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An Estimate on Riemannian Manifolds of Dimension 4

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Abstract. We give an estimate of type $\sup \times \inf$ on Riemannian manifold of dimension 4 for a Yamabe type equation.

Key Words: $\sup \times \inf$, Riemannian manifold, dimension 4.

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1 Introduction and main results

In this paper, we deal with the following Yamabe type equation in dimension $n=4$:

$$\Delta_g u + hu = 8u^3, \quad u > 0. \tag{1.1}$$

Here, $\Delta_g = -\nabla^i(\nabla_i)$ is the Laplace-Beltrami operator and h is an arbitrary bounded function.

The Eq. (1.1) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2–4, 11, 15]. In this case we have a $\sup \times \inf$ inequality. The corresponding equation in two dimensions on open set Ω of $\mathbb{R}^2$, is:

$$\Delta u = V(x) e^u. \tag{1.2}$$

The Eq. (1.2) was studied by many authors and we can find very important result about a priori estimates in [8,9,12,16] and [19]. In particular in [9], we have the following interior estimate:

$$\sup_K u \leq c = c\Big(\inf_\Omega V, \|V\|_{L^\infty(\Omega)}, \inf_\Omega u, K, \Omega\Big).$$

And, precisely, in [8,12,16] and [20], we have:

$$C \sup_K u + \inf_\Omega u \leq c = c\Big(\inf_\Omega V, \|V\|_{L^\infty(\Omega)}, K, \Omega\Big),$$

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and

$$\sup_K u+\inf_\Omega u\le c=c\Big(\inf_\Omega V,\|V\|_{C^\alpha(\Omega)},K,\Omega\Big),$$

where K is a compact subset of Ω, C is a positive constant which depends on

$$\frac{\inf_\Omega V}{\sup_\Omega V}$$

and $\alpha\in(0,1]$. When $6h=R_g$ the scalar curvature, and M compact, the Eq. (1.1) is Yamabe equation. T. Aubin and R. Schoen have proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary. When M is a compact Riemannian manifold, there exist some compactness results for Eq. (1.1) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeormorfic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose M a Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product $\sup\times\inf$ is bounded. Here we extend the result of [5]. Our proof is an extension Li-Zhang result in dimension 3, see [3] and [17], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [3,6,10,11,16,17], some applications of this method, for example an uniqueness result. We refer to [7] for the uniqueness result on the sphere and in dimension 3. Here, we give an equality of type $\sup\times\inf$ for the Eq. (1.1) in dimension 4. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3,5]. There are other estimates of type $\sup+\inf$ on complex Monge-Ampere equation on compact manifolds, see [20,21]. They consider, on compact Kahler manifold (M,g), the following equation:

$$\begin{cases}(\omega_g+\partial\bar{\partial}\varphi)^n=e^{f-t\varphi}\omega_g^n,\\ \omega_g+\partial\bar{\partial}\varphi>0\quad\text{on } M.\end{cases}\tag{1.3}$$

And, they prove some estimates of type $\sup_M+m\inf_M\le C$ or $\sup_M+m\inf_M\ge C$ under the positivity of the first Chern class of M. Here, we have,

Theorem 1.1. *For all compact set K of M, there is a positive constant c, which depends only on, $h_0=\|h\|_{L^\infty(M)}$, K, M, g such that:*

$$\Big(\sup_K u\Big)^{1/3}\times\inf_M u\le c,$$

for all u solution of (1.1).

Here we consider more general equation and this theorem extends a result of Li and Zhang, see [17]. Li and Zhang considered precisely the Yamabe equation and here we consider a general equation ($h\neq\frac{1}{6}R_g$ with R_g the scalar curvature). Here, we use a different method than the method of Li and Zhang in [17]. Also, we extend a result of [5].

Corollary 1.1. For all compact set K of M there is a positive constant c, such that:

$$\sup_K u \leq c = c(g,m,h_0,K,M) \quad \text{if } \inf_M u \geq m > 0,$$

for all u solution of (1.1).

2 Proof of the results

Proof of Theorem 1.1. Let x_0 be a point of M. We want to prove a uniform estimate around x_0. We argue by contradiction, we assume that the $\sup \times \inf$ is not bounded.

