An Estimate on Riemannian Manifolds of Dimension 4

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Abstract. We give an estimate of type sup × inf on Riemannian manifold of dimension 4 for a Yamabe type equation.

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1 Introduction and main results

In this paper, we deal with the following Yamabe type equation in dimension \( n = 4 \):

\[
\Delta_g u + hu = 8u^3, \quad u > 0. \tag{1.1}
\]

Here, \( \Delta_g = -\nabla^i(\nabla_i) \) is the Laplace-Beltrami operator and \( h \) is an arbitrary bounded function.

The Eq. (1.1) was studied a lot, when \( M = \Omega \subset \mathbb{R}^n \) or \( M = S_n \) see for example, [2–4, 11, 15]. In this case we have a sup × inf inequality. The corresponding equation in two dimensions on open set \( \Omega \) of \( \mathbb{R}^2 \), is:

\[
\Delta u = V(x)e^u. \tag{1.2}
\]

The Eq. (1.2) was studied by many authors and we can find very important result about a priori estimates in [8,9,12,16] and [19]. In particular in [9], we have the following interior estimate:

\[
\sup_K u \leq c = c\left(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, \inf_{\Omega} u, K, \Omega\right).
\]

And, precisely, in [8,12,16] and [20], we have:

\[
C\sup_K u + \inf_{\Omega} u \leq c = c\left(\inf_{\Omega} V, \|V\|_{L^\infty(\Omega)}, K, \Omega\right),
\]

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\[
\sup_K u + \inf \Omega u \leq c \left( \inf \Omega V \parallel V \parallel_{C^\alpha(\Omega), K, \Omega} \right),
\]
where \( K \) is a compact subset of \( \Omega \), \( C \) is a positive constant which depends on
\[
\inf \Omega V \sup \Omega V
\]
and \( \alpha \in (0,1] \). When \( 6h = R_g \) the scalar curvature, and \( M \) compact, the Eq. (1.1) is Yamabe equation. T. Aubin and R. Schoen have proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary. When \( M \) is a compact Riemannian manifold, there exist some compactness results for Eq. (1.1) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose \( M \) not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose \( M \) a Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product \( \sup \times \inf \) is bounded. Here we extend the result of [5]. Our proof is an extension Li-Zhang result in dimension 3, see [3] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [3,6,10,11,16,17], some applications of this method, for example an uniqueness result. We refer to [7] for the uniqueness result on the sphere and in dimension 3. Here, we give an equality of type \( \sup \times \inf \) for the Eq. (1.1) in dimension 4. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3,5]. There are other estimates of type \( \sup + \inf \) on complex Monge-Ampere equation on compact manifolds, see [20,21]. They consider, on compact Kahler manifold \((M,g)\), the following equation:
\[
\begin{align*}
(\omega_g + \partial \bar{\partial} \varphi)^n &= e^{f-t\varphi} \omega^n_g, \\
\omega_g + \partial \bar{\partial} \varphi &> 0 \quad \text{on} \ M.
\end{align*}
\]
And, they prove some estimates of type \( \sup_M + \inf_M \leq C \) or \( \sup_M + \inf_M \geq C \) under the positivity of the first Chern class of \( M \). Here, we have,

**Theorem 1.1.** For all compact set \( K \) of \( M \), there is a positive constant \( c \), which depends only on, \( h_0 = \|h\|_{L^\infty(M)} \), \( K \), \( M \), \( g \) such that:
\[
\left( \sup_K u \right)^{1/3} \inf_M u \leq c,
\]
for all \( u \) solution of (1.1).

Here we consider more general equation and this theorem extends a result of Li and Zhang, see [17]. Li and Zhang considered precisely the Yamabe equation and here we consider a general equation \((h \neq \frac{1}{3} R_g \text{ with } R_g \text{ the scalar curvature})\). Here, we use a different method than the method of Li and Zhang in [17]. Also, we extend a result of [5].
Corollary 1.1. For all compact set \( K \) of \( M \) there is a positive constant \( c \), such that:

\[
\sup_K u \leq c = c(g, m, h_0, K, M) \quad \text{if} \quad \inf_M u \geq m > 0,
\]

for all \( u \) solution of (1.1).

