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On the Connection between the Order of Riemann-Liouvile Fractional Calculus and Hausdorff Dimension of a Fractal Function

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Abstract. This paper investigates the fractal dimension of the fractional integrals of a fractal function. It has been proved that there exists some linear connection between the order of Riemann-Liouvile fractional integrals and the Hausdorff dimension of a fractal function.

Key Words: Fractional calculus, Hausdorff dimension, Riemann-Liouvile fractional integral.
AMS Subject Classifications: MR28A80, MR26A33, MR26A30

1 Introduction

Fractional calculus, both of theoretical and practical importance, is an important tool being used to investigate fractal functions and curves. Fractional calculus, such as Riemann-Liouvile fractional integrals, can be effective applied to certain fractals like the Weierstrass function [1]. With the help of the K-dimension, Yao [8,9], Su, and Zhou [11] proved that there exist some linear connection between the order of fractional calculus and the Box dimension, K-dimension, and Packing dimension of graphs of the Weierstrass function. A natural problem is, does this connection still hold for the Hausdorff dimension which is very important in fractal theory? Firstly, we recall the definition of Riemann-Liouvile fractional integral.

Definition 1.1 (see [5])**.** Let f be a function piecewisely continuous on $(0,\infty)$ and integrable on any finite subinterval of $(0,\infty)$. Then we call

$$D^{-v}f(t)=\frac{1}{\Gamma(v)}\int_0^t (t-x)^{v-1}f(x)dx.$$

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Riemann-Liouvile fractional integral of f of order v for $t>0$ and $Re(v)>0$.

This paper considers the Weierstrass function with random phase added to each term, i.e.,

$$f_\Theta(x)=\sum_{n=0}^{\infty}\lambda^{-\alpha n}\sin(2\pi(\lambda^n x+\theta_n)),\quad x\in I, \tag{1.1}$$

where $\lambda>1$, $0<\alpha<1$, $I=[0,1]$, $\Theta=\{\theta_0,\theta_1,\theta_2,\cdots\}$. More details about the type of the Weierstrass function can be found in [1,7].

Definition 1.2. Denote Riemann-Liouvile fractional integral of $\sin(2\pi(\lambda^n x+\theta_n))$ and $\cos(2\pi(\lambda^n x+\theta_n))$ of order v as following

$$\begin{aligned}S_t(v,\lambda,\theta)&=D^{-v}\sin(\lambda^n x+\theta_n)=\frac{1}{\Gamma(v)}\int_0^t(t-\xi)^{v-1}\sin(2\pi(\lambda^n\xi+\theta_n)),\\ C_t(v,\lambda,\theta)&=D^{-v}\cos(\lambda^n x+\theta_n)=\frac{1}{\Gamma(v)}\int_0^t(t-\xi)^{v-1}\cos(2\pi(\lambda^n\xi+\theta_n)).\end{aligned}$$

Then define

$$F_\theta(x)=D^{-v}(f_\theta(x))=\sum_{n=0}^{\infty}\lambda^{-\alpha n}S_t(v,\lambda,\theta) \tag{1.2}$$

be R-L fractional integral of $f_\theta(x)$ of order v.

Definition 1.3 (see [2])**.** Let a Borel set $F\in\mathscr{R}^n$ be given as follows. For $s\geq 0$ and $\delta>0$, define

$$\mathscr{H}_\delta^s(F)=\inf\Big\{\sum_{i=1}^{\infty}|U_i|^s:\{U_i\}\text{ is a }\delta\text{-cover of }F\Big\},$$

where $|U|=\sup\{|x-y|:x,y\in U\}$ denotes the diameter of a nonempty set U and the infimum is taken over all countable collections $\{U_i\}$ of sets for which $F\subset\bigcup_{i=1}^{\infty}U_i$ and $0<|U_i|\leq\delta$. As δ decreases, $\mathscr{H}_\delta^s(F)$ cannot decrease, and therefore it has a limit (possibly infinite) as $\delta\to 0$, define

