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Classical Fourier Analysis over Homogeneous Spaces of Compact Groups

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Abstract. This paper introduces a unified operator theory approach to the abstract Fourier analysis over homogeneous spaces of compact groups. Let G be a compact group and H be a closed subgroup of G. Let G/H be the left coset space of H in G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then, we present a generalized abstract framework of Fourier analysis for the Hilbert function space $L^2(G/H,\mu)$.

Key Words: Compact group, homogeneous space, dual space, Fourier transform, Plancherel (trace) formula, Peter-Weyl Theorem.

AMS Subject Classifications: 20G05, 43A85, 43A32, 43A40, 43A90

1 Introduction

The abstract aspects of harmonic analysis over homogeneous spaces of compact non-Abelian groups or precisely left coset (resp. right coset) spaces of non-normal subgroups of compact non-Abelian groups is placed as building blocks for coherent states analysis [2–4, 12], theoretical and particle physics [1, 9–11, 13]. Over the last decades, abstract and computational aspects of Plancherel formulas over symmetric spaces have achieved significant popularity in geometric analysis, mathematical physics and scientific computing (computational engineering), see [6, 7, 13–18] and references therein.

Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. The left coset space G/H is considered as a compact homogeneous space, which G acts on it via the left action. This paper which contains 5 sections, is organized as follows. Section 2 is devoted to fix notations and preliminaries including a brief summary on Hilbert-Schmidt operators,

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non-Abelian Fourier analysis over compact groups, and classical results on abstract harmonic analysis over locally compact homogeneous spaces. We present some abstract harmonic analysis aspects of the Hilbert function space $L^2(G/H,\mu)$, in Section 3. Then we define the abstract notion of dual space $\widehat{G/H}$ for the homogeneous space G/H and we will show that this definition is precisely the standard dual space for the compact quotient group G/H, when H is a closed normal subgroup of G. We then introduce the definition of abstract operator-valued Fourier transform over the Banach function space $L^1(G/H,\mu)$ and also generalized version of the abstract Plancherel (trace) formula for the Hilbert function space $L^2(G/H,\mu)$. The paper closes by a presentation of Peter-Weyl Theorem for the Hilbert function space $L^2(G/H,\mu)$.

2 Preliminaries and notations

Let $\mathcal{H}$ be a separable Hilbert space. An operator $T\in\mathcal{B}(\mathcal{H})$ is called a Hilbert-Schmidt operator if for one, hence for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$ we have $\sum_k\|Te_k\|^2<\infty$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathrm{HS}(\mathcal{H})$ and for $T\in\mathrm{HS}(\mathcal{H})$ the Hilbert-Schmidt norm of T is $\|T\|_{\mathrm{HS}}^2=\sum_k\|Te_k\|^2$. The set $\mathrm{HS}(\mathcal{H})$ is a self adjoint two sided ideal in $\mathcal{B}(\mathcal{H})$ and if $\mathcal{H}$ is finite-dimensional we have $\mathrm{HS}(\mathcal{H})=\mathcal{B}(\mathcal{H})$. An operator $T\in\mathcal{B}(\mathcal{H})$ is trace-class, whenever $\|T\|_{\mathrm{tr}}=\mathrm{tr}[|T|]<\infty$, if $\mathrm{tr}[T]=\sum_k\langle Te_k,e_k\rangle$ and $|T|=(TT^*)^{1/2}$ [20].

