sup_{x\in\mathbb{H}^n}(1+|x|)^N|X^\alpha\varphi(x)|,$$

where $N\in\mathbb{N}\cup\{0\}$ and $d(\alpha)$ stands for the homogeneous degree of α, i.e.,

$$d(\alpha)=\sum_{j=1}^{2n}\alpha_j+2\alpha_{2n+1}. \tag{2.1}$$

The Schrödinger representation π_λ of $\mathbb{H}^n$ is defined by

$$\pi_\lambda(x,y,t)\varphi(\xi)=e^{i\lambda t}e^{i\lambda(x.\xi+\frac{1}{2}x.y)}\varphi(\xi+y),\quad \lambda\neq 0,$$

where $\varphi \in L^2(\mathbb{R}^n)$. For $f \in L^1(\mathbb{H}^n)$, the Fourier transform of f can be defined by

$$\widehat{f}(\lambda)\varphi = \int_{\mathbb{H}^n} f(x,y,t)\pi_\lambda(x,y,t)\varphi dxdydt.$$

Since the Schrödinger representations are irreducible unitary representations of $\mathbb{H}^n$, we can conclude that

$$|(\pi_\lambda(x,y,t)\varphi,\psi)| \le \|\varphi\|_2\|\psi\|_2$$

it follows that

$$|(\widehat{f}(\lambda)\varphi,\, \psi)| \le \|\varphi\|_2\|\psi\|_2\|f\|_1.$$

Hence the Fourier transform $\widehat{f}(\lambda)$ is a bounded operator and $\|\widehat{f}(\lambda)\|_{op} \le \|f\|_1$.

By Plancheral formula

$$\|f\|_2^2 = (2\pi)^{-n-1}\int_{\mathbb{R}} \|\widehat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda,$$

where $||\cdot||_{HS}$ denotes the Hilbert-Schmidt norm of the operator, we can extend the definition of the Fourier transform to all $f \in L^2(\mathbb{H}^n)$. For all Schwartz class functions on $\mathbb{H}^n$, there holds the inversion formula for the group Fourier transform:

$$f(x,y,z) = (2\pi)^{-n-1}\int_{\mathbb{R}} tr(\pi_\lambda(x,y,t)^*\widehat{f}(\lambda))|\lambda|^n d\lambda.$$

As usual, the convolution of measurable functions f and g on $\mathbb{H}^n$ is defined by

$$f*g(x) = \int_{\mathbb{H}^n} f(y)g(y^{-1}x)dy = \int_{\mathbb{H}^n} f(xy^{-1})g(y)dy.$$

Write

$$f_j(x) \equiv 2^{(2n+2)j} f(2^j x).$$

Then it is easy to calculate that Fourier transform satisfies

1) $\widehat{f*g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda)$,
2) $\widehat{f_j}(\lambda) = \widehat{f}(2^{-2j}\lambda)$.

The function $p(\cdot) : \mathbb{H}^n \to (0,\infty)$ is called the variable exponent. For a measurable subset $E \subset \mathbb{H}^n$, we write

$$p_+(E) \equiv \sup_{x\in E} p(x), \quad p_-(E) \equiv \inf_{x\in E} p(x).$$

We abbreviate $p_+(\mathbb{H}^n)$ and $p_-(\mathbb{H}^n)$ to p_+ and p_- respectively, and in this paper, if there is no additional description, we always assume that

$$0 < p_- \le p_+ < \infty. \tag{2.2}$$

For a measurable function f, like [12,37], etc., we define

$$\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \equiv \inf\Big\{\lambda>0: \int_{\mathbb{H}^n}\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx\le 1\Big\}.$$

The symbol $A\lesssim B$ indicates the inequality $A\le CB$ for some constant C and $A\sim B$ stand for $A\lesssim B\lesssim A$.

Denote by $\mathcal{S}'(\mathbb{H}^n)$ the dual space of $\mathcal{S}(\mathbb{H}^n)$, as usual we call $\mathcal{S}'(\mathbb{H}^n)$ the space of tempered distributions on $\mathbb{H}^n$. Define

$$\mathcal{F}_N\equiv\{\varphi\in\mathcal{S}(\mathbb{H}^n): p_N(\varphi)\le 1\}.$$

The Hardy-Littlewood maximal function is defined by

$$Mf(x)=\sup\Big\{\frac{1}{|B|}\int_B |f|d\mu : B \text{ is a ball, } x\in B\Big\}$$

and the centered maximal function is defined by

$$M_c f(x)=\sup\Big\{\frac{1}{|B_r(x)|}\int_{B_r(x)}|f|d\mu : r>0\Big\}.$$

By [9, pp. 625] they are equivalent.

For $\varphi\in\mathcal{S}(\mathbb{H}^n)$, define the maximal function with respect to φ by

$$M_\varphi f(x)\equiv\sup_{j\in\mathbb{Z}}|f*\varphi_j(x)|.$$

Like [37], we give the definition of variable Hardy spaces and variable atomic Hardy spaces as follows.

Definition 2.1. Choose N to be a large integer. The grand maximal function is defined by

$$\mathcal{M}f(x)\equiv\sup_{t>0,\psi\in\mathcal{F}_N}|f*[t^{-n}\psi(t^{-1}\cdot)](x)|,$$

where $f\in\mathcal{S}'(\mathbb{H}^n)$.

We call $H^{p(\cdot)}(\mathbb{H}^n)\equiv\{f\in\mathcal{S}'(\mathbb{H}^n): \|\mathcal{M}f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}<\infty\}$ the variable Hardy spaces, which is equipped with the norm $\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}\equiv\|\mathcal{M}f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}$.

Definition 2.2 (($p(\cdot),q)-Atom$)**.** Suppose $p(\cdot):\mathbb{H}^n\to(0,\infty)$, $0<p_-\le p_+\le\infty$ and $q\ge 1$. Fix an integer $\mathcal{D}\ge\mathcal{D}_{p(\cdot)}\equiv\min\{\lambda\in\mathbb{N}\cup\{0\}:(2n+\lambda+3)p_->2n+2\}$. we call the function a on $\mathbb{H}^n$ a $(p(\cdot),q)-atom$ if there exists a ball B such that

$(a1)$ $supp(a)\subset B$,

$(a2)$ $\|a\|_q\le\dfrac{|B|^{\frac{1}{q}}}{\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}$,

$(a3)$ $\displaystyle\int_{\mathbb{H}^n}a(x)x^\alpha dx=0 \ for\ all\ \alpha\ s.t.\ d(\alpha)\le\mathcal{D}.$

Denote the set of all such pairs (a,B) by $A(p(\cdot),q)$.

Definition 2.3. ($\mathcal{A}(\{k_j\}_{j=1}^{\infty},\{B_j\}_{j=1}^{\infty})$ and $H_{atom}^{p(\cdot),q}(\mathbb{H}^n)$) Let $\underline{p}\equiv\min(p_-,1)$. For nonnegative number sequences $\{k_j\}_{j=1}^{\infty}$ and ball sequences $\{B_j\}_{j=1}^{\infty}$, we define

$$\mathcal{A}(\{k_j\}_{j=1}^{\infty},\{B_j\}_{j=1}^{\infty})\equiv\inf\Big\{\lambda>0:\int_{\mathbb{H}^n}\Big\{\sum_{j=1}^{\infty}\Big(\frac{k_j\chi_{B_j}(x)}{\lambda\|\chi_{B_j}\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\Big)^{\underline{p}}\Big\}^{\frac{p(x)}{\underline{p}}}dx\le 1\Big\}. \tag{2.3}$$

We write $H_{atom}^{p(\cdot),q}(\mathbb{H}^n)$ to denote the set of all functions $f\in\mathcal{S}'(\mathbb{H}^n)$ which can be decomposed as

$$f=\sum_{j=1}^{\infty}k_ja_j \quad \text{in } \mathcal{S}'(\mathbb{H}^n), \tag{2.4}$$

where $\{k_j\}_{j=1}^{\infty}$ is a sequence of nonnegative numbers, $\{(a_j,B_j)\}_{j=1}^{\infty}\subset A(p(\cdot),q)$ and $\mathcal{A}(\{k_j\}_{j=1}^{\infty},\{B_j\}_{j=1}^{\infty})$ is finite. We define

$$\|f\|_{H_{atom}^{p(\cdot),q}(\mathbb{H}^n)}\equiv\inf\mathcal{A}(\{k_j\}_{j=1}^{\infty},\{B_j\}_{j=1}^{\infty}),$$

where the infimum is taken over all admissible expressions as in (2.4).

3 Characterization by maximal functions and heat kernel

In this section, we give some basic properties of variable Lebesgue space and prove the equivalence of variable Hardy spaces characterized by a series of maximal functions on the Heisenberg group when the function φ in the definition of M_φ is a radial function, i.e., $\varphi(z_1,t)=\varphi(z_2,t)$ as long as $|z_1|=|z_2|$.

Recall that for a measurable function f,

$$\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\equiv\inf\Big\{\lambda>0:\int_{\mathbb{H}^n}\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}dx\le 1\Big\}.$$

If $0<\iota\le\underline{p}=\min(p_-,1)$, there holds the following properties:
1. (Positivity) $\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\ge 0$, and $\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}=0\Leftrightarrow f\equiv 0$.
2. (Homogeneity) $\|cf\|_{L^{p(\cdot)}_{\mathbb{H}^n}}=|c|\cdot\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}$ for $c\in\mathbb{C}$.
3. (The ι-triangle inequality) $\|f+g\|^{\iota}_{L^{p(\cdot)}_{\mathbb{H}^n}}\le\|f\|^{\iota}_{L^{p(\cdot)}_{\mathbb{H}^n}}+\|g\|^{\iota}_{L^{p(\cdot)}_{\mathbb{H}^n}}$.

Since this is an analogue of the $\mathbb{R}^n$ case, we only note that the ι-triangle inequality follows from the following identity: For $0<p_-<1$, choosing ι such that $0<\iota\le p_-$, we have

$$\|f^{\iota}\|^{\frac{1}{\iota}}_{L^{\frac{p(\cdot)}{\iota}}_{\mathbb{H}^n}}=\inf\Big\{\lambda^{\frac{1}{\iota}}>0:\int_{\mathbb{H}^n}\Big(\frac{|f^{\iota}(x)|}{\lambda}\Big)^{\frac{p(x)}{\iota}}dx\le 1\Big\}=\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}. \tag{3.1}$$

Like [37], throughout this paper, we suppose $p(\cdot)$ have the following conditions:

$$|p(x)-p(y)|\lesssim\frac{1}{-\log(|x^{-1}y|)}\quad\text{for }|x^{-1}y|\le\frac{1}{2}\quad(\log-\text{Hölder continuity}),\tag{3.2a}$$

$$|p(x)-p(y)|\lesssim\frac{1}{\log(e+|x|)}\quad\text{for }|y|\ge|x|\quad(\text{decay condition}).\tag{3.2b}$$

Note that $p_\infty\equiv\lim_{x\to\infty}p(x)$ exists in view of (3.2b). The assumptions $p_+<\infty$ and (3.2) is followed by

$$|p(x)-p(y)|\lesssim\frac{1}{\log(e+\frac{1}{|x^{-1}y|})}\quad\text{for all }x,y\in\mathbb{H}^n.\tag{3.3}$$

Observe that (3.2b) is equivalent to the following estimate:

$$|p(x)-p_\infty|\lesssim\frac{1}{\log(e+|x|)}\quad\text{for all }x\in\mathbb{H}^n,\tag{3.4}$$

which is equivalent to $|\log[(e+|x|)^{(p(x)-p_\infty)}]|\lesssim 1$, i.e.,

$$\frac{(e+|x|)^{p(x)}}{(e+|x|)^{p_\infty}}\sim 1\quad\text{for all }x\in\mathbb{H}^n.\tag{3.5}$$

Now we show some conclusions from [19].