$$\mathscr{H}^s(F)=\lim_{\delta\to 0}\mathscr{H}_\delta^s(F).$$

The quantity $\mathscr{H}^s(F)$ is known as s-dimensional Hausdorff measure of F. For a given F there is a value $\dim_H(F)$ for which $\mathscr{H}^s(F)=\infty$ for $s<\dim_H(F)$ and $\mathscr{H}^s(F)=0$ for $s>\dim_H(F)$. Hausdorff dimension $\dim_H(F)$ is defined to be this value, that is:

$$\dim_H(F)=\inf\{s:\mathscr{H}^s(F)=0\}=\sup\{s:\mathscr{H}^s(F)=\infty\}.$$

For simplicity, let

$$\begin{aligned}&\widetilde{S}_t(v,\lambda,\theta)=\Gamma(v)S_t(v,\lambda,\theta),\\ &\widetilde{C}_t(v,\lambda,\theta)=\Gamma(v)C_t(v,\lambda,\theta),\\ &C^\alpha(I)=\{f(x):|f(x)-f(y)|\leq c|x-y|^\alpha,\forall x,y\in I\}.\end{aligned}$$

Let $H=[0,1]^\infty$, endowed with the uniform probability measure,and let $\Theta=\{\theta_0,\theta_1,\cdots\}$ denote a point in H. Let $Graph(f,[a,b])=\{(x,f(x))|a\le x\le b,f:[a,b]\to\mathbb{R}^2\}$ be the graph of f.

The remainder of this paper is arranged as follows, in Section 2, we give some lemmas which are important for the proof of the linear relationship. In Section 3, there are two theorems discussed for the relationship between $\dim_H Graph(F_\Theta,I)$ and $\dim_H Graph(f_\Theta,I)$.

2 Lemmmas

To prove the main theorems about the linear relationship, we need the following lemmas. We first derive some simple but widely applicable estimation for Hausdorff dimension of continuous functions.

Lemma 2.1 (see [2]). *Let $f:[0,1]\to R$ be a continuous function. If*

$$|f(t)-f(u)|\le c|t-u|^{2-s}, \tag{2.1}$$

where $0\le t,u\le 1$, $c>0$, $1<s<2$, then we have $\mathscr{H}^s Graph(f,I)<\infty$ and $\dim_H Graph(f,I)\le s$.

Lemma 2.2. *Let $Z(\Theta)=F_\Theta(x)-F_\Theta(y)$, the variance of function $Z_\Theta(x)$ has a bounded density function.*

Proof. Let $\pi\lambda^n t=u$, we have

$$\begin{aligned}
Z(\Theta)=&\sum_{n=o}^{\infty}\lambda^{-\alpha n}\frac{1}{\Gamma(v)}\Big(\int_0^x\frac{\sin(2\pi(\lambda^n(x-t)+\theta_n)}{t^{1-v}}dt-\int_0^y\frac{\sin(2\pi(\lambda^n(y-t)+\theta_n)}{t^{1-v}}dt\Big)\\
=&\sum_{n=o}^{\infty}\lambda^{-\alpha n}\frac{1}{\Gamma(v)}\Big(\int_0^{\pi\lambda^n x}\frac{\sin(2\pi\lambda^n x-2u+2\pi\theta_n)}{(\pi\lambda^n)^v t^{1-v}}dt-\int_0^{\pi\lambda^n y}\frac{\sin(2\pi\lambda^n y-2u+2\pi\theta_n)}{(\pi\lambda^n)^v t^{1-v}}dt\Big)\\
=&\sum_{n=o}^{\infty}\pi^{-v}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}\Big(\int_0^{\pi\lambda^n x}\frac{\sin(2\pi\lambda^n x-2u+2\pi\theta_n)}{u^{1-v}}dt-\int_0^{\pi\lambda^n y}\frac{\sin(2\pi\lambda^n y-2u+2\pi\theta_n)}{u^{1-v}}dt\Big)\\
=&\sum_{n=o}^{\infty}\pi^{-v}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}\Big(\int_0^{\pi\lambda^n x}\frac{\sin(2\pi\lambda^n x-2u)\cos(2\pi\theta_n)+\cos(2\pi\lambda^n x-2u)\sin(2\pi\theta_n)}{u^{1-v}}du\\
&-\int_0^{\pi\lambda^n y}\frac{\sin(2\pi\lambda^n y-2u)\cos(2\pi\theta_n)+\cos(2\pi\lambda^n y-2u)\sin(2\pi\theta_n)}{u^{1-v}}du\Big)\\
=&\sum_{n=o}^{\infty}\pi^{-v}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}\Big(\int_0^{\pi\lambda^n x}\frac{\sin(2\pi\lambda^n x-2u)}{u^{1-v}}du\cos(2\pi\theta_n)\\
&+\int_0^{\pi\lambda^n x}\frac{\cos(2\pi\lambda^n x-2u)}{u^{1-v}}du\sin(2\pi\theta_n)-\int_0^{\pi\lambda^n y}\frac{\sin(2\pi\lambda^n y-2u)}{u^{1-v}}du\cos(2\pi\theta_n)\\
&-\int_0^{\pi\lambda^n y}\frac{\cos(2\pi\lambda^n y-2u)}{u^{1-v}}du\sin(2\pi\theta_n)\Big).
\end{aligned}$$