Let G be a compact group with the probability Haar measure dx. Then each irreducible representation of G is finite dimensional and every unitary representation of G is a direct sum of irreducible representations, see [1,10]. The set of of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. This definition of $\widehat{G}$ is in essential agreement with the classical definition when G is Abelian, since each character of an Abelian group is a one dimensional representation of G. If π is any unitary representation of G, for $\zeta,\xi\in\mathcal{H}_\pi$ the functions $\pi_{\zeta,\xi}(x)=\langle\pi(x)\zeta,\xi\rangle$ are called matrix elements of π. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then π_{ij} means π_{e_i,e_j}. The notation $\mathcal{E}_\pi$ is used for the linear span of the matrix elements of π and the notation $\mathcal{E}$ is used for the linear span of $\bigcup_{[\pi]\in\widehat{G}}\mathcal{E}_\pi$. Then Peter-Weyl Theorem [1,10] guarantees that if G is a compact group, $\mathcal{E}$ is uniformly dense in $\mathcal{C}(G)$, $L^2(G)=\bigoplus_{[\pi]\in\widehat{G}}\mathcal{E}_\pi$, and $\{d_\pi^{-1/2}\pi_{ij}:i,j=1,\cdots,d_\pi,[\pi]\in\widehat{G}\}$ is an orthonormal basis for $L^2(G)$. For $f\in L^1(G)$ and $[\pi]\in\widehat{G}$, the Fourier transform of f at π is defined in the weak sense as an operator in $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\widehat{f}(\pi)=\int_G f(x)\pi(x)^*dx. \tag{2.1}$$

If $\pi(x)$ is represented by the matrix $(\pi_{ij}(x))\in\mathbb{C}^{d_\pi\times d_\pi}$. Then $\widehat{f}(\pi)\in\mathbb{C}^{d_\pi\times d_\pi}$ is the matrix with entries given by $\widehat{f}(\pi)_{ij}=d_\pi^{-1}c_{ji}^\pi(f)$ which satisfies

$$\sum_{i,j=1}^{d_\pi}c_{ij}^\pi(f)\pi_{ij}(x)=d_\pi\sum_{i,j=1}^{d_\pi}\widehat{f}(\pi)_{ji}\pi_{ij}(x)=d_\pi\mathrm{tr}[\widehat{f}(\pi)\pi(x)],$$

where $c^{\pi}_{i,j}(f)=d_\pi\langle f,\pi_{ij}\rangle_{L^2(G)}$. Then as a consequence of Peter-Weyl Theorem we get [19, 21, 23]

$$\|f\|^2_{L^2(G)}=\sum_{[\pi]\in\widehat{G}} d_\pi\|\widehat{f}(\pi)\|^2_{\mathrm{HS}}. \tag{2.2}$$

Let H be a closed subgroup of G with the probability Haar measure dh. The left coset space G/H is considered as a compact homogeneous space that G acts on it from the left and $q:G\to G/H$ given by $x\mapsto q(x):=xH$ is the surjective canonical map. The classical aspects of abstract harmonic analysis on locally compact homogeneous spaces are quite well studied by several authors, see [5,8,10,11,22] and references therein. If G is compact, each transitive G-space can be considered as a left coset space G/H for some closed subgroup H of G. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where $f\in\mathcal{C}(G)$ and

$$T_H(f)(xH)=\int_H f(xh)dh. \tag{2.3}$$

Let μ be a Radon measure on G/H and $x\in G$. The translation μ_x of μ is defined by $\mu_x(E)=\mu(xE)$, for all Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x=\mu$, for all $x\in G$. The homogeneous space G/H has a normalized G-invariant measure μ, which satisfies the following Weil's formula [1,22]

$$\int_{G/H} T_H(f)(xH)d\mu(xH)=\int_G f(x)dx \quad \text{for all } f\in L^1(G), \tag{2.4}$$

and also the following norm-decreasing formula

$$\|T_H(f)\|_{L^1(G/H,\mu)}\le\|f\|_{L^1(G)} \quad \text{for all } f\in L^1(G).$$

3 Abstract harmonic analysis of Hilbert function spaces over homogeneous spaces of compact groups

Throughout this paper we assume that G is a compact group with the probability Haar measure dx, H is a closed subgroup of G with the probability Haar measure dh, and also μ is the normalized G-invariant measure on the homogeneous space G/H which satisfies (2.4).

In this section, we present some properties of the Hilbert function space $L^2(G/H,\mu)$ in the framework of abstract harmonic analysis.

First we shall show that the linear map T_H has a unique extension to a bounded linear map from $L^2(G)$ onto $L^2(G/H,\mu)$.