Theorem 3.1 (Stratified Mean Value Theorem)**.** *Let G be a stratified group. Then there exist $C,b>0$ such that the following inequality holds for all $f\in C^1(G)$ and all $x,y\in G$,*

$$|f(xy)-f(x)|\le C|y|\sup_{|z|\le b|y|,\ 1\le j\le\nu}|X_jf(xz)|.$$

We note that $\nu=2n$ when G is the Heisenberg group.

Corollary 3.1 (see [19])**.** If $f\in C^{k+1}$ then

$$|f(xy)-P_x(y)|\le C_k'|y|^{k+1}\sup_{|z|\le b^{k+1}|y|,\ d(I)=k+1}|X^If(xz)|,$$

where P_x is the left Taylor polynomial of f at x of homogeneous degree k, b is as in Theorem 3.1.

The following lemma is an analogue of [21, pp. 466].

Lemma 3.1. *Suppose that $a,b\in\mathbb{H}^n$, $M,N>0$ and L is a nonnegative integer. ϕ_μ and ϕ_ν are two functions on the Heisenberg group satisfying the following conditions:*

$$|X^\alpha\phi_\mu(x)|\lesssim\frac{2^{\mu(2n+2)+\mu L}}{(1+2^\mu|x_\mu^{-1}x|)^M}\quad\textit{for all }\alpha\textit{ with }d(\alpha)=L,$$

$$|\phi_\nu(x)|\lesssim\frac{2^{\nu(2n+2)}}{(1+2^\nu|x_\nu^{-1}x|)^N},$$

and

$$\int_{\mathbb{H}^n} X^\beta \phi_\nu(x)dx = 0 \quad \textit{for all } \beta \textit{ with } d(\beta) \le L-1,$$

where $x_\mu, x_\nu \in \mathbb{H}^n$ *and* $d(\cdot)$ *is the homogeneous degree which we have introduced in* (2.1).

Then for $N > M+L+2n+2$ *and* $\nu \ge \mu$, *we have*

$$\Big|\int_{\mathbb{H}^n} \phi_\mu(x)\phi_\nu(x)dx\Big| \lesssim \frac{2^{\mu(2n+2)-(\nu-\mu)L}}{(1+2^\mu|x_\nu^{-1}x_\mu|)^M}.$$

Proof. Denote the left Taylor polynomial of ϕ_μ at x_ν of homogeneous degree $L-1$ by P. By Corollary 3.1, we get

$$|\phi_\mu(x)-P(x)| \lesssim |x_\nu^{-1}x|^L \sup_{|x_\nu^{-1}y|\le b^L|x_\nu^{-1}x|,\ d(I)=L} |X^I\phi_\mu(y)|.$$

Then it is followed by

$$\begin{aligned}\Big|\int_{\mathbb{H}^n} \phi_\mu(x)\phi_\nu(x)dx\Big| &\le \int_{\mathbb{H}^n} |\phi_\mu(x)-P(x)||\phi_\nu(x)|dx\\ &\lesssim \int_{\mathbb{H}^n} \frac{|x_\nu^{-1}x|^L 2^{\mu(2n+2)+\mu L}}{(1+2^\mu|x_\mu^{-1}y|)^M}\cdot\frac{2^{\nu(2n+2)}}{(1+2^\nu|x_\nu^{-1}x|)^N}dx.\end{aligned}$$

The inequality

$$\begin{aligned}\frac{1}{1+2^\mu|x_\mu^{-1}y|}-\frac{1}{1+2^\mu|x_\nu^{-1}x_\mu|} &\le \frac{2^\mu(|x_\nu^{-1}x_\mu|-|x_\mu^{-1}y|)}{1+2^\mu|x_\nu^{-1}x_\mu|} \le \frac{2^\mu|x_\nu^{-1}y|}{1+2^\mu|x_\nu^{-1}x_\mu|}\\ &\le \frac{2^\mu b^L|x_\nu^{-1}x|}{1+2^\mu|x_\nu^{-1}x_\mu|} \lesssim \frac{2^\nu|x_\nu^{-1}x|}{1+2^\mu|x_\nu^{-1}x_\mu|}\end{aligned}$$

implies that

$$\frac{1}{1+2^\mu|x_\mu^{-1}y|} \lesssim \frac{1+2^\nu|x_\nu^{-1}x|}{1+2^\mu|x_\nu^{-1}x_\mu|},$$

which indicates that

$$\Big|\int_{\mathbb{H}^n} \phi_\mu(x)\phi_\nu(x)dx\Big| \lesssim \frac{2^{\mu(2n+2)-(\nu-\mu)L}2^{\nu(2n+2)}}{(1+2^\mu|x_\nu^{-1}x_\mu|)^M}\cdot\int_{\mathbb{H}^n}\frac{2^{\nu L}|x|^L}{(1+2^\nu|x|)^{N-M}}dx.$$

Denote the surface area of the ball $B_r(0)=\{x\in\mathbb{H}^n : |x|<r\}$ by $\omega_{2n+1}^{\mathbb{H}^n}(r)$ and write $\omega_{2n+1}^{\mathbb{H}^n}$ short for $\omega_{2n+1}^{\mathbb{H}^n}(1)$. Let us calculate $\omega_{2n+1}^{\mathbb{H}^n}(r)$:

$$\begin{aligned}\omega_{2n+1}^{\mathbb{H}^n}(r) &= 2\int_0^{r^2}\omega_{2n-1}\cdot(r^4-16t^2)^{\frac{2n-1}{4}}dt = 2\int_0^{\frac{\pi}{2}}\omega_{2n-1}\cdot(r^4\cos^2 x)^{\frac{2n-1}{4}}d\Big(\frac{r^2\sin x}{4}\Big)\\ &= \frac{1}{2}\omega_{2n-1}\cdot r^{2n+1}\int_0^{\frac{\pi}{2}}\cos^{\frac{2n-1}{2}} x d\sin x,\end{aligned}$$

where ω_{2n-1} denotes the surface area of the unit ball on $\mathbb{R}^{2n}$. Thus, we have $\omega_{2n+1}^{\mathbb{H}^n}(r)\sim r^{2n+1}$.

$$2^{\nu(2n+2)}\int_{\mathbb{H}^n}\frac{2^{\nu L}|x|^L}{(1+2^\nu|x|)^{N-M}}dx=\int_{\mathbb{H}^n}\frac{|x|^L}{(1+|x|)^{N-M}}dx=\omega_{2n+1}^{\mathbb{H}^n}\int_0^\infty\frac{r^L}{(1+r)^{N-M}}r^{2n+1}dx<\infty,$$

since $N-M>L+2n+2$.

Consequently, we have

$$\Big|\int_{\mathbb{H}^n}\phi_\mu(x)\phi_\nu(x)dx\Big|\lesssim\frac{2^{\mu(2n+2)-(\nu-\mu)L}}{(1+2^\mu|x_\nu^{-1}x_\mu|)^M}.$$

Thus, we complete the proof. $\square$

Lemma 3.2 (Local Reproducing Formula). *Choose a radial function $\varphi_0\in C_c^\infty(\mathbb{H}^n)$ with nonzero integral, and let*

$$\varphi(x)=\varphi_0(x)-2^{-(2n+2)}\varphi_0\Big(\frac{1}{2}x\Big).$$

Then for any given integer $L_\psi\geq 0$ there exist $\psi_0,\psi\in C_c^\infty(\mathbb{H}^n)$ so that $\int_{\mathbb{H}^n}x^\alpha\psi(x)dx=0$ for all multi-indices α with $d(\alpha)\leq L_\psi$ and

$$\sum_{j=0}^\infty\psi_j*\varphi_j=\sum_{j=0}^\infty\varphi_j*\psi_j=\delta,$$

where δ is the dirac delta function.

Since this can be proven in a similar way to [40, Theorem 1.6] with a more complicated calculation, we omit the details. We only note that by [35, Section 4], the Fourier transform of a radial Schwartz function f on $\mathbb{H}^n$ is diagonal on the Hermite basis for $L_2(\mathbb{R}^n)$, which means that $f*g=g*f$ whenever f and g are radial functions on $\mathbb{H}^n$.

Lemma 3.3. *For a function φ on $\mathbb{H}^n$, we define its radial decreasing dominating function by*

$$\Phi(x)=\sup_{|y|\geq|x|}|\varphi(y)|.$$

If $\Phi(x)\in L^1(\mathbb{H}^n)$, then there exists a constant C_n such that for $\forall f\in L^p(\mathbb{H}^n)$ and $\forall x\in\mathbb{H}^n$ we have

$$\sup_{r\in\mathbb{Z}}|(f*\varphi_r)(x)|\leq C_n\|\Phi\|_1 Mf(x),$$

where M is the Hardy-Littlewood maximal function.