2 Proof of the results

Proof of Theorem 1.1. Let \( x_0 \) be a point of \( M \). We want to prove a uniform estimate around \( x_0 \). We argue by contradiction, we assume that the \( \sup \times \inf \) is not bounded.

\[
\forall c, R > 0, \exists u_{c, R} \text{ solution to (1.1) such that:}
\]

\[
R^2 \left( \sup_{B(x_0, R)} u_{c, R} \right)^{1/3} \times \inf_M u_{c, R} \geq c. \tag{2.1}
\]

Proposition 2.1 (Blow-Up Analysis). There is a sequence of points \((y_i)_i, y_i \to x_0\) and two sequences of positive real numbers \((l_i)_i, (L_i)_i, l_i \to 0, L_i \to +\infty\), such that if we set

\[
v_i(y) = \frac{u_i[\exp_{y_i}(y/[u_i(y_i)])]}{u_i(y_i)},
\]

we have:

\[
0 < v_i(y) \leq \beta_i \leq 2, \quad \beta_i \to 1,
\]

\[
v_i(y) \to \frac{1}{1 + |y|^2} \text{ uniformly on compact sets of } \mathbb{R}^4,
\]

\[
l_i^2(u_i(y_i))^{1/3} \min_M u_i \to +\infty.
\]

Proof. We use the hypothesis (2.1), we take two sequences, \( R_i > 0, R_i \to 0 \) and \( c_i \to +\infty \), such that,

\[
R_i^2 \left( \sup_{B(x_0, R_i)} u_{c_i, R_i} \right)^{1/3} \times \inf_M u_{c_i, R_i} \geq c_i \to +\infty. \tag{2.2}
\]

Let, \( x_i \in B(x_0, R_i) \), such that \( \sup_{B(x_0, R_i)} u_i = u_i(x_i) \) and \( s_i(x) = [R_i - d(x, x_i)](u_i(x))^{1/6}, x \in B(x_i, R_i) \). Then, \( x_i \to x_0 \). We have:

\[
\sup_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i(u_i(x_i))^{1/6} \geq c_i^{1/2} \to +\infty
\]

with \( y_i \in B(x_i, R_i) \).

We set:

\[
l_i = R_i - d(y_i, x_i), \quad \tilde{u}_i(y) = u_i[\exp_{y_i}(y)], \quad v_i(z) = \frac{u_i[\exp_{y_i}(z/[u_i(y_i)])]}{u_i(y_i)}.
\]
Clearly, we have, $y_i \to x_0$. We obtain:
\[
 L_i = \frac{l_i}{(c_i)^{1/4}} \left[ u_i(y_i) \right] \geq \frac{c_i^{1/2}}{c_i^{1/4}} = c_i^{1/4} \to +\infty.
\]

If $|z| \leq L_i$, then $y = \exp_{y_i} \left[ z / |u_i(y_i)| \right] \in B(y_i, \delta_i l_i)$ with $\delta_i = \frac{1}{(c_i)^{1/4}}$ and $d(y_i, x_i) < R_i - d(y_i, x_i)$, thus, $d(y, x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$. We can write,
\[
 (u_i(y))^{1/6} [R_i - d(y, x_i)] \leq (u_i(y_i))^{1/6} l_i.
\]

But, $d(y, y_i) \leq \delta_i l_i$, $R_i > l_i$ and $R_i - d(y, x_i) \geq R_i - d(x_i, y_i) - \delta_i l_i > l_i - \delta_i l_i = l_i (1 - \delta_i)$, hence, we obtain,
\[
 0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left( \frac{l_i}{l_i(1 - \delta_i)} \right)^{6} \leq 2^{6}.
\]

We set, $\beta_i = \frac{1}{1 - v_i}$, clearly $\beta_i \to 1$.