Theorem 3.1. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. The linear map $T_H:\mathcal{C}(G)\to\mathcal{C}(G/H)$ has a unique extension to a bounded linear map from $L^2(G)$ onto $L^2(G/H,\mu)$.*

Proof. Let μ be the normalized G-invariant measure on the homogeneous space G/H which satisfies (2.4) and $f\in\mathcal{C}(G)$. Then we claim that

$$\|T_H(f)\|_{L^2(G/H,\mu)}\leq\|f\|_{L^2(G)}. \tag{3.1}$$

To this end, using compactness of H, we have

$$\begin{aligned}\|T_H(f)\|^p_{L^2(G/H,\mu)}&=\int_{G/H}|T_H(f)(xH)|^2d\mu(xH)=\int_{G/H}\left|\int_H f(xh)dh\right|^2d\mu(xH)\\&\leq\int_{G/H}\left(\int_H|f(xh)|dh\right)^2d\mu(xH)\leq\int_{G/H}\int_H|f(xh)|^2dhd\mu(xH).\end{aligned}$$

Then, by the Weil's formula, we get

$$\begin{aligned}\int_{G/H}\int_H|f(xh)|^2dhd\mu(xH)&=\int_{G/H}\int_H|f|^2(xh)dhd\mu(xH)\\&=\int_{G/H}T_H(|f|^2)(xH)d\mu(xH)=\int_G|f(x)|^2dx=\|f\|^2_{L^2(G)},\end{aligned}$$

which implies (3.1). Thus, we can extend T_H to a bounded linear operator from $L^2(G)$ onto $L^2(G/H,\mu)$, which we still denote it by T_H and satisfies

$$\|T_H(f)\|_{L^2(G/H,\mu)}\leq\|f\|_{L^2(G)}\quad\text{for all }\ f\in L^2(G).$$

Thus, we complete the proof. □

Let $\mathcal{J}^2(G,H):=\{f\in L^2(G):T_H(f)=0\}$ and $\mathcal{J}^2(G,H)^\perp$ be the orthogonal completion of the closed subspace $\mathcal{J}^2(G,H)$ in $L^2(G)$.

As an immediate consequence of Theorem 3.1 we deduce the following result.

Proposition 3.1. Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then $T_H:L^2(G)\to L^2(G/H,\mu)$ is a partial isometric linear map.

Proof. Let $\varphi\in L^2(G/H,\mu)$ and $\varphi_q:=\varphi\circ q$. Then $\varphi_q\in L^2(G)$ with

$$\|\varphi_q\|_{L^2(G)}=\|\varphi\|_{L^2(G/H,\mu)}. \tag{3.2}$$

Indeed, using the Weil's formula we can write

$$\begin{aligned}\|\varphi_q\|^2_{L^2(G)}&=\int_G|\varphi_q(x)|^2dx=\int_{G/H}T_H\left(|\varphi_q|^2\right)(xH)d\mu(xH)\\&=\int_{G/H}\left(\int_H|\varphi_q(xh)|^2dh\right)d\mu(xH),\end{aligned}$$

and since H is compact and dh is a probability measure, we get

$$\begin{aligned}\int_{G/H}\left(\int_H|\varphi_q(xh)|^2dh\right)d\mu(xH)&=\int_{G/H}\left(\int_H|\varphi(xhH)|^2dh\right)d\mu(xH)\\&=\int_{G/H}\left(\int_H|\varphi(xH)|^2dh\right)d\mu(xH)\\&=\int_{G/H}|\varphi(xH)|^2\left(\int_H dh\right)d\mu(xH)\\&=\int_{G/H}|\varphi(xH)|^2d\mu(xH)=\|\varphi\|^2_{L^2(G/H,\mu)},\end{aligned}$$

which implies (3.2). Then $T_H^*(\varphi)=\varphi_q$ and $T_HT_H^*(\varphi)=\varphi$. Because using the Weil's formula we have