$\forall c,R>0$, $\exists u_{c,R}$ solution to (1.1) such that:

$$R^2\Big(\sup_{B(x_0,R)} u_{c,R}\Big)^{1/3} \times \inf_M u_{c,R} \geq c. \tag{2.1}$$

Proposition 2.1 (Blow-Up Analysis)**.** There is a sequence of points $(y_i)_i$, $y_i \to x_0$ and two sequences of positive real numbers $(l_i)_i$, $(L_i)_i$, $l_i \to 0$, $L_i \to +\infty$, such that if we set

$$v_i(y) = \frac{u_i[\exp_{y_i}(y/[u_i(y_i)])]}{u_i(y_i)},$$

we have:

$$0 < v_i(y) \leq \beta_i \leq 2, \quad \beta_i \to 1,$$
$$v_i(y) \to \frac{1}{1+|y|^2} \text{ uniformly on compact sets of } \mathbb{R}^4,$$
$$l_i^2 (u_i(y_i))^{1/3} \min_M u_i \to +\infty.$$

Proof. We use the hypothesis (2.1), we take two sequences, $R_i>0$, $R_i \to 0$ and $c_i \to +\infty$, such that,

$$R_i{}^2\Big(\sup_{B(x_0,R_i)} u_{c_i,R_i}\Big)^{1/3} \times \inf_M u_{c_i,R_i} \geq c_i \to +\infty. \tag{2.2}$$

Let, $x_i \in B(x_0,R_i)$, such that $\sup_{B(x_0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = [R_i - d(x,x_i)](u_i(x))^{1/6}$, $x \in B(x_i,R_i)$. Then, $x_i \to x_0$. We have:

$$\sup_{B(x_i,R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i(u_i(x_i))^{1/6} \geq c_i{}^{1/2} \to +\infty$$

with $y_i \in B(x_i,R_i)$.

We set :

$$l_i = R_i - d(y_i,x_i), \quad \bar{u}_i(y) = u_i[\exp_{y_i}(y)], \quad v_i(z) = \frac{u_i[\exp_{y_i}(z/[u_i(y_i)])]}{u_i(y_i)}.$$

Clearly, we have, $y_i \to x_0$. We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/4}}[u_i(y_i)] \geq \frac{c_i^{1/2}}{c_i^{1/4}} = c_i^{1/4} \to +\infty.$$

If $|z| \leq L_i$, then $y = \exp_{y_i}[z/[u_i(y_i)]] \in B(y_i, \delta_i l_i)$ with $\delta_i = \frac{1}{(c_i)^{1/4}}$ and $d(y,y_i) < R_i - d(y_i,x_i)$, thus, $d(y,x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$. We can write,

$$(u_i(y))^{1/6}[R_i - d(y,x_i)] \leq (u_i(y_i))^{1/6} l_i.$$

But, $d(y,y_i) \leq \delta_i l_i$, $R_i > l_i$ and $R_i - d(y,x_i) \geq R_i - d(x_i,y_i) - \delta_i l_i > l_i - \delta_i l_i = l_i(1-\delta_i)$, hence, we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \Big(\frac{l_i}{l_i(1-\delta_i)}\Big)^6 \leq 2^6.$$

We set, $\beta_i = \frac{1}{1-\delta_i}$, clearly $\beta_i \to 1$.

Because $R_i \geq l_i$ we have $u_i(y_i) \geq u_i(x_i)$ using the fact that $s_i(y_i) \geq s_i(x_i)$ we obtain:

$$l_i^2 (u_i(y_i))^{1/3} \times \inf_M u_i \to +\infty.$$

Thus, we complete the proof. □

Remark 2.1. We can consider $s_i(x) = (R_i - d(y,x_i))u_i(x)$ and in this case we can replace l_i by R_i to have the last assertion of the proposition (our computations do not change):

$$R_i^2 (u_i(y_i))^{1/3} \times \inf_M u_i \to +\infty.$$

The function v_i satisfies the following equation:

$$-g^{jk}(z)\partial_{jk} v_i - \partial_k \Big[g^{jk}\sqrt{|g|}\Big](z)\partial_j v_i + \frac{h(z)}{[u_i(y_i)]^2} v_i = 8{v_i}^3 \tag{2.3}$$

with $g^{jk}(z) = g^{jk}(\exp_{y_i}(z/u_i(y_i)))$.