Because $R_i \geq l_i$ we have $u_i(y_i) \geq u_i(x_i)$ using the fact that $s_i(y_i) \geq s_i(x_i)$ we obtain:
\[
 l_i^{2} (u_i(y_i))^{1/3} \times \inf_{M} u_i \to +\infty.
\]

Thus, we complete the proof. \qed

**Remark 2.1.** We can consider $s_i(x) = (R_i - d(y, x_i)) u_i(x)$ and in this case we can replace $l_i$ by $R_i$ to have the last assertion of the proposition (our computations do not change):
\[
 R_i^{2} (u_i(y_i))^{1/3} \times \inf_{M} u_i \to +\infty.
\]

The function $v_i$ satisfies the following equation:
\[
 -g^{jk}(z) \partial_{j} v_i - \partial_{k} \left[ g^{jk} \sqrt{|g|} \right] (z) \partial_{j} v_i + \frac{h(z)}{u_i(y_i)^{2}} v_i = 8 v_i^{3} \tag{2.3}
\]

with $g^{jk}(z) = g^{jk}(\exp_{y_i}(z / u_i(y_i)))$.

We use Ascoli and Ladyzenskaya theorems (see [1]) to obtain the local uniform convergence (on every compact set of $\mathbb{R}^{4}$) of $(v_i)_i$ to $v$ solution on $\mathbb{R}^{4}$ to:
\[
 \Delta v = 8 v^{3}, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2.
\]

By the maximum principle, we have $v > 0$ on $\mathbb{R}^{n}$. According to Caffarelli-Gidas-Spruck result (see [10]), we have, $v(y) = \frac{1}{1 + |y|^{p}}$. 

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Polar geodesic coordinates

Let $\nu$ be a function on $M$. We denote $g_{x,i,j}$ the local expression of the metric $g$ in the exponential chart centered at $x$. We set,

$$w_i(t, \theta) = e^t u_i \exp \left( e^t \theta \right),$$

$$a(y_i, t, \theta) = \log J(y_i, e^t, \theta) = \log \left( \sqrt{\det(g_{y,i,j})} \right).$$

We can write the Laplace-Beltrami operator in polar geodesic coordinates:

$$-\Delta u = \partial_{rr} u + 3 r \partial_r u - \frac{1}{r^2} \Delta_{\theta} u. \quad (2.4)$$

We deduce the two following lemmas:

**Lemma 2.1.** The function $w_i$ is a solution to:

$$- \partial_{tt} w_i - \partial_t a \partial_t w_i - \Delta_{\theta} w_i + cw_i = 8 w_i^3 \quad (2.5)$$

with

$$c(y_i, t, \theta) = 1 + \partial_t a + he^{2t}.$$

**Proof.** We write:

$$\partial_t w_i = e^t \partial_r u_i + w_i,$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta),$$

$$\partial_t a \partial_t w_i = e^{3t} \left[ \partial_r \log J(y_i, e^t, \theta) \right].$$

Lemma 2.1 follows. \qed

Let $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$. We can write:

$$- \frac{1}{\sqrt{b_1}} \partial_{tt} (\sqrt{b_1} w_i) - \Delta_{\theta} w_i + \left[ c(t) + b_1^{-1/2} b_2(t, \theta) \right] w_i = 8 w_i^3,$$

where,

$$b_2(t, \theta) = \partial_{tt} (\sqrt{b_1}) = \frac{1}{2 \sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4 (b_1)^{3/2}} (\partial_t b_1)^2.$$

We set,

$$\tilde{w}_i = \sqrt{b_1} w_i.$$

**Lemma 2.2.** The function $\tilde{w}_i$ is a solution to:

$$- \partial_{tt} \tilde{w}_i + \Delta_{\theta} \tilde{w}_i + 2 \nabla_{\theta} (\tilde{w}_i) \cdot \nabla_{\theta} \log (\sqrt{b_1}) + \left( c + b_1^{-1/2} b_2 - c_2 \right) \tilde{w}_i = 8 \left( \frac{1}{b_1} \right) \tilde{w}_i^3, \quad (2.6)$$

where, $c_2$ is a function to be determined.
Proof. We have:

$$-\partial_{tt}\tilde{w}_i - \sqrt{b_1}\Delta_\theta w_i + (c + b_2)\tilde{w}_i = 8\left(\frac{1}{b_1}\right)\tilde{w}_i^3,$$