$$\begin{aligned}\langle T_H^*(\varphi),f\rangle_{L^2(G)}=\langle\varphi,T_H(f)\rangle_{L^2(G/H,\mu)}&=\int_{G/H}\varphi(xH)\overline{T_H(f)(xH)}d\mu(xH)\\&=\int_{G/H}\varphi(xH)T_H(\overline{f})(xH)d\mu(xH)=\int_{G/H}T_H(\varphi_q.\overline{f})(xH)d\mu(xH)\\&=\int_G\varphi_q(x)\overline{f(x)}dx=\langle\varphi_q,f\rangle_{L^2(G)},\end{aligned}$$

for all $f\in L^2(G)$, which implies that $T_H^*(\varphi)=\varphi_q$. Now a straightforward calculation shows that $T_H=T_HT_H^*T_H$. Then by Theorem 2.3.3 of [20], T_H is a partial isometric operator. □

We then can conclude the following corollaries as well.

Corollary 3.1. Let H be a closed subgroup of a compact group G. Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^\perp$ respectively. Then, for each $f\in L^2(G)$ and a.e. $x\in G$, we have

1. $P_{\mathcal{J}^2(G,H)^\perp}(f)(x)=T_H(f)(xH)$.
2. $P_{\mathcal{J}^2(G,H)}(f)(x)=f(x)-T_H(f)(xH)$.

Corollary 3.2. Let H be a compact subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then

1. $\mathcal{J}^2(G,H)^\perp=\{\psi_q:\psi\in L^2(G/H,\mu)\}$.
2. For $f\in\mathcal{J}^2(G,H)^\perp$ and $h\in H$ we have $R_hf=f$.
3. For $\psi\in L^2(G/H,\mu)$ we have $\|\psi_q\|_{L^2(G)}=\|\psi\|_{L^2(G/H,\mu)}$.
4. For $f,g\in\mathcal{J}^2(G,H)^\perp$ we have $\langle T_H(f),T_H(g)\rangle_{L^2(G/H,\mu)}=\langle f,g\rangle_{L^2(G)}$.

We finish this section by the following remark.

Remark 3.1. Invoking Corollary 3.2 one can regard the Hilbert function space $L^2(G/H,\mu)$ as a closed linear subspace of the Hilbert function space $L^2(G)$, that is the closed linear subspace consists of all $f\in L^2(G)$ which satisfies $R_hf=f$ for all $h\in H$. Then Theorem 3.1 and Proposition 3.1 guarantees that the bounded linear map

$$T_H:L^2(G)\to L^2(G/H,\mu)\subset L^2(G)$$

is an orthogonal projection.

4 Abstract trace formulas over homogeneous spaces of compact groups

In this section, we present the abstract notions of dual spaces and Plancherel (trace) formulas over homogeneous spaces of compact groups.

For a closed subgroup H of G, let

$$H^{\perp}=\Big\{[\pi]\in\widehat{G}:\pi(h)=I \quad \text{for all } h\in H\Big\}. \tag{4.1}$$

Then by definition we have

$$H^{\perp}\subseteq\widehat{G}. \tag{4.2}$$

If G is Abelian, each closed subgroup H of G is normal and the compact group G/H is Abelian and so $\widehat{G/H}$ is precisely the set of all characters (one dimensional irreducible representations) of G which are constant on H, that is precisely $H^{\perp}$. If G is a non-Abelian group and H is a closed normal subgroup of G, then the dual space $\widehat{G/H}$ which is the set of all unitary equivalence classes of unitary representations of the quotient group G/H, has meaning and it is well-defined. Indeed, G/H is a non-Abelian group. In this case, the map $\Phi:\widehat{G/H}\to H^{\perp}$ defined by $\sigma\mapsto\Phi(\sigma):=\sigma\circ q$ is a Borel isomorphism and $\widehat{G/H}=H^{\perp}$, see [1, 19, 23]. Thus if H is normal, $H^{\perp}$ coincides with the classic definitions of the dual space either when G is Abelian or non-Abelian.