We use Ascoli and Ladyzenskaya theorems (see [1]) to obtain the local uniform convergence (on every compact set of $\mathbb{R}^4$) of $(v_i)_i$ to v solution on $\mathbb{R}^4$ to:

$$\Delta v = 8v^3, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2.$$

By the maximum principle, we have $v > 0$ on $\mathbb{R}^n$. According to Caffarelli-Gidas-Spruck result (see [10]), we have, $v(y) = \frac{1}{1+|y|^2}$.

Polar geodesic coordinates

Let u be a function on M. We denote $g_{x,ij}$ the local expression of the metric g in the exponential chart centered at x. We set,

$$w_i(t,\theta)=e^t u_i[\exp_{y_i}(e^t\theta)],$$
$$a(y_i,t,\theta)=\log J(y_i,e^t,\theta)=\log[\sqrt{det(g_{y_i,ij})}].$$

We can write the Laplace-Beltrami operator in polar geodesic coordinates:

$$-\Delta u=\partial_{rr}\bar{u}+\frac{3}{r}\partial_r\bar{u}+\partial_r[\log J(x,r,\theta)]\partial_r\bar{u}-\frac{1}{r^2}\Delta_\theta\bar{u}. \tag{2.4}$$

We deduce the two following lemmas:

Lemma 2.1. *The function w_i is a solution to:*

$$-\partial_{tt}w_i-\partial_t a\partial_t w_i-\Delta_\theta w_i+cw_i=8w_i^3 \tag{2.5}$$

with

$$c=c(y_i,t,\theta)=1+\partial_t a+he^{2t}.$$

Proof. We write:

$$\partial_t w_i=e^{2t}\partial_r\bar{u}_i+w_i, \qquad \partial_{tt}w_i=e^{3t}\left[\partial_{rr}\bar{u}_i+\frac{3}{e^t}\partial_r\bar{u}_i\right]+w_i,$$
$$\partial_t a=e^t\partial_r\log J(y_i,e^t,\theta), \qquad \partial_t a\partial_t w_i=e^{3t}[\partial_r\log J\partial_r\bar{u}_i]+\partial_t a w_i.$$

Lemma 2.1 follows. □

Let $b_1(y_i,t,\theta)=J(y_i,e^t,\theta)>0$. We can write:

$$-\frac{1}{\sqrt{b_1}}\partial_{tt}(\sqrt{b_1}w_i)-\Delta_\theta w_i+[c(t)+b_1^{-1/2}b_2(t,\theta)]w_i=8{w_i}^3,$$

where,

$$b_2(t,\theta)=\partial_{tt}(\sqrt{b_1})=\frac{1}{2\sqrt{b_1}}\partial_{tt}b_1-\frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2.$$

We set,

$$\tilde{w}_i=\sqrt{b_1}w_i.$$

Lemma 2.2. *The function $\tilde{w}_i$ is a solution to:*

$$-\partial_{tt}\tilde{w}_i+\Delta_\theta(\tilde{w}_i)+2\nabla_\theta(\tilde{w}_i).\nabla_\theta\log(\sqrt{b_1})+(c+b_1^{-1/2}b_2-c_2)\tilde{w}_i==8\left(\frac{1}{b_1}\right)\tilde{w}_i^3, \tag{2.6}$$

where, c_2 is a function to be determined.

Proof. We have:

$$-\partial_{tt}\tilde{w}_i-\sqrt{b_1}\Delta_\theta w_i+(c+b_2)\tilde{w}_i=8\left(\frac{1}{b_1}\right)\tilde{w}_i^3,$$