But,

$$\Delta_\theta(\sqrt{b_1}w_i) = \sqrt{b_1}\Delta_\theta w_i - 2\nabla_\theta w_i \cdot \nabla_\theta \sqrt{b_1} + w_i \Delta_\theta(\sqrt{b_1}),$$

and

$$\nabla_\theta(\sqrt{b_1}w_i) = w_i \nabla_\theta \sqrt{b_1} + \sqrt{b_1} \nabla_\theta w_i,$$

we can write,

$$\nabla_\theta w_i \cdot \nabla_\theta \sqrt{b_1} = \nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) - \tilde{w}_i |\nabla_\theta \log(\sqrt{b_1})|^2,$$

we deduce,

$$\sqrt{b_1}\Delta_\theta w_i = \Delta_\theta(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) - c_2\tilde{w}_i,$$

with

$$c_2 = \left[\frac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2\right].$$

Lemma 2.2 is proved.

The Moving-Plane method

Let $\xi_i$ be a real number, we assume $\xi_i \leq t$. We set $t^{\xi_i} = 2\xi_i - t$ and $\tilde{w}_i(\xi_i) = \tilde{w}_i(t^{\xi_i},\theta)$. Set, $\lambda_i = -\log u_i(y_i)$.

Proposition 2.2. We claim: there exists a positive constant $\tilde{k}$ such that:

$$\tilde{w}_i(\lambda_i,\theta) - \tilde{w}_i(\lambda_i + 4,\theta) \geq \tilde{k} > 0, \quad \forall \theta \in S_3. \quad (2.7)$$

For all $\beta > 0$, there exists $c_\beta > 0$ such that:

$$\frac{1}{c_\beta} \leq \tilde{w}_i(\lambda_i + t,\theta) \leq c_\beta \tilde{e}_i, \quad \forall t \leq \beta, \quad \forall \theta \in S_3. \quad (2.8)$$

Proof. As in [2], there exists a positive constant $k$ such that, $w_i(\lambda_i,\theta) - w_i(\lambda_i + 4,\theta) \geq k > 0$ for $i$ large, $\forall \theta$. We can remark that $b_1(y_i,\lambda_i,\theta) \to 1$ and $b_1(y_i,\lambda_i + 4,\theta) \to 1$ uniformly in $\theta$, we obtain the first claim of Proposition 2.2. For the second claim we use Proposition 2.1, see also [2]. We set:

$$\bar{Z}_i = -\partial_{tt}(\bar{\cdots}) + \Delta_\theta(\bar{\cdots}) + 2\nabla_\theta(\bar{\cdots}) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)(\bar{\cdots}). \quad (2.9)$$

We complete the proof.

Remark 2.2. In the operator $\bar{Z}_i$, we can remark that:

$$c + b_1^{-1/2}b_2 - c_2 \geq k' > 0 \quad \text{for} \quad t \ll 0,$$

we can apply the maximum principle and the Hopf lemma.
Goal

Like in [2], we have an elliptic second order operator. Here it is $Z_i$, the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$Z_i (\tilde{w}^i \bar{t} - \tilde{w}_i) \leq 0, \quad \text{if} \quad \tilde{w}^i - \tilde{w}_i \leq 0. \tag{2.10}$$

We write, $\Delta = \Delta^s_{\theta, e} s_{n-1}$. We obtain:

$$Z_i (\tilde{w}^i \bar{t} - \tilde{w}_i) = (\Delta^s_{\theta, e} s_{n-1}) (\tilde{w}^i)$$

$$+ 2(\nabla_{\theta, e} \tilde{w}_i - \nabla_{\theta, e} \tilde{w}_i) (\tilde{w}^i) \cdot \nabla_{\theta, e} \tilde{w}_i \log(\sqrt{b_1^i}) + 2\nabla_{\theta, e} (\tilde{w}^i) \cdot \nabla_{\theta, e} \tilde{w}_i \log(\sqrt{b_1^i}) - \log \sqrt{b_1}$$

$$+ 2\nabla_{\theta, e} \tilde{w}_i (\nabla_{\theta, e} \tilde{w}_i - \nabla_{\theta, e} \tilde{w}_i) \log \sqrt{b_1} - [(c + b_1^{-1/2} b_2 - c_2) \tilde{w}_i - (c + b_1^{-1/2} b_2 - c_2)] \tilde{w}^i$$

$$+ 8 \left(1 \over b_1^i\right) (\tilde{w}^i \bar{t} - \tilde{w}_i) \left[\left(1 \over b_1^i\right) \tilde{w}^i \bar{t} - \tilde{w}_i \right]. \tag{2.11}$$