For a given closed subgroup H of G and also a continuous unitary representation $(\pi,\mathcal{H}_\pi)$ of G, define

$$T_H^{\pi}:=\int_H\pi(h)dh, \tag{4.3}$$

where the operator valued integral (4.3) is considered in the weak sense. In other words,

$$\langle T_H^{\pi}\zeta,\xi\rangle=\int_H\langle\pi(h)\zeta,\xi\rangle dh \quad \text{for } \zeta,\xi\in\mathcal{H}_\pi. \tag{4.4}$$

The function $h\mapsto\langle\pi(h)\zeta,\xi\rangle$ is bounded and continuous on H. Since H is compact, the right integral is the ordinary integral of a function in $L^1(H)$. Hence, T_H^{π} is a bounded linear operator on $\mathcal{H}_\pi$ with $\|T_H^{\pi}\|\le 1$.

Definition 4.1. Let H be a compact subgroup of a compact group G. The dual space $\widehat{G/H}$ of the left coset space G/H, is defined as the subset of $\widehat{G}$ given by

$$\widehat{G/H} := \left\{[\pi]\in\widehat{G}: T_H^\pi \neq 0\right\} = \left\{[\pi]\in\widehat{G}: \int_H \pi(h)dh \neq 0\right\}. \tag{4.5}$$

Then evidently we have

$$H^\perp \subseteq \widehat{G/H}. \tag{4.6}$$

First we shall present an interesting property of (4.5), when the left coset space G/H has the canonical quotient group structure.

Next theorem shows that the reverse inclusion of (4.6) holds, if H is a normal subgroup of G.

Theorem 4.1. *Let H be a closed normal subgroup of a compact group G. Then,*

$$\widehat{G/H} = H^\perp.$$

Proof. Let H be a closed normal subgroup of a compact group G. Invoking the inclusion (4.6), it is sufficient to show that $\widehat{G/H}\subseteq H^\perp$. Let $[\pi]\in\widehat{G/H}$ be given. Due to normality of H in G the map $\tau_x: H\to H$ given by $h\mapsto \tau_x(h) := x^{-1}hx$ belongs to $Aut(H)$ and also we have $x^{-1}Hx = H$, for all $x\in G$. Let $x\in G$. Then by compactness of G we have $d(\tau_x(h)) = dh$ and hence we can write

$$\begin{aligned}\int_H \pi(h)dh &= \int_{xHx^{-1}} \pi(\tau_x(h))d(\tau_x(h)) = \int_H \pi(\tau_x(h))dh\\ &= \int_H \pi(x)^*\pi(h)\pi(x)dh = \pi(x)^*\left(\int_H \pi(h)dh\right)\pi(x)\\ &= \pi(x)^* T_H^\pi \pi(x),\end{aligned}$$

which implies that $\pi(x)T_H^\pi = T_H^\pi\pi(x)$. Since $x\in G$ was arbitrary we deduce that $T_H^\pi\in \mathcal{C}(\pi)$. Irreducibility of π guarantees that $T_H^\pi = \alpha I$ for some constant $\alpha\in\mathbb{C}$ with $|\alpha|\leq 1$. By definition of $\widehat{G/H}$ we have $T_H^\pi\neq 0$ and hence we get $\alpha\neq 0$. Now let $t\in H$ be arbitrary. Then we can write

$$\pi(t) = \alpha^{-1}\pi(t)\alpha I = \alpha^{-1}\pi(t)T_H^\pi = \alpha^{-1}\int_H \pi(th)dh = \alpha^{-1}\int_H \pi(h)dh = \alpha^{-1}T_H^\pi = I,$$

which implies $[\pi]\in H^\perp$. □

Let

$$\mathcal{K}_\pi^H := \{\zeta\in\mathcal{H}_\pi : \pi(h)\zeta = \zeta\ \forall h\in H\}. \tag{4.7}$$

Then, $\mathcal{K}_\pi^H$ is a closed linear subspace of $\mathcal{H}_\pi$ and $\mathcal{R}(T_H^\pi) = \mathcal{K}_\pi^H$, where

$$\mathcal{R}(T_H^\pi) = \{T_H^\pi\zeta : \zeta\in\mathcal{H}_\pi\}.$$

It is easy to see that $[\pi]\in H^\perp$ if and only if $\mathcal{K}_\pi^H = \mathcal{H}_\pi$.