Proof. Write $\Phi_0(s)=\Phi(|x|)=\Phi(x)$, where $(s=|x|)$. Then

$$\begin{aligned}\|\Phi\|_1&=\int_0^\infty \omega_{2n+1}^{\mathbb{H}^n}\Phi_0(s)s^{2n+1}ds=\frac{\omega_{2n+1}^{\mathbb{H}^n}}{r^{2n+2}}\int_0^\infty \Phi_0\Big(\frac{s}{r}\Big)s^{2n+1}ds\\&=\frac{\omega_{2n+1}^{\mathbb{H}^n}}{r^{2n+2}}\sum_{k=-\infty}^{\infty}\int_{2^{k-1}}^{2^k}\Phi_0\Big(\frac{s}{r}\Big)s^{2n+1}ds\\&\geq\omega_{2n+1}^{\mathbb{H}^n}\sum_{k=-\infty}^{\infty}\Phi_0\Big(\frac{2^k}{r}\Big)\frac{2^{(2n+2)k}-2^{(2n+2)(k-1)}}{(2n+2)r^{2n+2}},\end{aligned}$$

where $\omega_{2n+1}^{\mathbb{H}^n}$ has the same definition as in the proof of Lemma 3.1. It follows that

$$\begin{aligned}(|f|*\Phi_r)(x)&=\sum_{k=-\infty}^{\infty}\frac{1}{r^{2n+2}}\int_{2^k<|y|\leq 2^{k+1}}|f(xy^{-1})|\Phi\Big(\frac{y}{r}\Big)dy\\&\leq\sum_{k=-\infty}^{\infty}\frac{1}{r^{2n+2}}\Phi_0\Big(\frac{2^k}{r}\Big)\int_{2^k<|y|\leq 2^{k+1}}|f(xy^{-1})|dy\\&\lesssim Mf(x)\sum_{k=-\infty}^{\infty}\Phi_0\Big(\frac{2^k}{r}\Big)\frac{2^{(2n+2)k}-2^{(2n+2)(k-1)}}{(2n+2)r^{2n+2}}\lesssim\|\Phi\|_1Mf(x).\end{aligned}$$

Then by the inequality $|(f*\varphi_r)(x)|\leq(|f|*|\varphi_r|)(x)\leq(|f|*\Phi_r)(x)$, we get the desired conclusion. $\square$

Given an integer $L\gg 1$ and a radial function $\varphi\in\mathcal{S}(\mathbb{H}^n)$ such that $\int_{\mathbb{H}^n}\varphi(x)dx\neq 0$. We write

$$M_\varphi^*f(x)=M_{\varphi,L}^*f(x)\equiv\sup_{j\in\mathbb{Z}}\sup_{y\in\mathbb{H}^n}\frac{|f*\varphi_j(y)|}{(1+4^j|x^{-1}y|^2)^L}.$$

Lemma 3.4. *If $f\in\mathcal{S}'(\mathbb{H}^n)$, $0<\theta<1$ and φ is a radial function in $\mathcal{S}(\mathbb{H}^n)$ with nonzero integral, then there exists L_θ such that for all $L\geq L_\theta$, we have*

$$M_{\varphi,L}^*f(x)\lesssim M\Big[\sup_{k\in\mathbb{Z}}|f*\varphi_k|^\theta\Big](x)^{\frac{1}{\theta}}=M\Big[(M_\varphi f)^\theta\Big](x)^{\frac{1}{\theta}}.$$

Proof. Let $L\gg 1$. By local reproducing formula (Lemma 3.2), there exist $\psi_0,\psi\in\mathcal{S}(\mathbb{H}^n)$ such that

$$\varphi*\psi_0+\sum_{k=1}^{\infty}(\varphi_k-\varphi_{k-1})*\psi_k=\delta,$$

and

$$\int_{\mathbb{H}^n}x^\alpha\psi(x)dx=0\quad\text{ for all }\ \alpha\ \text{ with }\ d(\alpha)\leq 2n+3L+4.$$

Then, by dilation, we have $\varphi_j*\psi_{0_j}+\sum_{k=j+1}^{\infty}(\varphi_k-\varphi_{k-1})*\psi_k=\delta$. Combining this formula with the triangle inequality, we get

$$\frac{|(f*\varphi_j)(y)|}{(1+4^j|x^{-1}y|^2)^L}\leq\frac{|f*\varphi_j*\psi_{0_j}*\varphi_j(y)|}{(1+4^j|x^{-1}y|^2)^L}+\sum_{k=j+1}^{\infty}\frac{|f*(\varphi_k-\varphi_{k-1})*\psi_k*\varphi_j(y)|}{(1+4^j|x^{-1}y|^2)^L}.\tag{3.6}$$

We estimate each term of the right hand side; when $k\geq j+1$, we have

$$\frac{|f*(\varphi_k-\varphi_{k-1})*\psi_k*\varphi_j(y)|}{(1+4^j|x^{-1}y|^2)^L}\leq\int_{\mathbb{H}^n}\frac{|[f*(\varphi_k-\varphi_{k-1})(z)][\psi_k*\varphi_j(z^{-1}y)]|}{(1+4^j|x^{-1}y|^2)^L}dz.$$

By applying Lemma 3.1, we have

$$|\psi_k*\varphi_j(z^{-1}y)|\leq 2^{(2n+2)j-3(k-j)L}(1+4^j|z^{-1}y|^2)^{-L}.$$

Then it implies that

$$\begin{aligned}&\frac{|f*(\varphi_k-\varphi_{k-1})*\psi_k*\varphi_j(y)|}{(1+4^j|x^{-1}y|^2)^L}\\ \lesssim&\int_{\mathbb{H}^n}\frac{2^{(2n+2)j-3(k-j)L}|f*(\varphi_k-\varphi_{k-1})(z)|}{(1+4^j|z^{-1}y|^2)^L(1+4^j|x^{-1}y|^2)^L}dz\\ \leq&\int_{\mathbb{H}^n}\frac{2^{(2n+2)j-3(k-j)L}|f*(\varphi_k-\varphi_{k-1})(z)|}{(1+4^j|x^{-1}z|^2)^L}dz\\ \lesssim&M^*_{\varphi,L}f(x)^{1-\theta}\int_{\mathbb{H}^n}\frac{2^{(2n+2)j-3(k-j)\theta L}}{(1+4^j|x^{-1}z|^2)^{\theta L}}(|f*\varphi_k(z)|^\theta+|f*\varphi_{k-1}(z)|^\theta)dz\end{aligned}$$

and by a similar discussion we have

$$\frac{|f*\varphi_j*\psi_{0_j}*\varphi_j(y)|}{(1+4^j|x^{-1}y|^2)^L}\lesssim M^*_{\varphi,L}f(x)^{1-\theta}\int_{\mathbb{H}^n}\frac{2^{(2n+2)j}}{(1+4^j|x^{-1}z|^2)^{\theta L}}|f*\varphi_j(z)|^\theta dz.$$

Now we substitute the above inequality into (3.6), we can deduce that

$$\frac{|f*\varphi_j(y)|}{(1+4^j|x^{-1}y|^2)^L}\lesssim M^*_{\varphi,L}f(x)^{1-\theta}\sum_{k=j}^{\infty}2^{-3(k-j)\theta L}\int_{\mathbb{H}^n}\frac{2^{(2n+2)j}|f*\varphi_k(z)|^\theta}{(1+4^j|x^{-1}z|^2)^{\theta L}}dz,$$

which indicates that

$$M^*_{\varphi,L}f(x)\lesssim M^*_{\varphi,L}f(x)^{1-\theta}\sup_{j\in\mathbb{Z}}\sum_{k=j}^{\infty}2^{-3(k-j)\theta L}\int_{\mathbb{H}^n}\frac{2^{(2n+2)j}|f*\varphi_k(z)|^\theta}{(1+4^j|x^{-1}z|^2)^{\theta L}}dz.$$

Hence we get

$$M^*_{\varphi,L}f(x)^\theta\lesssim\sup_{j\in\mathbb{Z}}\sum_{k=j}^{\infty}2^{-3(k-j)\theta L}\int_{\mathbb{H}^n}\frac{2^{(2n+2)j}|f*\varphi_k(z)|^\theta}{(1+4^j|x^{-1}z|^2)^{\theta L}}dz.$$

Since the function $\frac{2^{(2n+2)j}}{(1+4^j|x|^2)^{\theta L}}$ defined on $\mathbb{H}^n$ can be regarded as the radial decreasing dominating function of itself, by Lemma 3.3 it follows that

$$\begin{aligned}\int_{\mathbb{H}^n}\frac{2^{(2n+2)j}|f*\varphi_k(z)|^\theta}{(1+4^j|x^{-1}z|^2)^{\theta L}}dz\leq&\Big\|\frac{2^{(2n+2)j}}{(1+4^j|\cdot|^2)^{\theta L}}\Big\|_1M[(f*\varphi_k)^\theta](x)\\ =&\Big\|\frac{1}{(1+|\cdot|^2)^{\theta L}}\Big\|_1M[(f*\varphi_k)^\theta](x)\lesssim M[(f*\varphi_k)^\theta](x).\end{aligned}$$

Thus, we have $M^*_{\varphi,L}f(x)^\theta \lesssim M[\sup_{k\in\mathbb{Z}}(f*\varphi_k)^\theta](x)$. $\square$

Now we turn to show the equivalence of Hardy norms given by different kinds of maximal functions in the following theorem.

Theorem 3.2. *For an radial function $\varphi\in\mathcal{S}(\mathbb{H}^n)$ with nonzero integral, we have*

$$\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}} \sim \|M^*_\varphi f\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \sim \|M_\varphi f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}$$

for all $f\in\mathcal{S}'(\mathbb{H}^n)$, where we choose N in the definition of $\mathcal{M}f$ and L in the definition of M^*_φ to be large enough.

Proof. We know that M is a strong (p,p) operator when $p>1$. Since from (3.1) we have

$$\|f^\iota\|^{\frac{1}{\iota}}_{L^{\frac{p(\cdot)}{\iota}}_{\mathbb{H}^n}} = \|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}},$$

combining with Lemma 3.4, if the number θ is chosen small enough we have

$$\begin{aligned}\left\|M^*_\varphi f\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}} &\lesssim \left\|M\left[(M_\varphi f)^\theta\right](\cdot)^{\frac{1}{\theta}}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}} = \left\|M\left[(M_\varphi f)^\theta\right]\right\|^{\frac{1}{\theta}}_{L^{\frac{p(\cdot)}{\theta}}_{\mathbb{H}^n}} \\ &\lesssim \left\|(M_\varphi f)^\theta\right\|^{\frac{1}{\theta}}_{L^{\frac{p(\cdot)}{\theta}}_{\mathbb{H}^n}} = \left\|M_\varphi f\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}.\end{aligned}$$

By the definition of these maximal functions, we can conclude that $\mathcal{M}f\geq M_\varphi f$ and $M^*_\varphi f\geq M_\varphi f$. Hence, we have

$$\left\|M^*_\varphi f\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \sim \left\|M_\varphi f\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}.$$

Then we only need to show that $\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \left\|M^*_\varphi f\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}$. Again, we apply Lemma 3.2. Choose $\psi_0,\psi\in\mathcal{S}(\mathbb{H}^n)$ to be as in the previous lemma and let $\tau\in\mathcal{S}(\mathbb{H}^n)$ satisfy $p_N(\tau)\leq 1$. Then

$$f*\tau_j = f*\varphi_j*\psi_{0_j}*\tau_j + \sum_{k=j+1}^{\infty} f*(\varphi_k-\varphi_{k-1})*\psi_k*\tau_j \quad \text{for all } j\in\mathbb{Z}.$$

When $k\geq j$, we have

$$\begin{aligned}|f*\varphi_k*\psi_k*\tau_j| &\leq M^*_\varphi f(x)\int_{\mathbb{H}^n}|\psi_k*\tau_j(y^{-1}x)|(1+4^k|x^{-1}y|^2)^L dy \\ &\leq M^*_\varphi f(x)\int_{\mathbb{H}^n}|\psi_k*\tau_j(y^{-1}x)|(4^{k-j}+4^k|x^{-1}y|^2)^L dy \\ &\leq 2^{2(k-j)L}M^*_\varphi f(x)\int_{\mathbb{H}^n}|\psi_k*\tau_j(y^{-1}x)|(1+4^j|x^{-1}y|^2)^L dy.\end{aligned}$$

By applying Lemma 3.1, we have

$$|\psi_k * \tau_j(y^{-1}x)| \lesssim \frac{2^{(2n+2)j-3(k-j)L}}{(1+2^j|y^{-1}x|)^{2n+3L+3}} \quad \text{for all} \quad k \geq j+1,$$

and

$$|\psi_{0_j} * \tau_j(y^{-1}x)| \lesssim \frac{2^{(2n+2)j}}{(1+2^j|y^{-1}x|)^{2n+3L+3}}.$$

Since $|y^{-1}x| = |x^{-1}y|$, it follows that

$$|f * \tau_j(x)| \lesssim \sum_{k=j}^{\infty} 2^{(2n+2)j-(k-j)L} M_\varphi^* f(x) \int_{\mathbb{H}^n} (1+2^j|x^{-1}y|)^{-2n-L-3} dy \lesssim M_\varphi^* f(x).$$

Hence by the arbitrariness of τ and j we obtain $\mathcal{M}f \lesssim M_\varphi^* f(x)$. $\square$

From Theorem 3.2, we can easily conclude the following corollary.