But,

$$\Delta_\theta(\sqrt{b_1}w_i)=\sqrt{b_1}\Delta_\theta w_i-2\nabla_\theta w_i\cdot\nabla_\theta\sqrt{b_1}+w_i\Delta_\theta(\sqrt{b_1}),$$

and

$$\nabla_\theta(\sqrt{b_1}w_i)=w_i\nabla_\theta\sqrt{b_1}+\sqrt{b_1}\nabla_\theta w_i,$$

we can write,

$$\nabla_\theta w_i\cdot\nabla_\theta\sqrt{b_1}=\nabla_\theta(\tilde{w}_i)\cdot\nabla_\theta\log(\sqrt{b_1})-\tilde{w}_i|\nabla_\theta\log(\sqrt{b_1})|^2,$$

we deduce,

$$\sqrt{b_1}\Delta_\theta w_i=\Delta_\theta(\tilde{w}_i)+2\nabla_\theta(\tilde{w}_i).\nabla_\theta\log(\sqrt{b_1})-c_2\tilde{w}_i,$$

with

$$c_2=\Big[\frac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1})+|\nabla_\theta\log(\sqrt{b_1})|^2\Big].$$

Lemma 2.2 is proved. □

The Moving-Plane method

Let ξ_i be a real number, we assume $\xi_i\le t$. We set $t^{\xi_i}=2\xi_i-t$ and $\tilde{w}_i^{\xi_i}(t,\theta)=\tilde{w}_i(t^{\xi_i},\theta)$. Set, $\lambda_i=-\log u_i(y_i)$.

Proposition 2.2. We claim: there exists a positive constant $\tilde{k}$ such that:

$$\tilde{w}_i(\lambda_i,\theta)-\tilde{w}_i(\lambda_i+4,\theta)\ge\tilde{k}>0,\quad\forall\theta\in\mathbb{S}_3. \tag{2.7}$$

For all $\beta>0$, there exists $c_\beta>0$ such that:

$$\frac{1}{c_\beta}e^t\le\tilde{w}_i(\lambda_i+t,\theta)\le c_\beta e^t,\quad\forall t\le\beta,\quad\forall\theta\in\mathbb{S}_3. \tag{2.8}$$

Proof. As in [2], There exists a positive constant k such that, $w_i(\lambda_i,\theta)-w_i(\lambda_i+4,\theta)\ge k>0$ for i large, $\forall\theta$. We can remark that $b_1(y_i,\lambda_i,\theta)\to 1$ and $b_1(y_i,\lambda_i+4,\theta)\to 1$ uniformly in θ, we obtain the first claim of Proposition 2.2. For the second claim we use Proposition 2.1, see also [2]. We set:

$$\bar{Z}_i=-\partial_{tt}(\cdots)+\Delta_\theta(\cdots)+2\nabla_\theta(\cdots)\cdot\nabla_\theta\log(\sqrt{b_1})+(c+b_1^{-1/2}b_2-c_2)(\cdots). \tag{2.9}$$

We complete the proof. □

Remark 2.2. In the operator $\bar{Z}_i$, we can remark that:

$$c+b_1^{-1/2}b_2-c_2\ge k'>0\quad\text{for}\ \ t\ll 0,$$

we can apply the maximum principle and the Hopf lemma.

Goal

Like in [2], we have an elliptic second order operator. Here it is $\bar{Z}_i$, the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i}-\tilde{w}_i)\le 0, \quad \text{if } \tilde{w}_i^{\xi_i}-\tilde{w}_i\le 0. \tag{2.10}$$

We write, $\Delta_\theta=\Delta_{g_{y_i,e^t},\mathbb{S}_{n-1}}$. We obtain:

$$\begin{aligned}\bar{Z}_i(\tilde{w}_i^{\xi_i}-\tilde{w}_i)=&(\Delta_{g_{y_i,e^{t^{\xi_i}}},\mathbb{S}_3}-\Delta_{g_{y_i,e^t},\mathbb{S}_3})(\tilde{w}_i^{\xi_i})\\ &+2(\nabla_{\theta,e^{t^{\xi_i}}}-\nabla_{\theta,e^t})(w_i^{\xi_i})\cdot\nabla_{\theta,e^{t^{\xi_i}}}\log(\sqrt{b_1^{\xi_i}})+2\nabla_{\theta,e^t}(\tilde{w}_i^{\xi_i})\cdot\nabla_{\theta,e^{t^{\xi_i}}}[\log(\sqrt{b_1^{\xi_i}})-\log\sqrt{b_1}]\\ &+2\nabla_{\theta,e^t}w_i^{\xi_i}\cdot(\nabla_{\theta,e^{t^{\xi_i}}}-\nabla_{\theta,e^t})\log\sqrt{b_1}-[(c+b_1^{-1/2}b_2-c_2)^{\xi_i}-(c+b_1^{-1/2}b_2-c_2)]\tilde{w}_i^{\xi_i}\\ &+8\left(\frac{1}{b_1^{\xi_i}}\right)(\tilde{w}_i^{\xi_i})^3-8\left(\frac{1}{b_1}\right)\tilde{w}_i^3.\end{aligned} \tag{2.11}$$