Then, we can also present the following results.

Proposition 4.1. Let H be a closed subgroup of a compact group G and $(\pi,\mathcal{H}_\pi)$ be a continuous unitary representation of G. Then,

1. The operator T_H^π is an orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{K}_\pi^H$.
2. The operator T_H^π is unitary if and only if $[\pi]\in H^\perp$.

Proof. (1) Using compactness of H, we have

$$(T_H^\pi)^*=\left(\int_H \pi(h)dh\right)^*=\int_H \pi(h)^*dh=\int_H \pi(h^{-1})dh=T_H^\pi.$$

As well as, we can write

$$\begin{aligned}T_H^\pi T_H^\pi&=\left(\int_H \pi(h)dh\right)\left(\int_H \pi(t)dt\right)=\int_H \pi(h)\left(\int_H \pi(t)dt\right)dh\\&=\int_H\left(\int_H \pi(h)\pi(t)dt\right)dh=\int_H\left(\int_H \pi(ht)dt\right)dh=\int_H T_H^\pi dt=T_H^\pi.\end{aligned}$$

(2) The operator T_H^π is unitary if and only if $T_H^\pi=I$. The operator T_H^π is identity if and only if $\pi(h)=I$ for all $h\in H$. Thus, T_H^π is unitary if and only if $[\pi]\in H^\perp$. $\square$

Let $\varphi\in L^1(G/H,\mu)$ and $[\pi]\in\widehat{G/H}$. The Fourier transform of φ at $[\pi]$ is defined as the linear operator

$$\mathcal{F}(\varphi)(\pi)=\widehat{\varphi}(\pi):=\int_{G/H}\varphi(xH)\Gamma_\pi(xH)^*d\mu(xH), \tag{4.8}$$

on the Hilbert space $\mathcal{H}_\pi$, where for each $xH\in G/H$ the notation $\Gamma_\pi(xH)$ stands for the bounded linear operator defined on the Hilbert space $\mathcal{H}_\pi$ by $\Gamma_\pi(xH)=\pi(x)T_H^\pi$, that is

$$\langle\Gamma_\pi(xH)\zeta,\xi\rangle=\langle\pi(x)T_H^\pi\zeta,\xi\rangle \quad \text{for } \zeta,\xi\in\mathcal{H}_\pi. \tag{4.9}$$

Then we have

$$\langle\Gamma_\pi(xH)\zeta,\xi\rangle=T_H(\pi_{\zeta,\xi})(xH)$$

for all $\zeta,\xi\in\mathcal{H}_\pi$. Indeed,

$$\begin{aligned}\langle\Gamma_\pi(xH)\zeta,\xi\rangle&=\langle\pi(x)T_H^\pi\zeta,\xi\rangle=\left\langle\pi(x)\left(\int_H\pi(h)dh\right)\zeta,\xi\right\rangle\\&=\left\langle\left(\int_H\pi(x)\pi(h)dh\right)\zeta,\xi\right\rangle=\left\langle\left(\int_H\pi(xh)dh\right)\zeta,\xi\right\rangle\\&=\int_H\langle\pi(xh)\zeta,\xi\rangle dh=\int_H\pi_{\zeta,\xi}(xh)dh=T_H(\pi_{\zeta,\xi})(xH).\end{aligned}$$

Remark 4.1. Let H be a closed normal subgroup of a compact group G and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. Then it is easy to check that μ is a Haar measure of the compact quotient group G/H and by Theorem 4.1 we have $\widehat{G/H}=H^{\perp}$. Also, for each $\varphi\in L^1(G/H,\mu)$ and $[\pi]\in H^{\perp}$, we have

$$\widehat{\varphi}(\pi)=\int_{G/H}\varphi(xH)\pi(x)^*d\mu(xH).$$

Thus, we deduce that the abstract Fourier transform defined by (4.8) coincides with the classical Fourier transform over the compact quotient group G/H if H is normal in G.