Corollary 3.2. Suppose $\varphi \in \mathcal{S}(\mathbb{H}^n)$ is a radial function with nonzero integral. Write

$$M^\varphi f(x) \equiv \sup_{s>0} \left| s^{-2n-2} f * \varphi(s^{-1}\cdot) \right|.$$

We define

$$\|f\|_{H^{p(\cdot)}_{\varphi,*}} \equiv \|M^\varphi f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}, \quad f \in \mathcal{S}'(\mathbb{H}^n).$$

Then $\|f\|_{H^{p(\cdot)}_{\varphi,*}} \sim \|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}$, since $M_\varphi f \leq M^\varphi f \leq \mathcal{M}f$ and $\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}} \sim \left\|M_\varphi f\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}$, when N in the definition of $\mathcal{M}f$ is sufficiently large.

Now we turn to consider the characterization of variable Hardy spaces by heat kernel. Write $\varphi_j(x) = 2^{(2n+2)j}\varphi(2^j x)$ as before. The sub–Laplacian operator $\mathcal{L}$ of $\mathbb{H}^n$ is defined by $\mathcal{L} = -\sum_{j=1}^{2n} X_j^2$, where X_j denote the left invariant vector fields on the Heisenberg group introduced in Section 2.

The heat kernel on $\mathbb{H}^n$ is such a function that satisfies the heat equation:

$$\begin{aligned} &\frac{\partial h^s(x,y,t)}{\partial s} = -\mathcal{L}h^s(x,y,t), \quad s>0, \quad (x,y,t) \in \mathbb{H}^n, \\ &h^0(x,y,t) = \delta. \end{aligned}$$

Theorem 3.3. *Suppose $p(\cdot)$ satisfies* (2.2), (3.2) *and* (3.2b). *Let $f \in \mathcal{S}'(\mathbb{H}^n)$. Then*

$$\left\| \sup_{s>0} \left| e^{-s\mathcal{L}} f \right| \right\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \sim \|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}.$$

Proof. By [53, Theorem 3.2] or [42, pp. 399] we know that the heat kernel of $\mathbb{H}^n$ can be written as

$$h^s(z,t)=\frac{1}{(2\pi)(4\pi)^n}\int_{\mathbb{R}}\Big(\frac{|\lambda|}{\sinh|\lambda|s}\Big)^n\exp\Big\{-\frac{|\lambda||z|^2}{4}\coth|\lambda|s-i\lambda\cdot t\Big\}d\lambda,\quad \text{where } s>0.$$

Write h short for h^1. From this expression of heat kernel, we can see that h^s is a radial function on $\mathbb{H}^n$. By [19, Proposition 1.68(iv)] we know that h^s is the dilation of h, i.e.,

$$h^s(x)=\frac{1}{s^{2n+2}}h\Big(\frac{1}{s}x\Big),$$

where $s>0$, and by [19, Proposition 1.74] we know that h is a Schwartz function. Now the heat semigroup $\{e^{-s\mathcal{L}}\}_{s>0}$ for $\mathbb{H}^n$ can be given by $e^{-s\mathcal{L}}f(x)=[f*h^s](x)$. By Corollary 3.2, we have

$$\Big\|\sup_{s>0}\big|e^{-s\mathcal{L}}f\big|\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\sim\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}.$$

Thus, we complete the proof. □

4 Atomic decompositions and some applications

Before considering atomic decompositions, we need to give some further properties of variable Lebesgue space on $\mathbb{H}^n$.

Since the proofs of the following Hölder inequality and its corollary are similar to the corresponding ones on Euclidean spaces [12], we omit their proof.

Theorem 4.1 (Hölder Inequality)**.** *Let $p'(\cdot)$ be the conjugate exponent of $p(\cdot)$, i.e., there holds $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ for all $x\in\mathbb{H}^n$ and $p(\cdot)$, $p'(\cdot)\geq 1$. Then for all $f\in L^{p(\cdot)}_{\mathbb{H}^n}$ and $g\in L^{p'(\cdot)}_{\mathbb{H}^n}$, we have*

$$\int_{\mathbb{H}^n}|f(x)g(x)|dx\lesssim\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\|g\|_{L^{p'(\cdot)}_{\mathbb{H}^n}}.$$

Corollary 4.1. Suppose $\frac{1}{p(x)}=\frac{1}{q(x)}+\frac{1}{r(x)}$, for all $x\in\mathbb{H}^n$, where $p(\cdot)$, $q(\cdot)$ and $r(\cdot)$ satisfy $p(\cdot),q(\cdot),r(\cdot)\geq 1$. Then for all $f\in L^{p(\cdot)}_{\mathbb{H}^n}$ and $g\in L^{q(\cdot)}_{\mathbb{H}^n}$, there holds the following inequality

$$\|fg\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\lesssim\|f\|_{L^{q(\cdot)}_{\mathbb{H}^n}}\|g\|_{L^{r(\cdot)}_{\mathbb{H}^n}}.$$

Lemma 4.1. *Assume that the function $p(\cdot)$ satisfying* (2.2), (3.2) *and* (3.2b)*.*
1. For all balls $B=B_r(z)\in\mathbb{H}^n$ with $|B|\leq 1$, we have

$$|B|^{\frac{1}{p_-(B)}}\sim|B|^{\frac{1}{p_+(B)}}\sim|B|^{\frac{1}{p(z)}}\sim\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}. \tag{4.1}$$

2. For all balls $B=B_r(z)\in\mathbb{H}^n$ with $|B|\geq 1$, we have

$$\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\sim|B|^{\frac{1}{p_\infty}}. \tag{4.2}$$

Proof. Following [12, Lemma 3.24] with a more complicated calculation we know that given any ball B and $x\in B$,

$$|B|^{p(x)}\lesssim|B|^{p_+(B)},\qquad |B|^{p_-(B)}\lesssim|B|^{p(x)}.$$

If $|B|\le 1$, then the inverse inequalities hold, which means that

$$|B|^{p(x)}\sim|B|^{p_+(B)}\sim|B|^{p_-(B)}\sim|B|^{p(z)}.$$

Then it follows that

$$\int_{\mathbb{H}^n}\left(\frac{\chi_B(x)}{|B|^{\frac{1}{p(z)}}}\right)^{p(x)}dx=\int_B|B|^{-\frac{p(x)}{p(z)}}dx\sim\int_B|B|^{-1}dx=1.$$

Following from [12, Proposition 2.21], we have

$$\int_{\mathbb{H}^n}\left(\frac{|f(x)|}{\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\right)^{p(x)}dx=1$$

whenever $f\in L^{p(\cdot)}(\mathbb{H}^n)$ and $\|f\|_{L^{p(\cdot)}_{\mathbb{H}^n}}>0$. Then we obtain the equivalence relationship (4.1).

When $|B|\ge 1$, by the dyadic cubes given by [29, Theorem 2.2], Vitali covering Lemma and Zorn's Lemma, we can easily get

$$\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\sim\|\chi_B\|_{p(\cdot),(3B_j)}\sim|B|^{\frac{1}{p_\infty}}$$

for the case $p(\cdot):\mathbb{H}^n\to(1,\infty)$ satisfying (3.4) with a similar discussion to [26, Theorem 2.4]. Then we can get the same conclusion by (3.1) for $p(\cdot)$ satisfying (2.2) and (3.4). $\square$

By Definition 2.1, Corollary 4.1 and Lemma 4.1 we know that $\|a\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\lesssim 1$ whenever $(a,B)\in A(p(\cdot),q)$. This conclusion can be extended as follows:

Proposition 4.1. If $q>1$ and $p(\cdot)$ is a function satisfying (2.2), (3.2) and (3.2b), then, for all $(a,B)\in A(p(\cdot),q)$, we have

$$\|a\|_{H^{p(\cdot)}_{\mathbb{H}^n}}\lesssim 1.$$

Proof. Suppose $\varphi\in\mathcal{S}(\mathbb{H}^n)\setminus\{0\}$ is a nonnegative radial function with the properties that the value of $\varphi(x)$ only depends on $|x|$ and $\varphi(x)\le\varphi(y)$ whenever $|x|\ge|y|$ and $supp\ \varphi\subset B_1(0)$. Write $B=B_r(z)$. Define another variable exponent $\widetilde{q}(\cdot)$ by

$$\frac{1}{p(x)}=\frac{1}{q}+\frac{1}{\widetilde{q}(x)}\quad (x\in\mathbb{H}^n).$$

Then a direct deduction follows from [12, Proposition 2.3(5)] is that $\widetilde{q}(\cdot)$ also satisfies (3.2) and (3.2b).