Clearly, we have the following lemma:

Lemma 2.3. *It holds*

$$\begin{aligned}&b_1(y_i,t,\theta)=1-\frac{1}{3}Ricci_{y_i}(\theta,\theta)e^{2t}+\cdots,\\ &R_g(e^t\theta)=R_g(y_i)+<\nabla R_g(y_i)|\theta>e^t+\cdots.\end{aligned}$$

By the previous computations and Lemma 2.3, we have:

Proposition 2.3. It holds

$$\begin{aligned}\bar{Z}_i(\tilde{w}_i^{\xi_i}-\tilde{w}_i)\le&8(b_1^{\xi_i})[(\tilde{w}_i^{\xi_i})^3-\tilde{w}_i^3]+C|e^{2t}-e^{2t^{\xi_i}}|(|\nabla_\theta\tilde{w}_i^{\xi_i}|+|\nabla_\theta^2(\tilde{w}_i^{\xi_i})|)\\ &+C|e^{2t}-e^{2t^{\xi_i}}|(|Ricci_{y_i}|+|h|)\tilde{w}_i^{\xi_i}+C'w_i^{\xi_i}|e^{3t^{\xi_i}}-e^{3t}|.\end{aligned} \tag{2.12}$$

Proof. In polar geodesic coordinates (and the Gauss lemma):

$$g=dt^2+r^2\tilde{g}_{ij}^k d\theta^i d\theta^j \quad \text{at} \quad \sqrt{|\tilde{g}^k|}=\alpha^k(\theta)\sqrt{[det(g_{x,ij})]}, \tag{2.13}$$

where α^k is the volume element of the unit sphere associated to U^k.

We can write (with Lemma 2.2):

$$|\partial_t b_1(t)|+|\partial_{tt}b_1(t)|+|\partial_{tt}a(t)|\le Ce^{2t},$$

and

$$|\partial_{\theta_j}b_1|+|\partial_{\theta_j,\theta_k}b_1|+\partial_{t,\theta_j}b_1|+|\partial_{t,\theta_j,\theta_k}b_1|\le Ce^{2t}.$$

But,

$$\Delta_\theta = \Delta_{g_{y_i,e^t,\mathbb{S}_3}} = -\frac{\partial_{\theta^l}[\tilde{g}^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde{g}^k(e^t,\theta)|}\partial_{\theta^j}]}{\sqrt{|\tilde{g}^k(e^t,\theta)|}}.$$

Then,

$$A_i := \left[\left[\frac{\partial_{\theta^l}(\tilde{g}^{\theta^l\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j})}{\sqrt{|\tilde{g}^k|}}\right]^{\xi_i} - \left[\frac{\partial_{\theta^l}(\tilde{g}^{\theta^l\theta^j}\sqrt{|\tilde{g}^k|}\partial_{\theta^j})}{\sqrt{|\tilde{g}^k|}}\right]\right](\tilde{w}_i^{\xi_i}) = B_i + D_i, \tag{2.14}$$

where,

$$B_i = \left[\tilde{g}^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta) - \tilde{g}^{\theta^l\theta^j}(e^t,\theta)\right]\partial_{\theta^l\theta^j}\tilde{w}_i^{\xi_i}, \tag{2.15}$$

and

$$D_i = \left[\frac{\partial_{\theta^l}[\tilde{g}^{\theta^l\theta^j}(e^{t^{\xi_i}},\theta)\sqrt{|\tilde{g}^k|}(e^{t^{\xi_i}},\theta)]}{\sqrt{|\tilde{g}^k|}(e^{t^{\xi_i}},\theta)} - \frac{\partial_{\theta^l}[\tilde{g}^{\theta^l\theta^j}(e^t,\theta)\sqrt{|\tilde{g}^k|}(e^t,\theta)]}{\sqrt{|\tilde{g}^k|}(e^t,\theta)}\right]\partial_{\theta^j}\tilde{w}_i^{\xi_i}. \tag{2.16}$$