The operator-valued integral (4.8) is considered in the weak sense. That is

$$\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle=\int_{G/H}\varphi(xH)\langle\zeta,\Gamma_\pi(xH)^*\xi\rangle d\mu(xH)\quad\text{for }\ \zeta,\xi\in\mathcal{H}_\pi. \tag{4.10}$$

In other words, for $[\pi]\in\widehat{G/H}$ and $\zeta,\xi\in\mathcal{H}_\pi$ we have

$$\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle=\int_{G/H}\varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH). \tag{4.11}$$

Because, we can write

$$\begin{aligned}\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle&=\int_{G/H}\varphi(xH)\langle\zeta,\Gamma_\pi(xH)^*\xi\rangle d\mu(xH)\\&=\int_{G/H}\varphi(xH)\langle\Gamma_\pi(xH)\zeta,\xi\rangle d\mu(xH)\\&=\int_{G/H}\varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH).\end{aligned}$$

If $\zeta,\xi\in\mathcal{H}_\pi$, then we have

$$\begin{aligned}&|\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle|\\&=\left|\int_{G/H}\varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH)\right|\le\int_{G/H}|\varphi(xH)|\big|T_H(\pi_{\zeta,\xi})(xH)\big|d\mu(xH)\\&=\int_{G/H}|\varphi(xH)|\left|\int_H\pi_{\zeta,\xi}(xh)dh\right|d\mu(xH)\le\int_{G/H}|\varphi(xH)|\left(\int_H|\pi_{\zeta,\xi}(xh)|dh\right)d\mu(xH)\\&\le\int_{G/H}|\varphi(xH)|\left(\int_H\|\pi(xh)\zeta\|\cdot\|\xi\|dh\right)d\mu(xH)=\int_{G/H}|\varphi(xH)|\left(\int_H\|\zeta\|\cdot\|\xi\|dh\right)d\mu(xH)\\&=\|\zeta\|\cdot\|\xi\|\cdot\left(\int_{G/H}|\varphi(xH)|\left(\int_H dh\right)d\mu(xH)\right)=\|\zeta\|\cdot\|\xi\|\cdot\|\varphi\|_{L^1(G/H,\mu)},\end{aligned}$$

so we deduce that $\widehat{\varphi}(\pi)$ is a bounded linear operator on $\mathcal{H}_\pi$ with

$$\|\widehat{\varphi}(\pi)\|\le\|\varphi\|_{L^1(G/H,\mu)}.$$

The following proposition presents the canonical connection of the abstract Fourier transform defined in (4.8) with the classical Fourier transform (2.1).

Proposition 4.2. Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then, for $\varphi \in L^1(G/H,\mu)$ and $[\pi]\in\widehat{G/H}$, we have

$$\widehat{\varphi}(\pi)=\widehat{\varphi_q}(\pi). \tag{4.12}$$

Proof. Using the Weil's formula and also (4.11), for $\zeta,\xi\in\mathcal{H}_\pi$, we can write

$$\begin{aligned}\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle&=\int_{G/H}\varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH)=\int_{G/H}T_H(\varphi_q\cdot\pi_{\zeta,\xi})(xH)d\mu(xH)\\&=\int_G\varphi_q(x)\pi_{\zeta,\xi}(x)dx=\int_G\varphi_q(x)\langle\pi(x)\zeta,\xi\rangle dx\\&=\int_G\varphi_q(x)\langle\zeta,\pi(x)^*\xi\rangle dx=\langle\zeta,\widehat{\varphi_q}(\pi)\xi\rangle,\end{aligned}$$

which implies (4.12). □

In the next theorem we show that the abstract Fourier transform defined in (4.8) satisfies a generalized version of the Plancherel (trace) formula.