Let

$$A=\int_0^{\pi\lambda^n x}\frac{\sin(2\pi\lambda^n x-2u)}{u^{1-v}}du, \qquad B=\int_0^{\pi\lambda^n x}\frac{\cos(2\pi\lambda^n x-2u)}{u^{1-v}}du,$$
$$C=\int_0^{\pi\lambda^n y}\frac{\sin(2\pi\lambda^n y-2u)}{u^{1-v}}du, \qquad D=\int_0^{\pi\lambda^n y}\frac{\cos(2\pi\lambda^n y-2u)}{u^{1-v}}du.$$

We have

$$Z(\Theta)=Q_n\sin(2\pi\theta_n+\varphi_n), \tag{2.2}$$

where

$$\tan\varphi_n=\frac{A-C}{B-D},$$
$$Q_n=\sum_{n=o}^{\infty}\pi^{-v}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}((A-C)^2+(B-D)^2)^{\frac{1}{2}}.$$

Since $z_1,z_2,\cdots$, are independent random variables with density functions

$$h_n(\theta_n)=\begin{cases}\dfrac{1}{\pi(Q_n^2-Z_n^2)^{\frac{1}{2}}}, & |Z_n|<|Q_n|,\\ 0, & |Z_n|\geq|Q_n|.\end{cases}$$

For $z=z_0+z_1+z_2+\cdots$, we get

$$h(z)=h_0*h_1*h_2*\cdots.$$

Because the maximum value of a probability density can not increase under convolution with another probability density, any upper bound we obtain on a finite convolution $h_j*\cdots*h_k$ is an upper bound on $h_n(\theta_n)$ as well.

Notice that

$$\begin{aligned}A&=\int_0^{b}\frac{\sin(2\pi\lambda^n x-2u)}{u^{1-v}}du+\int_b^{\pi\lambda^n x}\frac{\sin(2\pi\lambda^n x-2u)}{u^{1-v}}du\\&=:\Sigma_1+\Sigma_2,\end{aligned}$$

here $0<b<\pi\lambda^n x$.

By Cauchy's test we get Σ_1 is absolute convergence. At the same time Σ_2 is convergence too by Dirichlet test. Thus A is convergence. In a similar way B, C, D is convergence too,

$$((A-C)^2+(B-D)^2)^{\frac{1}{2}}=L.$$

Let integer $K>2$ satisfy $\lambda^{-(K+1)}<|x-y|<\lambda^K$. We have

$$|Q_n|>\pi^{-v}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}L>\pi^{-v}\frac{1}{\Gamma(v)}L|x-y|^{\alpha+v}. \tag{2.3}$$

It holds that $|Q_n| > M_1|x-y|^{\alpha+v}$. Now let $\|\bullet\|_p$ denote the L^p norm. obviously $h_n \in L^p$ $(0<p<2)$, so

$$\|h_n(Z_n)\|_{\frac{3}{2}} = M_1|q_n|^{-\frac{1}{3}} \le M_2|x-y|^{\frac{v+\alpha}{3}}.$$

Here M, M_1, M_2 are constants independent of x, y. By Young's inequality

$$\|h_{k-1}*h_k\|_3 \le \|h_{k-1}\|_{\frac{3}{2}} * \|h_k\|_{\frac{3}{2}},$$

and Holder inequality

$$\|h_{k-2}*h_{k-1}*h_k\|_3 \le \|h_{k-2}\|_{\frac{3}{2}}\|h_{k-1}*h_k\|_3 \le \|h_{k-2}\|_{\frac{3}{2}}\|h_{k-1}\|_{\frac{3}{2}}\|h_k\|_{\frac{3}{2}} \le M_2^3|x-y|^{v+\alpha}.$$

It follows that

$$h(z) \le M_2^3|x-y|^{v+\alpha}. \tag{2.4}$$

Thus, we complete the proof. □

3 Theorems

In this section, we will prove the main theorems.