Clearly, we can choose $\epsilon_1 > 0$ such that:

$$|\partial_r \tilde{g}^k_{ij}(x,r,\theta)| + |\partial_r\partial_{\theta^m}\tilde{g}^k_{ij}(x,r,\theta)| \le Cr, \quad x \in B(x_0,\epsilon_1), \quad r \in [0,\epsilon_1], \quad \theta \in U^k. \tag{2.17}$$

Finally,

$$A_i \le C_k|e^{2t} - e^{2t^{\xi_i}}|\left[|\nabla_\theta\tilde{w}_i^{\xi_i}| + |\nabla_\theta^2(\tilde{w}_i^{\xi_i})|\right]. \tag{2.18}$$

We take, $C = \max\{C_i, 1 \le i \le q\}$ and we use (2.11). Proposition 2.3 is proved. □

We have,

$$c(y_i,t,\theta) = 1 + \partial_t a + he^{2t}, \tag{2.19a}$$

$$b_2(t,\theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}}\partial_{tt}b_1 - \frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2, \tag{2.19b}$$

$$c_2 = [\frac{1}{\sqrt{b_1}}\Delta_\theta(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2]. \tag{2.19c}$$

We assume that $\lambda \le \lambda_i + 2 = -\log u_i(y_i) + 2$, which will be chosen later. We work on $[\lambda, t_i] \times \mathbb{S}_3$ with

$$\tilde{t}_i = \log l_i > t_i = \frac{\lambda_i}{3} = -\frac{1}{3}\log u_i(y_i) \to -\infty,$$

l_i as in the Proposition 2.1. For i large $t_i \gg \lambda_i + 2$.

The functions v_i tend to a radially symmetric function, then, $\partial_{\theta_j} w_i^\lambda \to 0$ if $i \to +\infty$ and,

$$\frac{\partial_{\theta_j} w_i^\lambda(t,\theta)}{w_i^\lambda} = \frac{e^{(n-2)[(\lambda-\lambda_i)+(\xi_i-t)]/2}e^{[(\lambda-\lambda_i)+(\xi_i-t)]}(\partial_{\theta_j}v_i)(e^{[(\lambda-\lambda_i)+(\lambda-t)]}\theta)}{e^{(n-2)[(\lambda-\lambda_i)+(\lambda-t)]/2}v_i[e^{(\lambda-\lambda_i)+(\lambda-t)}\theta]} \le \bar{C}_i,$$

where $\bar{C}_i$ does not depend on λ and tends to 0. We have also,

$$|\partial_\theta w_i^\lambda(t,\theta)|+|\partial_{\theta,\theta} w_i^\lambda(t,\theta)|\le \tilde{C}_i w_i^\lambda(t,\theta), \quad \tilde{C}_i\to 0, \tag{2.20}$$

and

$$|\partial_\theta \tilde{w}_i^\lambda(t,\theta)|+|\partial_{\theta,\theta} \tilde{w}_i^\lambda(t,\theta)|\le \tilde{C}_i \tilde{w}_i^\lambda(t,\theta), \quad \tilde{C}_i\to 0. \tag{2.21}$$

$\tilde{C}_i$ does not depend on λ.