By Lemma 3.3 we have $M_\varphi a(x) \lesssim \|\varphi\|_1 Ma(x)$. Since M is a strong (p, p) operator when $p>1$, it follows that

$$\begin{aligned}\left\|(M_\varphi a)\chi_{B_{2r}(z)}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}} &\lesssim \|Ma\|_q \left\|\chi_{B_{2r}(z)}\right\|_{L^{\widetilde{q}(\cdot)}_{\mathbb{H}^n}} \lesssim \|a\|_q \left\|\chi_{B_{2r}(z)}\right\|_{L^{\widetilde{q}(\cdot)}_{\mathbb{H}^n}} \\ &\le \frac{|B_r(z)|^{\frac{1}{q}} \left\|\chi_{B_{2r}(z)}\right\|_{L^{\widetilde{q}(\cdot)}_{\mathbb{H}^n}}}{\left\|\chi_{B_r(z)}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}.\end{aligned}$$

By applying Lemma 4.1, we have

$$\left\|(M_\varphi a)\chi_{B_{2r}(z)}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim 1.$$

Now we show that $\|(M_\varphi a)\chi_{\mathbb{H}^n\setminus B_{2r}(z)}\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim 1$. Let $x\notin B_{2r}(z)$. Since $supp(\varphi_j(\,\cdot\, x^{-1}))\subset B_{2^{-j}}(x)$, if $supp(a)\cap B_{2^{-j}}(x)=\varnothing$, then $|\varphi_j * a(x)|=0$. If $supp(a)\cap B_{2^{-j}}(x)\neq\varnothing$, then $|z^{-1}x|\le 2^{-j}+r$ and $r<2^{-j}$. Let $P(\cdot)$ be the left Taylor polynomial of $\varphi_j(x^{-1}\,\cdot\,)$ at z of homogeneous degree $\mathcal{D}$, where $\mathcal{D}$ is picked in the same way as in Definition 2.2. Suppose $y\in B_r(z)$, then

$$\begin{aligned}|\varphi_j(x^{-1}y)-P(y)| &\lesssim |z^{-1}y|^{\mathcal{D}+1} \sup_{d(I)=\mathcal{D}+1,|z^{-1}u|\le b^{\mathcal{D}+1}|z^{-1}y|} |X^I\varphi_j(x^{-1}z^{-1}u)| \\ &\lesssim 2^{j(2n+\mathcal{D}+3)}|z^{-1}y|^{\mathcal{D}+1} \lesssim \frac{r^{\mathcal{D}+1}}{|z^{-1}x|^{2n+\mathcal{D}+3}}.\end{aligned}$$

By applying conditions $(a2)$ and $(a3)$ in Definition 2.2 as well as the Hölder inequality, we have

$$\begin{aligned}|a*\varphi_j(x)| &= \left|\int_{B_r(z)} a(y)\left[\varphi_j(x^{-1}y)-P(y)\right]dy\right| \lesssim \frac{r^{\mathcal{D}+1}}{|z^{-1}x|^{2n+\mathcal{D}+3}}\|a\|_1 \\ &\le \frac{r^{\mathcal{D}+1}}{|z^{-1}x|^{2n+\mathcal{D}+3}}\left\|\chi_{B_r(z)}\right\|_{q'}\|a\|_q = \frac{r^{\mathcal{D}+1}}{|z^{-1}x|^{2n+\mathcal{D}+3}}|B_r(z)|^{1-\frac{1}{q}}\|a\|_q \\ &\lesssim \frac{r^{2n+\mathcal{D}+3}}{|z^{-1}x|^{2n+\mathcal{D}+3}}\left\|\chi_{B_r(z)}\right\|^{-1}_{L^{p(\cdot)}_{\mathbb{H}^n}}.\end{aligned}$$

We write

$$\theta=\frac{2n+\mathcal{D}+3}{2n+2}>\frac{1}{p_-}.$$

Since

$$M\chi_{B_{2r}(z)}(x)\sim M_c\chi_{B_{2r}(z)}(x)\ge \frac{1}{|B_{|z^{-1}x|+2r}(z)|}\int_{\mathbb{H}^n}\chi_{B_{2r}(z)}(y)dy=\frac{|2r|^{2n+2}}{(|z^{-1}x|+2r)^{2n+2}},$$

we obtain

$$r^{-(2n+\mathcal{D}+3)}(M\chi_{B_{2r}(z)}(x))^\theta \gtrsim \frac{1}{(|z^{-1}x|+2r)^{2n+\mathcal{D}+3}} \gtrsim \frac{1}{|z^{-1}x|^{2n+\mathcal{D}+3}}.$$

Hence we have

$$\begin{aligned}
&\Big\| |z^{-1}\cdot|^{-(2n+\mathcal{D}+3)}\chi_{\mathbb{H}^n\setminus B_{2r}(z)}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \\
\lesssim & r^{-(2n+\mathcal{D}+3)}\Big\|(M\chi_{B_{2r}(z)})^\theta\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} = r^{-(2n+\mathcal{D}+3)}\Big\|M\chi_{B_{2r}(z)}\Big\|^\theta_{L^{\theta p(\cdot)}_{\mathbb{H}^n}} \\
\lesssim & r^{-(2n+\mathcal{D}+3)}\Big\|\chi_{B_{2r}(z)}\Big\|^\theta_{L^{\theta p(\cdot)}_{\mathbb{H}^n}} = r^{-(2n+\mathcal{D}+3)}\Big\|\chi_{B_{2r}(z)}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}.
\end{aligned}$$

Thus we can conclude that

$$\begin{aligned}
\Big\|(M_\varphi a)\chi_{\mathbb{H}^n\setminus B_{2r}(z)}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim & \bigg\| \frac{r^{2n+\mathcal{D}+3}}{|z^{-1}\cdot|^{2n+\mathcal{D}+3}}\Big\|\chi_{B_r(z)}\Big\|^{-1}_{L^{p(\cdot)}_{\mathbb{H}^n}}\chi_{\mathbb{H}^n\setminus B_{2r}(z)}\bigg\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \\
=& r^{2n+\mathcal{D}+3}\Big\|\chi_{B_r(z)}\Big\|^{-1}_{L^{p(\cdot)}_{\mathbb{H}^n}}\Big\||z^{-1}\cdot|^{-(2n+\mathcal{D}+3)}\chi_{\mathbb{H}^n\setminus B_{2r}(z)}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim 1.
\end{aligned}$$

Hence we get the desired conclusion. ☐

Theorem 4.2. *Suppose that $p(\cdot)$ is a function on $\mathbb{H}^n$ satisfying* (3.2), (3.2b) *and* $1<p_-\le p_+<\infty$. *Then for every r, $1<r<\infty$, and sequence of measurable functions $\{f_k\}_{k=1}^\infty$ we have*

$$\bigg\|\Big(\sum_{k=1}^\infty (Mf_k)^r\Big)^{\frac{1}{r}}\bigg\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \bigg\|\Big(\sum_{k=1}^\infty |f_k|^r\Big)^{\frac{1}{r}}\bigg\|_{L^{p(\cdot)}_{\mathbb{H}^n}}.$$

Proof. Denote by A_p the Muckenhoupt A_p weights as in [29, pp. 19]. By [29, Proposition 7.13], for each $p_0>1$ and for all $w\in A_{p_0}$, the Hardy-Littlewood maximal function is a bounded operator on weighted Lebesgue spaces on the Heisenberg group, i.e.,

$$\int_{\mathbb{H}^n} Mf(x)^{p_0}w(x)dx \lesssim \int_{\mathbb{H}^n} f(x)^{p_0}w(x)dx.$$

Then by virtue of the dyadic cubes given by [29, Theorem 2.2], combining the boundedness of Hardy-Littlewood maximal functions on $L^{p(\cdot)}(\mathbb{H}^n)$ when $p_->1$ introduced by [1, Theorem 1.7] with a similar discussion to [12, Corollary 5.34], we get the desired conclusion. ☐

We have defined in Definition 2.3 the function spaces $H^{p(\cdot),q}_{atom}(\mathbb{H}^n)$. Now we turn to define the function spaces $H^{p(\cdot),q}_{atom,*}(\mathbb{H}^n)$.

Definition 4.1. ($H^{p(\cdot),q}_{atom,*}(\mathbb{H}^n)$) Let $p(\cdot):\mathbb{H}^n\to(0,\infty)$, $0<p_-\le p_+<q\le\infty$ and $q\ge 1$. Then $f\in\mathcal{S}'(\mathbb{H}^n)$ is in $H^{p(\cdot),q}_{atom,*}(\mathbb{H}^n)$ if and only if there exist nonnegative number sequences $\{k_j\}_{j=1}^\infty$ and $\{(a_j,B_j)\}_{j=1}^\infty\subset A(p(\cdot),q)$ such that

$$f=\sum_{j=1}^\infty k_j a_j \quad \text{in } \mathcal{S}'(\mathbb{H}^n) \quad \text{and that} \quad \sum_{j=1}^\infty\int_{B_j}\left(\frac{k_j}{\left\|\chi_{B_j}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\right)^{p(x)}dx<\infty.$$

Define

$$\mathcal{A}^*(\{k_j\}_{j=1}^\infty,\{B_j\}_{j=1}^\infty)\equiv\inf\left\{\lambda>0:\sum_{j=1}^\infty\int_{B_j}\left(\frac{k_j}{\lambda\left\|\chi_{B_j}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\right)^{p(x)}dx\le 1\right\}. \tag{4.3}$$

For sequences of nonnegative numbers $\{k_j\}_{j=1}^\infty$, measurable subsets $\{E_j\}_{j=1}^\infty$ and balls $\{B_j\}_{j=1}^\infty$, define

$$\mathcal{A}(\{k_j\}_{j=1}^\infty,\{E_j\}_{j=1}^\infty,\{B_j\}_{j=1}^\infty)\equiv\inf\left\{\lambda>0:\int_{\mathbb{H}^n}\left\{\sum_{j=1}^\infty\left(\frac{k_j\chi_{E_j}(x)}{\lambda\left\|\chi_{B_j}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\right)^{\underline{p}}\right\}^{\frac{p(\cdot)}{\underline{p}}}dx\le 1\right\}. \tag{4.4}$$

Remark 4.1. 1. It follows from the embedding $\ell^{\underline{p}}\hookrightarrow\ell^1$ that

$$\mathcal{A}^*(\{k_j\}_{j=1}^\infty,\{B_j\}_{j=1}^\infty)\le\mathcal{A}(\{k_j\}_{j=1}^\infty,\{B_j\}_{j=1}^\infty). \tag{4.5}$$

2. Let $(a,B)\in A(p(\cdot),q)$. Then by (4.5), we have

$$\|a\|_{H^{p(\cdot),q}_{atom,*}(\mathbb{H}^n)}\le\|a\|_{H^{p(\cdot),q}_{atom}(\mathbb{H}^n)}\le 1. \tag{4.6}$$

3. Suppose $f\in L^q(\mathbb{H}^n)$ is supported on a ball B and satisfies

$$\int_{\mathbb{H}^n}x^\alpha f(x)dx=0$$

for all $d(\alpha)\le\mathcal{D}$. Let

$$\widetilde{f}=\frac{f|B|^{\frac{1}{q}}}{\|f\|_q\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}.$$

Then by Definition 2.2 we have $(\widetilde{f},B)\in A(p(\cdot),q)$ and hence it follows from (4.6) that

$$\|f\|_{H^{p(\cdot),q}_{atom,*}(\mathbb{H}^n)}\le\|f\|_{H^{p(\cdot),q}_{atom}(\mathbb{H}^n)}\le\frac{\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}{|B|^{\frac{1}{q}}}\|f\|_q. \tag{4.7}$$

Before proving the equivalence of Hardy norms given by atomic decompositions, we need to show the Calderón-Zygmund Decomposition on homogeneous group and some related conclusions from [19]. Since $\mathbb{H}^n$ is a kind of homogeneous group, we can use these them directly.