Theorem 3.1. *For* $0<v<1$, $\lambda>1$, $0<\alpha+v<1$, *we have* $\dim_H Graph(F_\Theta, I) \le 2-\alpha-v$.

Proof. For $x,y \in I$, let $x>y$. By Lemma 2.2, we have

$$\begin{aligned}
|F_\Theta(x)-F_\Theta(y)| &\le \sum_{n=0}^{\infty}\lambda^{-\alpha n}\frac{1}{\Gamma(v)}\Big|\int_0^x \frac{\sin(\lambda^n(x-t)+\theta_n)}{t^{1-v}} - \int_0^y \frac{\sin(\lambda^n(y-t)+\theta_n)}{t^{1-v}}\Big| \\
&= \sum_{n=0}^{m-1}\lambda^{-\alpha n}\frac{1}{\Gamma(v)}\Big|\int_0^x \frac{\sin(\lambda^n(x-t)+\theta_n)}{t^{1-v}} - \int_0^y \frac{\sin(\lambda^n(y-t)+\theta_n)}{t^{1-v}}\Big| \\
&\quad + \sum_{n=m}^{\infty}\lambda^{-\alpha n}\frac{1}{\Gamma(v)}\Big|\int_0^x \frac{\sin(\lambda^n(x-t)+\theta_n)}{t^{1-v}} - \int_0^y \frac{\sin(\lambda^n(y-t)+\theta_n)}{t^{1-v}}\Big| \\
&=: \Sigma_1 + \Sigma_2.
\end{aligned}$$

For all $m>0$, if $|x-y| \le 1$. let m be the positive integer with

$$\lambda^{-m} < |x-y| < \lambda^{-(m-1)}.$$

For Σ_1, let $\lambda^{n+\frac{m}{v}}t=u$, then we have

$$\begin{aligned}\Sigma_1=&\sum_{n=0}^{m-1}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}\lambda^{-m}\Big|\int_0^{\lambda^{n+\frac{m}{v}}x}\frac{\sin(\lambda^n x-\lambda^{-\frac{m}{v}}u+\theta_n)}{u^{1-v}}du\\&-\int_0^{\lambda^{n+\frac{m}{v}}y}\frac{\sin(\lambda^n y-\lambda^{-\frac{m}{v}}u+\theta_n)}{u^{1-v}}du\Big|\\\le&\sum_{n=0}^{m-1}|x-y|\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}2K\\=&\frac{1-\lambda^{[1-(\alpha+v)]m}}{1-\lambda^{1-(\alpha+v)}}|x-y|\frac{1}{\Gamma(v)}2K\\\le&\frac{\lambda^{[1-(\alpha+v)]m}}{\lambda^{1-(\alpha+v)}-1}|x-y|\frac{1}{\Gamma(v)}2K\\\le&\frac{2K}{\Gamma(v)[\lambda^{1-(\alpha+v)}-1]}|x-y|^{\alpha+v}\\\le&C_1|x-y|^{\alpha+v}.\end{aligned}$$

For Σ_2, let $\lambda^n t=u$, we have

$$\begin{aligned}\Sigma_2=&\sum_{n=m}^{\infty}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}\Big|\int_0^{\lambda^n x}\frac{\sin(\lambda^n x-u+\theta_n)}{u^{1-v}}du-\int_0^{\lambda^n y}\frac{\sin(\lambda^n y-u+\theta_n)}{u^{1-v}}du\Big|\\\le&\sum_{n=m}^{\infty}\lambda^{-(\alpha+v)n}\frac{1}{\Gamma(v)}2K=\frac{\lambda^{-(\alpha+v)m}}{1-\lambda^{-(\alpha+v)}}\frac{2K}{\Gamma(v)}\\\le&\frac{2K}{\Gamma(v)(1-\lambda^{-(\alpha+v)})}|x-y|^{\alpha+v}\\=&C_2|x-y|^{\alpha+v}.\end{aligned}$$