Now, we set:

$$\bar{w}_i=\tilde{w}_i-\frac{\tilde{m}_i}{2}e^{2t}, \tag{2.22}$$

with $m_i=\frac{1}{2}u_i(x_i)^{1/3}\min_M u_i$. As in [2], we have,

Lemma 2.4. *There is $\nu<0$ such that for $\lambda\le\nu$:*

$$\bar{w}_i^\lambda(t,\theta)-\bar{w}_i(t,\theta)\le 0, \quad \forall(t,\theta)\in[\lambda,t_i]\times \mathbb{S}_3. \tag{2.23}$$

Let ξ_i be the following real number,

$$\xi_i=\sup\{\lambda\le\lambda_i+2, \bar{w}_i^{\lambda}(t,\theta)-\bar{w}_i(t,\theta)\le 0,\ \forall(t,\theta)\in[\lambda,t_i]\times\mathbb{S}_3\}.$$

By continuity we have in $[\lambda,t_i]\times\mathbb{S}_3$:

$$\bar{w}_i^{\xi_i}-\bar{w}_i\le 0.$$

According to the definition of $\bar{w}_i$ and $\tilde{w}_i$ (before Lemma 2.4 and Lemma 2.4), we have:

$$0<\tilde{w}_i^{\xi_i}\le 2e,\quad \tilde{w}_i\ge\frac{m_i}{2}e^{2t}\quad \text{and}\quad \tilde{w}_i^{\xi_i}-\tilde{w}_i\le\frac{m_i}{2}(e^{2t^{\xi_i}}-e^{2t}).$$

Like in [2], we use the previous lemma to show:

$$\bar{w}_i^{\xi_i}-\bar{w}_i\le 0\Rightarrow \bar{Z}_i(\bar{w}_i^{\xi_i}-\bar{w}_i)\le 0.$$

We have,

$$\begin{aligned}&\bar{Z}_i(\tilde{w}_i^{\xi_i}-\tilde{w}_i)\le 8(b_1^{\xi_i})^{-1}[(\tilde{w}_i^{\xi_i})^3-\tilde{w}_i^3]+O(1)(e^{2t}-e^{2t^{\xi_i}})+O(1)\tilde{w}_i^{\xi_i}(e^{2t}-e^{2t^{\xi_i}}),\\ &-\bar{Z}_i(e^{2t^{\xi_i}}-e^{2t})=(4-1-\partial_t a-he^{2t}+b_1^{-1/2}b_2-c_2)(e^{2t^{\xi_i}}-e^{2t})\le c_3(e^{2t^{\xi_i}}-e^{2t}).\end{aligned}$$

Thus,

$$\bar{Z}_i(\bar{w}_i^{\xi_i}-\bar{w}_i)\le 8(b_1^{\xi_i})^{-1}[(\tilde{w}_i^{\xi_i})^3-\tilde{w}_i^3]+(c_3m_i-c_4)(e^{2t^{\xi_i}}-e^{2t})$$

with, $c_3,c_4>0$. But,

$$0<\tilde{w}_i^{\xi_i}\le 2e,\quad \tilde{w}_i\ge\frac{m_i}{2}e^{2t}\quad \text{and}\quad \tilde{w}_i^{\xi_i}-\tilde{w}_i\le\frac{m_i}{2}(e^{2t^{\xi_i}}-e^{2t}),$$

and

$$(\tilde{w}_i^{\xi_i})^3-\tilde{w}_i^3=(\tilde{w}_i^{\xi_i}-\tilde{w}_i)[(\tilde{w}_i^{\xi_i})^2+\tilde{w}_i^{\xi_i}\tilde{w}_i+\tilde{w}_i^2]$$
$$\leq(\tilde{w}_i^{\xi_i}-\tilde{w}_i)(\tilde{w}_i^{\xi_i})^2+(\tilde{w}_i^{\xi_i}-\tilde{w}_i)\frac{m^2e^{2t}}{4}+(\tilde{w}_i^{\xi_i}-\tilde{w}_i)\frac{m}{2}e^t\tilde{w}_i^{\xi_i}, \tag{2.24}$$

then,

$$\bar{Z}_i(\bar{w}_i^{\xi_i}-\bar{w}_i)\leq\Big[\Big[\frac{am_i^3}{16}-O(1)\Big]+\Big[\frac{am_i^2}{8}-O(1)\Big]e^t\tilde{w}_i^{\xi_i}\Big](e^{2t^{\xi_i}}-e^{2t})\leq 0. \tag{2.25}$$

If we use the maximum principle and the Hopf lemma, we obtain (as in [2]):

$$\min_{\theta\in\mathbb{S}_3}\bar{w}_i(t_i,\theta)\leq\max_{\theta\in\mathbb{S}_3}\bar{w}_i(2\xi_i-t_i,\theta),$$

we can write (using Proposition 2.2):

$$w_i(2\xi_i-t_i,\theta)=w_i(\xi_i-t_i+\xi_i-\lambda_i+\lambda_i,\theta)\leq ce^{\xi_i-t_i},\quad \xi_i\leq\lambda_i+2,$$

and we take,

$$t_i=\frac{\lambda_i}{3}=-\frac{1}{3}\log u_i(y_i)$$

to have:

$$[u_i(y_i)]^{1/3}\min_M u_i\leq c, \tag{2.26}$$

which in contradiction with Proposition 2.1.