Clearly, we have the following lemma:

**Lemma 2.3.** It holds

$$b_1 (y_i, t, \theta) = 1 - \frac{1}{3} Ricci_{yi}(\theta, \theta) e^{2t} + \cdots,$$

$$R_s (e^t \theta) = R_s (y_i) + <\nabla R_s (y_i) \theta > e^t + \cdots.$$ 

By the previous computations and Lemma 2.3, we have:

**Proposition 2.3.** It holds

$$Z_i (\tilde{w}^i \bar{t} - \tilde{w}_i) \leq 8 \left(1 \over b_1^i\right) [(\tilde{w}^i \bar{t} - \tilde{w}^i \bar{t})^3 - \tilde{w}^3 \bar{t}] + C |e^{2t} - e^{2t} | |(|\nabla_{\theta, e} \tilde{w}^i \bar{t}| + |\nabla^2_{\theta, e} \tilde{w}^i \bar{t}|)|$$

$$+ C |e^{2t} - e^{2t} | |(|Ricci_{yi} | + |h)|) \tilde{w}^i \bar{t} + C \tilde{w}^i \bar{t} |e^{3t} | - e^{3t} |. \tag{2.12}$$

**Proof.** In polar geodesic coordinates (and the Gauss lemma):

$$g = dt^2 + \frac{r^2 g^k_{ij} d \theta^i d \theta^j}{a^k(\theta)} \text{ at } \sqrt{\left|g^k\right|} = a^k(\theta) \sqrt{[\text{det} \{g_{x,ij}\}]}, \tag{2.13}$$

where $a^k$ is the volume element of the unit sphere associated to $U^k$.

We can write (with Lemma 2.2):

$$|\partial_t b_1 (t)| + |\partial_{tt} b_1 (t)| + |\partial_{tt} a (t)| \leq C e^{2t},$$

and

$$|\partial_{\theta} b_1 | + |\partial_{\theta, \rho} b_1 | + |\partial_{t, \theta} b_1 | + |\partial_{t, \theta, \rho} b_1 | \leq C e^{2t}.$$
But,

$$\Delta_\theta = \Delta_{\mathcal{S}_3} = - \frac{\partial_{\theta\theta} \left( g^{\phi\phi} (e', \theta) \sqrt{|g^{kk}(e', \theta)|} \right)}{\sqrt{|g^{kk}(e', \theta)|}}.$$  

Then,

$$A_i := \left[ \frac{\partial_{\theta\theta} \left( g^{\phi\phi} (e', \theta) \sqrt{|g^{kk}(e', \theta)|} \right)}{\sqrt{|g^{kk}(e', \theta)|}} \right] \left( \bar{\omega}^{\phi\phi}_i \right) = B_i + D_i, \quad \text{(2.14)}$$

where,

$$B_i = \left[ g^{\phi\phi} (e', \theta) - g^{\phi\phi} (e', \theta) \right] \partial_{\theta\phi} \bar{\omega}^{\phi\phi}_i,$$  

and

$$D_i = \left[ \frac{\partial_{\theta\theta} \left( g^{\phi\phi} (e', \theta) \sqrt{|g^{kk}(e', \theta)|} \right)}{\sqrt{|g^{kk}(e', \theta)|}} - \frac{\partial_{\phi\phi} \left( g^{\phi\phi} (e', \theta) \sqrt{|g^{kk}(e', \theta)|} \right)}{\sqrt{|g^{kk}(e', \theta)|}} \right] \partial_{\phi\phi} \bar{\omega}^{\phi\phi}_i. \quad \text{(2.16)}$$