Theorem 4.3 (Calderón-Zygmund Decomposition)**.** *Let G be a homogeneous group. Suppose $f\in\mathcal{S}'(G)$ such that $|\{x:\mathcal{M}(x)>\alpha\}|<\infty$ for all $\alpha>0$. Set $\Omega=\{x:\mathcal{M}(x)>\alpha\}$. Then f can be decomposed as $f=g+\sum_i b_i$ with the following properties:*
(1) $\Omega=\bigcup_{j=1}^{\infty}B_j$,
(2) The balls $\frac{1}{4\gamma}B_j$ are disjoint,
(3) $T_2B_j\cap\Omega^c=\varnothing$, but $T_3B_j\cap\Omega^c\neq\varnothing$,
(4) There exists $L\in\mathbb{N}$ such that no point of Ω lies in more than L of the balls T_2B_j,
(5) $\int_{\mathbb{H}^n}b_i(x)x^\beta dx=0$ for all β with $d(\beta)\le a$,
(6) b_i is supported in $2B_i$,
where T_1, T_2, T_3, a and γ are constants from [19]. We call it Calderón-Zygmund decomposition of f of degree a and height α associated to $\mathcal{M}f$.

Lemma 4.2 (see [19])**.** *Let b_i be from the Calderón-Zygmund decomposition of f as in Theorem 4.3 and $\widetilde{B}_i=T_1B_i$. The constant a is from Theorem 4.3 and the constant N is given in the definition of $\mathcal{M}$. Then we have*

$$\mathcal{M}b_i(x)\lesssim\mathcal{M}f(x)\quad for\quad x\in\widetilde{B}_i$$

and

$$\mathcal{M}b_i(x)\lesssim\alpha\left(\frac{r_i}{|z_i^{-1}x|}\right)^{2n+2+b}\qquad for\quad x\notin\widetilde{B}_i,$$

where $b=min\{b'\in\triangle:b'>a\}$ if $a<N$ and $b=N$ if $a\ge N$ (Note that $\triangle=\mathbb{N}$ if G is the Heisenberg group and since the constants N and a can be chosen sufficiently large, we can choose b to be a large integer).

Lemma 4.3 (see [19])**.** *Suppose G is a homogeneous group and $\sum_i b_i$ converges in $\mathcal{S}'$. Then for all $x\in G$,*

$$\mathcal{M}g(x)\lesssim\alpha\sum_i\left(\frac{r_i}{|z_i^{-1}x|+r_i}\right)^{2n+2+b}+\mathcal{M}f(x)\chi_{\Omega^c}(x),$$

where b is as in Lemma 4.2.

Lemma 4.4. *Assume that g is given by Calderón-Zygmund decomposition of f as in Theorem 4.3. If $f\in L^p$, $1\le p<\infty$, then we have $\|g\|_\infty\lesssim\alpha$.*

The proof is the same as [19, Theorem 3.20(ii)]. Thus we omit it.

Now we show the equivalence of atomic Hardy norms.

Theorem 4.4. *If $p(\cdot)$ satisfies* (2.2), (3.2) *and* (3.2b), *then for all $f\in\mathcal{S}'(\mathbb{H}^n)$,*

$$\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}\sim\|f\|_{H^{p(\cdot),\infty}_{atom}(\mathbb{H}^n)}\sim\|f\|_{H^{p(\cdot),\infty}_{atom,*}(\mathbb{H}^n)}.$$

Proof. Since it is a direct deduction from Remark 4.1 that $\|f\|_{H^{p(\cdot),\infty}_{atom}(\mathbb{H}^n)} \geq \|f\|_{H^{p(\cdot),\infty}_{atom,*}(\mathbb{H}^n)}$. We show the other parts of this proof below.

Part 1. $\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \|f\|_{H^{p(\cdot),\infty}_{atom,*}(\mathbb{H}^n)}$.

Since the proof of this inequality is like [37, pp. 3690], we omit it.

Part 2. Let us prove $\|f\|_{H^{p(\cdot),\infty}_{atom}(\mathbb{H}^n)} \lesssim \|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}$.

Let $f\in H^{p(\cdot)}(\mathbb{H}^n)\cap L^{p_++1}(\mathbb{H}^n)$. For each $j\in\mathbb{Z}$, we write $\mathcal{O}_j\equiv\{x\in\mathbb{H}^n:\mathcal{M}f(x)>2^j\}$. Then it follows that $\mathcal{O}_{j+1}\subset\mathcal{O}_j$. Now we use the Calderón-Zygmund Decomposition introduced in Theorem 4.3 with $\alpha=2^j$ to decompose f as follow,

$$f=g_j+b_j, \qquad b_j=\sum_k b_{j,k}, \quad b_{j,k}=(f-P_{j,k})\zeta_{j,k}.$$

By Lemma 4.2 we have

$$\begin{aligned}\|M_\varphi b_j\|_{L^{p(\cdot)}_{\mathbb{H}^n}} &\leq \Big\|\sum_k M_\varphi b_{j,k}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \Big\|\sum_k \mathcal{M}f\cdot\chi_{\widetilde{B}_{j,k}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} + \Big\|\sum_k 2^j\left(\frac{r_{j,k}}{|z_{j,k}^{-1}\cdot|}\right)^{2n+2+b}\chi_{\mathbb{H}^n\setminus\widetilde{B}_{j,k}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\\ &\lesssim \Big\|\chi_{\mathcal{O}_j}\mathcal{M}f\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} + \Big\|\sum_k 2^j\left(M\chi_{\widetilde{B}_{j,k}}\cdot\right)^{\frac{2n+2+b}{2n+2}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}.\end{aligned}$$

We can choose $b=\mathcal{D}$. Then by Theorem 4.2 we have

$$\Big\|\sum_k 2^j\left(M\chi_{\widetilde{B}_{j,k}}\right)^{\frac{2n+2+b}{2n+2}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \Big\|\sum_k 2^j\chi_{\widetilde{B}_{j,k}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \Big\|2^j\chi_{\mathcal{O}_j}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \Big\|\chi_{\mathcal{O}_j}\mathcal{M}f\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}.$$

Thus it follows that

$$\|f-g_j\|_{H^{p(\cdot)}_{\mathbb{H}^n}} = \|b_j\|_{H^{p(\cdot)}_{\mathbb{H}^n}} \sim \|M_\varphi b_j\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \lesssim \Big\|\chi_{\mathcal{O}_j}\mathcal{M}f\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \to 0 \quad \text{as} \quad j\to\infty.$$

Next, by Lemma 4.4 we have $g_j\to 0$ uniformly as $j\to-\infty$. Hence it follows that

$$f=\sum_{j=-\infty}^{\infty}(g_{j+1}-g_j) \quad \text{in } \mathcal{S}'(\mathbb{H}^n). \tag{4.8}$$

By [19, pp. 101-102] we have $f=\sum_{j,k}h_{j,k}$ in $\mathcal{S}'(\mathbb{H}^n)$, where each $h_{j,k}$ is supported in $T_2B_{j,k}$ and satisfies the estimate $|h_{j,k}(x)|\leq C_2 2^k$ for some universal constant C_2 and the moment condition $\int_{\mathbb{H}^n}h_{j,k}(x)x^\beta dx$ for all $d(\beta)\leq\mathcal{D}$. Set

$$a_{j,k}\equiv\frac{h_{j,k}}{k_{j,k}}, \quad k_{j,k}\equiv C_2 2^k\Big\|\chi_{T_2B_{j,k}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}.$$

Then it follows that each $a_{j,k}$ is a $(p(\cdot),\infty)$-atom and that $f=\sum_{j,k}k_{j,k}a_{j,k}$ in $L^{p_++1}(\mathbb{H}^n)\approx H^{p_++1}(\mathbb{H}^n)$. By Theorem 4.3(4) we get

$$\begin{aligned}\mathcal{A}(\{k_{j,k}\}_{j,k},\{T_2B_{j,k}\}_{j,k})=&\inf\left\{\lambda>0:\int_{\mathbb{H}^n}\left\{\sum_{j,k}\left(\frac{k_{j,k}\chi_{T_2B_{j,k}}(x)}{\lambda\left\|\chi_{T_2B_{j,k}}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\right)^{\underline{p}}\right\}^{\frac{p(x)}{\underline{p}}}dx\leq 1\right\}\\ \sim&\inf\left\{\lambda>0:\int_{\mathbb{H}^n}\left\{\sum_{j=-\infty}^{\infty}\left(\frac{2^j\chi_{\mathcal{O}_j}(x)}{\lambda}\right)^{\underline{p}}\right\}^{\frac{p(x)}{\underline{p}}}dx\leq 1\right\}.\end{aligned}$$

Since the rest part of this proof is similar to [37, pp. 3692–3693], we omit it. $\square$

Then the following theorem shows the atomic decomposition for $A(p(\cdot),q)$ $(q\gg 1)$.

Theorem 4.5. *Under the assumptions that $p(\cdot)$ satisfies* (3.2), (3.2b) *and $0<p_-\leq p_+<q\leq\infty$ and $q\gg 1$ is large enough. Then for all $f\in\mathcal{S}'(\mathbb{H}^n)$,*

$$\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}\sim\|f\|_{H^{p(\cdot),q}_{atom}(\mathbb{H}^n)}.$$

Since the proof of this theorem is an analogue of [37, pp. 3694] by using the dyadic cubes on $\mathbb{H}^n$ given by [29, Theorem 2.2], we omit it and we only note that the inequality (4.4) used in the discussion of [37, pp. 3694] has already been generalized to the $\mathbb{H}^n$ version in the proof of Proposition 4.1.

As an application of Theorem 4.4 and Theorem 4.5 we study the molecular decomposition. Here we give the definition of molecules like in [37].

Definition 4.2 (Molecules)**.** Let $0<p_-\leq p_+<q\leq\infty$, $q\geq 1$ and $\mathcal{D}\in\mathbb{Z}\cap[\mathcal{D}_{p(\cdot)},\infty)$ be fixed. We call $\mathfrak{M}$ a $(p(\cdot),q)$-molecule centered at a ball $B=B_r(z)$ if the following conditions hold.

1. On B, $\mathfrak{M}$ satisfies

$$\|\mathfrak{M}\|_{L^q(B)}\leq\frac{|B|^{\frac{1}{q}}}{\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}.$$

2. Outside B, $\mathfrak{M}$ satisfies

$$|\mathfrak{M}(x)|\leq\frac{1}{\|\chi_B\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\left(1+\frac{|z^{-1}x|}{r}\right)^{-4n-2\mathcal{D}-7},$$

which is called the decay condition.

3. For all $d(\alpha)\leq\mathcal{D}$ we have $\int_{\mathbb{H}^n}\mathfrak{M}(x)x^\alpha dx=0$, which is called the moment condition.

By this definition $(p(\cdot),q)$-atoms are $(p(\cdot),q)$-molecules.

Theorem 4.6. *Under the assumptions that $0<p_-\le p_+<q\le\infty$, $q\ge 1$, $\mathcal{D}\in\mathbb{Z}\cap[\mathcal{D}_{p(\cdot)},\infty)$, we choose $q\ge 1$ to be large enough and $p(\cdot)$ satisfying* (3.2) *and* (3.2b).