References

[1] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer-Verlag, 1998.

[2] S. S Bahoura, Majorations du type $\sup u\times\inf u\leq c$ pour l'équation de la courbure scalaire sur un ouvert de $\mathbb{R}^n$, $n\geq 3$, J. Math. Pures. Appl., (9)(83) (2004), 1109–1150.

[3] S. S. Bahoura, Harnack inequalities for Yamabe type equations, Bull. Sci. Math., 133(8) (2009), 875–892.

[4] S. S. Bahoura, Lower bounds for sup+inf and sup × inf and an extension of Chen-Lin result in dimension 3, Acta Math. Sci. Ser. B Engl. Ed., 28(4) (2008), 749–758.

[5] S. S. Bahoura, Estimations uniformes pour l'equation de Yamabe en dimensions 5 et 6, J. Funct. Anal., 242(2) (2007), 550–562.

[6] S. S. Bahoura, $\sup\times\inf$ inequality on manifold of dimension 3, to appear in Mathematica Aeterna.

[7] H. Brezis and Y. Y. Li, Some nonlinear elliptic equations have only constant solutions, J. Partial Differential Equations, 19(3) (2006), 208–217.

[8] H. Brezis, Y. Y. Li and I. Shafrir, A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities, J. Funct. Anal., 115 (1993), 344–358.

[9] H. Brezis and F. Merle, Uniform estimates and blow-up bihavior for solutions of $-\Delta u=Ve^u$ in two dimensions, Commun. Partial Differential Equations, 16 (1991), 1223–1253.

[10] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Commun. Pure Appl. Math., 37 (1984), 369–402.

[11] C-C.Chen and C-S. Lin, Estimates of the conformal scalar curvature equation via the method of moving planes, Commun. Pure Appl. Math., L (1997), 0971–1017.

[12] C-C.Chen and C-S. Lin, A sharp sup+inf inequality for a nonlinear elliptic equation in $\mathbb{R}^2$, Commun. Anal. Geom., 6(1) (1998), 1–19.

[13] B. Gidas, W-Y. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Commun. Math. Phys., 68(3) (1979), 209–243.

[14] J. M. Lee and T. H. Parker, The Yamabe problem, Bull. Amer. Math. Soc., 17(1) (1987), 37–91.

[15] Y. Y. Li, Prescribing scalar curvature on $\mathbb{S}_n$ and related problems, C.R. Acad. Sci. Paris, 317 (1993), 159–164. Part I: J. Differ. Equations, 120 (1995), 319–410. Part II: Existence and compactness, Commun. Pure Appl. Math., 49 (1996), 541–597.

[16] Y. Y. Li, Harnack Type inequality: the method of moving planes, Commun. Math. Phys., 200 (1999), 421–444.

[17] Y. Y. Li and L. Zhang, A Harnack type inequality for the Yamabe equation in low dimensions, Calc. Var. Partial Differential Equations, 20(2) (2004), 133–151.

[18] Y. Y. Li and M. Zhu, Yamabe type equations on three dimensional Riemannian manifolds, Commun. Contem. Math., 1(1) (1999), 1–50.

[19] I. Shafrir, A sup+inf inequality for the equation $-\Delta u = Ve^u$, C. R. Acad.Sci. Paris Sér. I Math., 315(2) (1992), 159–164.

[20] Y-T. Siu, The existence of Kahler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group, Ann. Math., 127(3) (1988), 585–627.

[21] G. Tian, A Harnack type inequality for certain complex Monge-Ampere equations, J. Differential Geom., 29(3) (1989), 481–488.