Clearly, we can choose $\varepsilon_1 > 0$ such that:

$$|\partial_r g^{k i}_i (x, r, \theta)| + |\partial_r \partial_\theta g^{k i}_i (x, r, \theta)| \leq C r, \quad x \in B(x_0, \varepsilon_1), \quad r \in [0, \varepsilon_1], \quad \theta \in \mathbb{U}^k. \quad \text{(2.17)}$$

Finally,

$$A_i \leq C_k c_i = e^{2t_i - e^{2t_i}} \left[ |\nabla_\theta \bar{\omega}^{\phi\phi}_i| + |\nabla_\theta \log (\sqrt{b_1})| \right]. \quad \text{(2.18)}$$

We take, $C = \max \{ C_i \leq 1 \leq q \}$ and we use (2.11). Proposition 2.3 is proved.

We have,

$$c(y_i, t_i, \theta) = 1 + \partial_t a + c_i a^{2t_i}, \quad \text{(2.19a)}$$

$$b_2 (t, \theta) = \partial_{tt} (\sqrt{b_1}) = \frac{1}{2} \sqrt{b_1} \partial_{tt} b_1 - \frac{1}{4} (b_1 b_1)^{3/2} (\partial_t b_1)^2, \quad \text{(2.19b)}$$

$$c_2 = \left[ \frac{1}{\sqrt{b_1}} \Delta_\theta (\sqrt{b_1}) + |\nabla_\theta \log (\sqrt{b_1})|^2 \right]. \quad \text{(2.19c)}$$

We assume that $\lambda \leq \lambda_i + 2 = - \log u_i (y_i) + 2$, which will be chosen later. We work on $[\lambda_i, t_i] \times \mathcal{S}_3$ with

$$\tilde{t}_i = \log t_i \leq \tilde{t}_i = \frac{\lambda_i - 2}{3} \rightarrow -\infty, \quad \text{for large } t_i \gg \lambda_i + 2.$$

The functions $\psi_i$ tend to a radially symmetric function, then, $\partial_\theta w_i \rightarrow 0$ if $i \to +\infty$ and,

$$\frac{\partial_\theta w_i (t, \theta)}{w_i} = \frac{e^{(n-2)((\lambda - \lambda_i) + (\tilde{t}_i - t))/2} e^{((\lambda - \lambda_i) + (\tilde{t}_i - t)) (\partial_\theta \psi_i)} e^{(\lambda - \lambda_i) + (\lambda - t) \theta}}{e^{(n-2)((\lambda - \lambda_i) + (\lambda - t)) / 2 \psi_i} e^{(\lambda - \lambda_i) + (\lambda - t) \theta}} \leq C_i.$$
where $\tilde{C}_i$ does not depend on $\lambda$ and tends to 0. We have also,

$$|\partial_{\theta} \tilde{w}_i^1(t, \theta)| + |\partial_{\theta, \theta} \tilde{w}_i^1(t, \theta)| \leq \tilde{C}_i w_i^1(t, \theta), \quad \tilde{C}_i \to 0,$$

(2.20)

and

$$|\partial_{\theta} \tilde{w}_i^3(t, \theta)| + |\partial_{\theta, \theta} \tilde{w}_i^3(t, \theta)| \leq \tilde{C}_i w_i^3(t, \theta), \quad \tilde{C}_i \to 0.$$

(2.21)

$\tilde{C}_i$ does not depend on $\lambda$.

Now, we set:

$$\bar{w}_i = \bar{w}_i - \frac{m_i}{2} e^\gamma,$$

(2.22)

with $m_i = \frac{1}{2} u_i (x_i)^{1/3} \min_M u_i$. As in [2], we have,

**Lemma 2.4.** There is $\nu < 0$ such that for $\lambda \leq \nu$ :

$$\bar{w}_i^1(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \quad \forall (t, \theta) \in [\lambda, t_i] \times S_3.$$

(2.23)

Let $\zeta_i$ be the following real number,

$$\zeta_i = \sup \{ \lambda : \lambda \leq \lambda_i + 2, w_i^\lambda(t, \theta) - w_i(t, \theta) \leq 0, \forall (t, \theta) \in [\lambda, t_i] \times S_3 \}.$$