Let $\{B_j\}_{j=1}^{\infty}=\Big\{B_{r_j}(z_j)\Big\}_{j=1}^{\infty}$ be a sequence of balls and, for each $j\in\mathbb{N}$, there is a $(p(\cdot),q)$-molecule $\mathfrak{M}_j$ centered at B_j. If a sequence of positive numbers $\{k_j\}_{j=1}^{\infty}$ satisfies

$$\mathcal{A}(\{k_j\}_{j=1}^{\infty},\{B_j\}_{j=1}^{\infty})=1,$$

which implies that

$$\int_{\mathbb{H}^n}\left\{\sum_{j=1}^{\infty}\left(\frac{k_j\chi_{B_j}(x)}{\left\|\chi_{B_j}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}}\right)^{\underline{p}}\right\}^{\frac{p(x)}{\underline{p}}}dx\le 1.$$

Then we have

$$\left\|\sum_{j=1}^{\infty}k_j\mathfrak{M}_j\right\|_{H^{p(\cdot)}_{\mathbb{H}^n}}\lesssim 1.$$

By using Theorem 3.1 and Corollary 3.1, we can prove the theorem in a similar way to [37, Theorem 5.2] with a more complicated calculation. Thus we omit the proof.

We turn to consider the boundedness of a class of singular integral operators as an application of molecular decomposition. Here and below we denote by b the constant from Theorem 3.1.

Theorem 4.7. *Let $T:L^2(\mathbb{H}^n)\to L^2(\mathbb{H}^n)$ be a singular integral operator with a kernel $k:\mathbb{H}^n\setminus\{0\}\to\mathbb{R}$ possessing the following properties:*

1.

$$A_m\equiv\sup_{x\in\mathbb{H}^n\setminus\{0\},d(\vartheta)=m}|x|^{2n+2+m}|X^{\vartheta}k(x)|<\infty$$

for every $m\in\mathbb{N}\cup\{0\}$, where $\vartheta\in(\mathbb{N}\cup\{0\})^{2n+1}$.

2. If f is a $L^2(\mathbb{H}^n)$-function with compact support, we have

$$Tf(x)=\int_{\mathbb{H}^n}f(xy^{-1})k(y)dy$$

for a.e. $x\notin supp(f)$.

Assume that $p(\cdot)$ satisfies (3.2), (3.2b) *and $0<p_-\le p_+<\infty$. The operator T can not only be extended to a $H^{\mu}(\mathbb{H}^n)-H^{\mu}(\mathbb{H}^n)$ bounded operator for all $0<\mu<\infty$ (see [19, Theorem 6.10]), but also be extended to a $H^{p(\cdot)}(\mathbb{H}^n)-L^{p(\cdot)}(\mathbb{H}^n)$ bounded operator and the operator norm of T depends only on b, $\|T\|_{L^2(\mathbb{H}^n)\to L^2(\mathbb{H}^n)}$ and a sequence of numbers $A_1,A_2,\cdots,A_N$, where N is a finite number depending only on $p(\cdot)$.*

Since by using Corollary 3.1 this can be proven in a similar way to [37, Proposition 5.3] with a more complicated calculation, we omit the proof.

Theorem 4.8. *Suppose* $0<p_-\le p_+<q<\infty$ *and* $\mathcal{D}\in\mathbb{Z}\cup[\mathcal{D}_{p(\cdot)},\infty)$. *Assume that* $q\ge 1$ *is large enough and* $p(\cdot)$ *satisfies* (3.2) *and* (3.2b). *Let* $k\in\mathcal{S}(\mathbb{H}^n)$ *and write*

$$A_m\equiv\sup_{x\in\mathbb{H}^n\setminus\{0\},d(\vartheta)=m}|x|^{2n+2+m}|X^{\vartheta}k(x)|,\qquad (m\in\mathbb{N}\cup\{0\},\vartheta\in(\mathbb{N}\cup\{0\})^{2n+1}).$$

Suppose B is a ball and $a\in L_c^\infty(\mathbb{H}^n)$ *with the following properties:*
1. supp $a\in B$.
2. $\|a\|_{L^\infty_{\mathbb{H}^n}}\le\|\chi_B\|^{-1}_{L^{p(\cdot)}_{\mathbb{H}^n}}$.
3. For all multiindex α *which satisfies* $d(\alpha)\le 4n+2\mathcal{D}+6$, *we have* $\int_{\mathbb{H}^n}x^\alpha a(x)dx=0$.
If the singular integral operator T is induced by the kernel k and bounded on $L^2(\mathbb{H}^n)$, *Then there exists a constant* C_0' *depending only on b,* $\|T\|_{L^2(\mathbb{H}^n)\to L^2(\mathbb{H}^n)}$ *and a sequence of numbers* $A_1,A_2,\cdots,A_N$, *where N is a finite number depending only on* $p(\cdot)$ *such that* $\frac{a*k}{C_0'}$ *is a* $(p(\cdot),q)$*-molecule centered at B.*

By using Corollary 3.1 we can prove this theorem in a similar way to [37, Proposition 5.4] with a more complicated calculation. Thus we omit the proof.

Theorem 4.9. *Assume that* $p(\cdot)$ *satisfies* (2.2), (3.2) *and* (3.2b). *Let* $k\in\mathcal{S}(\mathbb{H}^n)$ *and write*

$$A_m\equiv\sup_{x\in\mathbb{H}^n\setminus\{0\},d(\vartheta)=m}|x|^{2n+2+m}|X^{\vartheta}k(x)|,\qquad (m\in\mathbb{N}\cup\{0\},\vartheta\in(\mathbb{N}\cup\{0\})^{2n+1}).$$

Define a convolution operator T by

$$Tf(x)\equiv f*k(x),\qquad (f\in L^2(\mathbb{H}^n)).$$

The operator T which is bounded on $L^2(\mathbb{H}^n)$ *can be extended to a* $H^{p(\cdot)}(\mathbb{H}^n)-H^{p(\cdot)}(\mathbb{H}^n)$ *bounded operator and the norm of T depends only on b,* $\|T\|_{L^2(\mathbb{H}^n)\to L^2(\mathbb{H}^n)}$ *and the numbers* $A_1,A_2,\cdots,A_N$ *with N, a finite number, depending only on* $p(\cdot)$.

Since we can prove this theorem in a similar way to [37, Proposition 5.5] with a more complicated calculation, we omit the proof here.

5 Littlewood-Paley characterization

In this section, as in Theorem 3.3, we denote by $\mathcal{L}$ the sub-Laplace operator. The gradient operator on $\mathbb{H}^n$ is given by

$$\nabla=(X_1,X_2,\cdots,X_{2n}).$$

Let $\mathbb{R}_+$ be the set $(0,\infty)$. Since by [7, pp. 76] or [2, pp. 975] the point 0 may be neglected in the spectral resolution of $\mathcal{L}$, we denote by ϕ the kernel function of $\phi^*(\mathcal{L})$ for each function

$\phi^*\in\mathcal{S}(\mathbb{R}_+)$, that is, $\widehat{f*\phi}(\lambda)=\widehat{\phi}(\lambda)\widehat{f}(\lambda)=\widehat{\phi^*(\mathcal{L})f}(\lambda)$. As in [32, pp. 97] we can calculate that

$$\phi(z,t)=(2\pi)^{-n-1}\sum_{k=0}^{\infty}\int_{-\infty}^{+\infty}e^{-i\lambda t}L_k^{n-1}\left(\frac{|\lambda||z|^2}{2}\right)e^{\frac{|\lambda||z|^2}{4}}\phi^*((2k+n)|\lambda|)|\lambda|^n d\lambda,$$

where L_k^{n-1} are Laguerre polynomials. We see that $\phi(z,t)$ depends only on $|z|$ and t. By [20, Proposition 6] we know that ϕ is a Schwartz function on $\mathbb{H}^n$. Let $\phi^{k*}(\lambda)=\phi^*(2^{-2k}\lambda)$. Then by a coordinate transformation we can easily see that ϕ_k is the kernel function of $\phi^{k*}(\mathcal{L})$.

We introduce the $\ell^2(\mathbb{Z})$ valued function space $H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))$.

Definition 5.1 ($H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))$)**.** Let $\psi^*\in\mathcal{S}(\mathbb{R}_+)$ be such that $\chi_{(0,1)}\le\psi^*\le\chi_{(0,2)}$ and ψ be the kernel function of $\psi^*(\mathcal{L})$. We define $H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))$ to be the function space equipped with the following norm:

$$\left\|\{f_j\}_{j=-\infty}^{\infty}\right\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}\equiv\left\|\sup_{k\in\mathbb{Z}}\left(\sum_{j=-\infty}^{\infty}|f_j*\psi_k|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}.$$

Theorem 5.1. *Suppose that $p(\cdot)$ satisfies* (2.2), (3.2) *and* (3.2b). *Let $T=\{T_k\}_{k\in\mathbb{Z}}$ be a collection of $L^2(\mathbb{H}^n;\ell^2(\mathbb{Z}))-L^2(\mathbb{H}^n)$ bounded operators such that there exists a sequence of functions $\{k_{i,j}\}_{i,j\in\mathbb{Z}}\subset\mathcal{S}(\mathbb{H}^n)$ possessing the following properties:*

1. $\sup\limits_{x\in\mathbb{H}^n,d(\vartheta)=m}|x|^{2n+2+m}\left\|\{X^{\vartheta}k_{i,j}(x)\}_{i,j\in\mathbb{Z}}\right\|_{\ell^2(\mathbb{Z})}<\infty$ for every $m\in\mathbb{N}\cup\{0\}$, where $\vartheta\in(\mathbb{N}\cup\{0\})^{2n+1}$.

2. If $\{f_j\}_{j=-\infty}^{\infty}$ is a $L^2(\mathbb{H}^n;\ell^2(\mathbb{Z}))$-function with compact support, then we have

$$T_i[\{f_j\}_{j=-\infty}^{\infty}](x)=\sum_{j=-\infty}^{\infty}f_j*k_{i,j}(x),\quad i\in\mathbb{Z}\ \text{for}\ x\in\mathbb{H}^n.$$

3. $k_{i,j}\equiv 0$ if $|i|+|j|$ is large enough. Then we have

$$\left\|\{T_i[\{f_j\}_{j=-\infty}^{\infty}]\}_{i=-\infty}^{\infty}\right\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}\lesssim\left\|\{f_j\}_{j=-\infty}^{\infty}\right\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}.\tag{5.1}$$

By re-examining some related assertions (see Theorems 4.5, 4.7 and 4.9) we can give its proof. We, therefore, omit the details (see [37, Theorem 5.6] for the $\mathbb{R}^n$ version).