Then, we get

$$|F_\Theta(x)-F_\Theta(y)|\le C|x-y|^{\alpha+v}, \tag{3.1}$$

where C_1, C_2, C is a constant. By Lemma 2.1, we get

$$\dim_H Graph(F_\Theta,I)\le 2-\alpha-v. \tag{3.2}$$

Thus, we complete the proof. □

Theorem 3.2. *If* $0<v<1$, $\lambda>1$, $0<\alpha+v<1$, *we have*

$$\dim_H Graph(F_\Theta,I)\ge 2-\alpha-v.$$

Proof. By Theorem 3.1, we only need to show $\dim_H Graph(F_\Theta,I)\ge 2-\alpha-v$. Let μ_Θ be the measure supported on graph of F_Θ that is induced by Lebesgue measure u on interval $I=[0,1]$. That is, for $S\subset\mathbb{R}^2$,

$$\mu_\Theta(S)=\nu(\{x\in I:(x,F_\Theta(x))\in S\}).$$

Then the t-energy of μ_Θ is

$$\begin{aligned} I_t(\mu_\Theta) &= \iint_{Graph(F_\Theta,I)} \frac{d\mu_\Theta(x)d\mu_\Theta(y)}{[(x-y)^2+(F_\Theta(x)-F_\Theta(y))^2]^{t/2}} \\ &= \int_I\int_I \frac{dxdy}{[(x-y)^2+(F_\Theta(x)-F_\Theta(y))^2]^{t/2}}. \end{aligned}$$

Fix $t\in(1,2-\alpha-v)$, let

$$E_t = \int_H I_t(\mu_\Theta)d\Theta, \tag{3.3}$$

then by Tonelli theorem,

$$E_t = \int_I\int_I\int_H \frac{d\Theta}{[(x-y)^2+(F_\Theta(x)-F_\Theta(y))^2]^{t/2}}dxdy. \tag{3.4}$$

By Lemma 2.2, we have $h(Z)\le C_1|x-y|^{-(v+\alpha)}$ for certain positive constant C that is independent of x and y. It follows that

$$\begin{aligned} &\int_H \frac{d\Theta}{[(x-y)^2+(F_\Theta(x)-F_\Theta(y))^2]^{t/2}} \\ =&\int_{-\infty}^{\infty} \frac{h(Z)}{[(x-y)^2+Z^2]^{t/2}}dZ \\ =&\int_{-\infty}^{\infty} \frac{h(|x-y|W)|x-y|}{|x-y|^t(1+W^2)^{t/2}}dW \\ \le&\sup h(Z)|x-y|^{1-t}\int_{-\infty}^{\infty}\frac{dW}{(1+W^2)^{t/2}} \\ \le& C_2|x-y|^{2-\alpha-v-t-1}. \end{aligned}$$

Since $t<2-\alpha-v$, it holds that $E_t<\infty$. Thus we have proven for $t<2-\alpha-v$ that $I_t(\mu_\Theta)$ is finite for almost every $\Theta\in H$, which implies that the Hausdorff dimension of the graph of F_Θ is at least t. Choosing a sequence of values of t approaching $t<2-\alpha-v$, we conclude that for almost every $\Theta\in H$, the Hausdorff dimension of the graph of F_Θ is at least $t<2-\alpha-v$. That is

$$\dim_H Graph(F_\Theta,I)\ge 2-\alpha-v.$$

Thus, we complete the proof. □

From Theorem 3.1 and Theorem 3.2, we get the result that

$$\dim_H Graph(F_\Theta,I) = 2-\alpha-v.$$

4 Conclusions

This paper mainly studies the fractal dimension of a certain fractional function and proves that there exists some linear connection between the order of Riemann-Liouvile fractional integrals and the Hausdorff dimension of a fractal function. However, the conclusion is only true for some special fractal functions, we believe the linear connection holds for general fractal functions all the time.

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