By continuity we have in $[\lambda, t_i] \times S_3$:

$$\hat{w}_i^\zeta - \bar{w}_i \leq 0.$$

According to the definition of $\hat{w}_i$ and $\bar{w}_i$ (before Lemma 2.4 and Lemma 2.4), we have:

$$0 < \hat{w}_i^2 \leq 2e, \quad \bar{w}_i \geq \frac{m_i}{2} e^\gamma \quad \text{and} \quad \hat{w}_i^2 - \bar{w}_i \leq \frac{m_i}{2} (e^{2\hat{w}_i} - e^{2\bar{w}_i}).$$

Like in [2], we use the previous lemma to show:

$$\hat{w}_i^\zeta - \bar{w}_i \leq 0 \Rightarrow \hat{Z}_i(\hat{w}_i^\zeta - \bar{w}_i) \leq 0.$$

We have,

$$\hat{Z}_i(\hat{w}_i^\zeta - \bar{w}_i) \leq 8(b_i^\zeta)^{-1} [(\hat{w}_i^\zeta)^3 - \bar{w}_i^3] + O(1)(e^{2\bar{w}_i} - e^{2\hat{w}_i}) + O(1)\bar{w}_i^\zeta (e^{2\hat{w}_i} - e^{2\hat{w}_i}),$$

and

$$-\hat{Z}_i(e^{2\hat{w}_i} - e^{2\bar{w}_i}) = (4 - |\partial_t a - \bar{a} h e^{2t} + b_1^{-1/2} b_2 - c_2)(e^{2\hat{w}_i} - e^{2\bar{w}_i}) \leq c_3(e^{2\hat{w}_i} - e^{2\bar{w}_i}).$$

Thus,

$$\hat{Z}_i(\hat{w}_i^\zeta - \bar{w}_i) \leq 8(b_i^\zeta)^{-1} [(\hat{w}_i^\zeta)^3 - \bar{w}_i^3] + (c_3 m_i - c_4)(e^{2\hat{w}_i} - e^{2\bar{w}_i})$$

with, $c_3, c_4 > 0$. But,

$$0 < \hat{w}_i^2 \leq 2e, \quad \bar{w}_i \geq \frac{m_i}{2} e^\gamma \quad \text{and} \quad \hat{w}_i^\zeta - \bar{w}_i \leq \frac{m_i}{2} (e^{2\hat{w}_i} - e^{2\bar{w}_i}),$$
and
\[(\bar{w}_i^x)^3 - \bar{w}_i^3 = (\bar{w}_i^x - \bar{w}_i) [(\bar{w}_i^x)^2 + \bar{w}_i \bar{w}_i^x + \bar{w}_i^2] \leq (\bar{w}_i^x - \bar{w}_i) (\bar{w}_i^x)^3 + (\bar{w}_i^x - \bar{w}_i) \frac{m^2}{4} e^{t \bar{w}_i^x}, \quad (2.24)\]
then,
\[\bar{Z}_i (\bar{w}_i^x - \bar{w}_i) \leq \left[ \frac{am_i^3}{16} - O(1) \right] + \left[ \frac{am_i^2}{8} - O(1) \right] e^{t \bar{w}_i^x} (e^{2t} - e^{2t}) \leq 0. \quad (2.25)\]
If we use the maximum principle and the Hopf lemma, we obtain (as in [2]):
\[\min_{\theta \in S_3} \bar{w}_i (t_i, \theta) \leq \max_{\theta \in S_3} \bar{w}_i (2 \xi_i - t_i, \theta),\]
we can write (using Proposition 2.2):
\[w_i (2 \xi_i - t_i, \theta) = w_i (\xi_i - t_i + \xi_i - \lambda_i + \lambda_i, \theta) \leq ce^{t_i - t_i}, \quad \xi_i \leq \lambda_i + 2,\]
and we take,
\[t_i = \frac{\lambda_i}{3} = -\frac{1}{3} \log u_i (y_i)\]
to have:
\[\left[ u_i (y_i) \right]^{1/3} \min_M u_i \leq c, \quad (2.26)\]
which in contradiction with Proposition 2.1.

References