Lemma 5.1. *Suppose that $\Omega\in\mathbb{R}_+$ is a compact set, $\varphi^*\in\mathcal{S}(\mathbb{R}_+)$ and $supp\ \varphi^*\subset\Omega$. Let φ be the kernel function of $\varphi^*(\mathcal{L})$. Then for $0<r<\infty$ and $f\in\mathcal{S}'(\mathbb{R}_+)$ we have*

$$\sup_{z\in\mathbb{H}^n}\frac{|\nabla f*\varphi(xz)|}{1+|z|^{\frac{2n+2}{r}}}\lesssim\sup_{z\in\mathbb{H}^n}\frac{|f*\varphi(xz)|}{1+|z|^{\frac{2n+2}{r}}}\lesssim((M|f*\varphi|^r)(x))^{\frac{1}{r}}.$$

Proof. 1° Let the function $\psi^*\in\mathcal{S}(\mathbb{R}_+)$ satisfy $\psi^*(x)=1$ whenever $x\in\Omega$. Then we have $\varphi^*=\varphi^*\cdot\psi^*$, which indicates that $\varphi=\varphi*\psi$. Hence it follows that

$$|X_i f*\varphi(x^{-1}z)|=\left|\int_{\mathbb{H}^n} f*\varphi(y)X_i\psi(y^{-1}x^{-1}z)dy\right|\lesssim\left|\int_{\mathbb{H}^n}|f*\varphi(y)|(1+|y^{-1}x^{-1}z|)^{-\lambda}dy\right|,$$

where $i=1,2,\cdots,2n$ and λ is chosen sufficiently large. We divide both sides of this modified estimate by $1+|z|^{\frac{2n+2}{r}}$ and use the inequality

$$\frac{1+|xy|^{\frac{2n+2}{r}}}{1+|z|^{\frac{2n+2}{r}}}\lesssim 1+|y^{-1}x^{-1}z|^{\frac{2n+2}{r}},\qquad (x,y,z\in\mathbb{H}^n).$$

Then it follows that

$$\sup_{z\in\mathbb{H}^n}\frac{|\nabla f*\varphi(x^{-1}z)|}{1+|z|^{\frac{2n+2}{r}}}\lesssim\sup_{z\in\mathbb{H}^n}\int_{\mathbb{H}^n}\frac{|f*\varphi(y)|}{1+|xy|^{\frac{2n+2}{r}}}(1+|y^{-1}x^{-1}z|^{\frac{2n+2}{r}})^{-\varepsilon}dy,$$

where ε is chosen large enough. Hence by a coordinate transformation we have

$$\sup_{z\in\mathbb{H}^n}\frac{|\nabla f*\varphi(xz)|}{1+|z|^{\frac{2n+2}{r}}}\lesssim\sup_{z\in\mathbb{H}^n}\frac{|f*\varphi(xz)|}{1+|z|^{\frac{2n+2}{r}}}. \tag{5.2}$$

2° Suppose $g(z)$ is a continuously differentiable function supported in the closed ball bB, where b is the constant from Theorem 3.1 and $B=\{x\in\mathbb{H}^n:|x|\le 1\}$ denotes the closed unit ball of $\mathbb{H}^n$. By the stratified mean value theorem (Theorem 3.1), for $z\in B$, we have

$$\begin{aligned}|g(z)|&\lesssim\min_{w\in B}|g(w)|+\sup_{w'\in bB}|\nabla g(w')|\\&\le\sup_{w'\in bB}|\nabla g(w')|+\left(\int_B|g(w)|^r dw\right)^{\frac{1}{r}}.\end{aligned}$$

Replacing $g(z)$ by $g(\delta z)$ we have

$$|g(z)|\lesssim\delta\sup_{w'\in\delta bB}|\nabla g(w')|+\delta^{-\frac{2n+2}{r}}\left(\int_{\delta B}|g(w)|^r dw\right)^{\frac{1}{r}},\qquad (z\in\delta B).$$

3° Let $g(z)=f*\varphi(xz)$ we have

$$|f*\varphi(xz)|\lesssim\delta\sup_{|y'|\le\delta b}|\nabla f*\varphi(xzy')|+\delta^{-\frac{2n+2}{r}}\left(\int_{|y|\le\delta}|f*\varphi(xzy)|^r dy\right)^{\frac{1}{r}}. \tag{5.3}$$

Let $0<\delta\le 1$. Then the last integral from the right-hand side of the above inequality can be estimated by

$$\left(\int_{|u|\le|z|+1}|f*\varphi(xu)|^r du\right)^{\frac{1}{r}}\lesssim(1+|z|^{\frac{2n+2}{r}})[(M|f*\varphi|^r)(x)]^{\frac{1}{r}}.$$

We substitute this estimate into (5.3) and divide both sides by $1+|z|^{\frac{2n+2}{r}}$. Taking the supremum with respect to $z\in\mathbb{H}^n$, we obtain

$$\sup_{z\in\mathbb{H}^n}\frac{|f*\varphi(xz)|}{1+|z|^{\frac{2n+2}{r}}}\lesssim\delta\sup_{z\in\mathbb{H}^n}\frac{|\nabla f*\varphi(xz)|}{1+|z|^{\frac{2n+2}{r}}}+\delta^{-\frac{2n+2}{r}}[(M|f*\varphi|^r)(x)]^{\frac{1}{r}}. \tag{5.4}$$

Substituting (5.4) in (5.2) and choosing δ sufficiently small, we obtain

$$\sup_{z\in\mathbb{H}^n}\frac{|f*\varphi(xz)|}{1+|z|^{\frac{2n+2}{r}}}\lesssim((M|f*\varphi|^r)(x))^{\frac{1}{r}}.$$

Thus, we complete the proof. $\square$

Now we consider the Littlewood-Paley characterization of variable Hardy spaces.

Theorem 5.2. *Assume that $\varphi^*\in\mathcal{S}(\mathbb{R}_+)$, $supp\varphi^*\subset\{x:\frac{1}{4}\le x\le 4\}$ and $\sum_{j=-\infty}^{\infty}|\varphi^*(\lambda)|^2>0$ for all $\lambda\in\mathbb{R}_+$. Let φ be the kernel function of $\varphi^*(\mathcal{L})$. Then the following norm is equivalent to the norm of $H^{p(\cdot)}$:*

$$\|f\|_{\dot{F}^0_{p(\cdot)2}}=\left\|\left(\sum_{j=-\infty}^{\infty}|f*\varphi_j|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}},\qquad f\in\mathcal{S}'(\mathbb{H}^n). \tag{5.5}$$

Proof. Let

$$\zeta^*(\lambda)=\frac{\varphi^*(\lambda)}{\sum_{j\in\mathbb{Z}}|\varphi^{j*}(\lambda)|^2}$$

and ζ be the kernel function of $\zeta^*(\mathcal{L})$. Then we can easily see that $\zeta^*\in\mathcal{S}(\mathbb{R}_+)$ and $\sum_{j=-\infty}^{\infty}\zeta^{j*}\varphi^{j*}\equiv\chi_{\mathbb{R}_+}$. Note that by the theorem given in [2, pp. 974], for each function $\phi^*\in\mathcal{S}(\mathbb{R}_+)$, $\phi^*(\mathcal{L})$ extends to a bounded operator on $L^p(\mathbb{H}^n)$, $1<p<\infty$. If we define $T=\{T_i\}_{i\in\mathbb{Z}}$ by $T_i[\{f_j\}_{j=-\infty}^{\infty}]=\varphi*f_0$, then T satisfies the conditions 1 and 2 in Theorem 5.1. Hence we have

$$\left\|\{f*\varphi_j\}_{j=-\infty}^{\infty}\right\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}\lesssim\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}.$$

Fix $N\in\mathbb{N}$. If we choose k to be sufficiently large, by the definition of ψ^* for all $|i|\le N$ we have $\psi^{k*}\varphi^{i*}=\varphi^{i*}$, which indicates that $\psi_k*\varphi_i=\varphi_i$. Hence we have

$$\left\|\left(\sum_{i=-N}^{N}|f*\varphi_i|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\lesssim\left\|\{f*\varphi_j\}_{j=-\infty}^{\infty}\right\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}$$

uniformly over N. Then we obtain

$$\|f\|_{\dot{F}^0_{p(\cdot)2}}=\left\|\left(\sum_{i=-\infty}^{\infty}|f*\varphi_i|^2\right)^{\frac{1}{2}}\right\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\lesssim\left\|\{f*\varphi_j\}_{j=-\infty}^{\infty}\right\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}\lesssim\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}.$$

If we define $T=\{T_i\}_{i\in\mathbb{Z}}$ by

$$T_i[\{f_j\}_{j=-\infty}^{\infty}](x)\equiv\begin{cases}\sum\limits_{k=-\infty}^{\infty} f_k*\zeta_k, & \text{if } i=0,\\ 0, & \text{if } i\neq 0.\end{cases}$$

By (5.1) we have

$$\Big\|\sum_{k=-\infty}^{\infty} f_k*\zeta_k\Big\|_{H^{p(\cdot)}_{\mathbb{H}^n}}\lesssim\Big\|\{f_j\}_{j=-\infty}^{\infty}\Big\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}.$$

If we let $f_j=f*\varphi_j$ $(j\in\mathbb{Z})$, then by [28, Lemma 3], we can see that

$$\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}}\lesssim\Big\|\{f*\varphi_j\}_{j=-\infty}^{\infty}\Big\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}.$$

Since $\chi_{(0,1)}\le\psi^*\le\chi_{(0,2)}$, we have

$$\psi_k*\varphi_j\begin{cases}\varphi_j, & \text{if } j<k-1,\\ 0, & \text{if } j>k+2.\end{cases}$$

Set $k-1\le j\le k+2$, $L>\frac{2n+2}{\theta}+2n+3$ and $\theta=\frac{1}{2}\underline{p}$. Then by Lemma 5.1 we have

$$\begin{aligned}|f*\varphi_j*\psi_k(x)|&\lesssim\int_{\mathbb{H}^n}|f*\varphi_j(y)\cdot 2^{k(2n+2)}\psi(2^k(y^{-1}x))|dy\\ &\lesssim\int_{\mathbb{H}^n}\frac{2^{k(2n+2)}}{(1+2^k|y^{-1}x|)^L}(1+2^j|y^{-1}x|)^{\frac{2n+2}{\theta}}M[|f*\varphi_j|^{\theta}](x)^{\frac{1}{\theta}}dy\lesssim M[|f*\varphi_j|^{\theta}](x)^{\frac{1}{\theta}}.\end{aligned}$$

Hence by using Theorem 4.2, we have

$$\begin{aligned}\Big\|\{f*\varphi_j\}_{j=-\infty}^{\infty}\Big\|_{H^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))}&=\Big\|\sup_{k\in\mathbb{Z}}\Big(\sum_{j=-\infty}^{\infty}|f*\varphi_j*\psi_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\\ &\lesssim\Big\|\Big(\sum_{j=-\infty}^{\infty}M[|f*\varphi_j|^{\theta}](x)^{\frac{2}{\theta}}\Big)^{\frac{1}{2}}\Big\|_{L^{p(\cdot)}_{\mathbb{H}^n}}\\ &\lesssim\Big\|\{f*\varphi_j\}_{j=-\infty}^{\infty}\Big\|_{L^{p(\cdot)}(\mathbb{H}^n;\ell^2(\mathbb{Z}))},\end{aligned}$$

which implies the desired result. □

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