Theorem 4.2. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then, each $\varphi\in L^2(G/H,\mu)$ satisfies the following Plancherel formula;*

$$\sum_{[\pi]\in\widehat{G/H}}d_\pi\|\widehat{\varphi}(\pi)\|^2_{\mathrm{HS}}=\|\varphi\|^2_{L^2(G/H,\mu)}. \tag{4.13}$$

Proof. Let $\varphi\in L^2(G/H,\mu)$ be given. If $[\pi]\in\widehat{G}$ with $[\pi]\notin\widehat{G/H}$, then we have $T_H^\pi=0$. Hence, for $\zeta,\xi\in\mathcal{H}_\pi$, we have $T_H(\pi_{\zeta,\xi})=0$. Therefore, we get

$$\widehat{\varphi_q}(\pi)=0. \tag{4.14}$$

Indeed, using the Weil's formula, for $\zeta,\xi\in\mathcal{H}_\pi$ we can write

$$\begin{aligned}\langle\zeta,\widehat{\varphi_q}(\pi)\xi\rangle&=\int_G\varphi_q(x)\langle\zeta,\pi(x)^*\xi\rangle dx=\int_G\varphi_q(x)\langle\pi(x)\zeta,\xi\rangle dx=\int_G\varphi(x)\pi_{\zeta,\xi}(x)dx\\&=\int_{G/H}T_H(\varphi_q.\pi_{\zeta,\xi})(xH)d\mu(xH)=\int_{G/H}\varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH)=0.\end{aligned}$$

Using Eqs. (4.12), (4.14), invoking Plancherel formula (2.2), and also Corollary 3.2 we achieve

$$\begin{aligned}\sum_{[\pi]\in\widehat{G/H}}d_\pi\|\widehat{\varphi}(\pi)\|^2_{\mathrm{HS}}&=\sum_{[\pi]\in\widehat{G/H}}d_\pi\|\widehat{\varphi_q}(\pi)\|^2_{\mathrm{HS}}\\&=\sum_{[\pi]\in\widehat{G/H}}d_\pi\|\widehat{\varphi_q}(\pi)\|^2_{\mathrm{HS}}+\sum_{\{[\pi]\in\widehat{G}:[\pi]\notin\widehat{G/H}\}}d_\pi\|\widehat{\varphi_q}(\pi)\|^2_{\mathrm{HS}}\\&=\sum_{[\pi]\in\widehat{G}}d_\pi\|\widehat{\varphi_q}(\pi)\|^2_{\mathrm{HS}}=\|\varphi_q\|^2_{L^2(G)}=\|\varphi\|^2_{L^2(G/H,\mu)},\end{aligned}$$

which implies (4.13). □

Remark 4.2. Let H be a closed normal subgroup of a compact group G and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. Then Theorem 4.1 implies that $\widehat{G/H}=H^{\perp}$ and hence the Plancherel (trace) formula (4.13) reads as follows;

$$\sum_{[\pi]\in H^{\perp}} d_{\pi}\|\widehat{\varphi}(\pi)\|_{\mathrm{HS}}^2=\|\varphi\|^2_{L^2(G/H,\mu)}$$

for all $\varphi\in L^2(G/H,\mu)$, where

$$\widehat{\varphi}(\pi)=\int_{G/H}\varphi(xH)\pi(x)^*d\mu(xH)$$

for all $[\pi]\in H^{\perp}$, see Remark 4.1.

5 Peter-Weyl theorem for homogeneous spaces of compact groups

In this section we present a version of Peter-Weyl Theorem [21] for the Hilbert function space $L^2(G/H,\mu)$.

Let $(\pi,\mathcal{H}_{\pi})$ be a continuous unitary representation of G such that $T_H^{\pi}\neq 0$. Then the functions $\pi^H_{\zeta,\xi}:G/H\to\mathbb{C}$ defined by

$$\pi^H_{\zeta,\xi}(xH):=\langle\pi(x)T_H^{\pi}\zeta,\xi\rangle\quad\text{for } xH\in G/H \tag{5.1}$$

for $\xi,\zeta\in\mathcal{H}_{\pi}$ are called H-matrix elements of $(\pi,\mathcal{H}_{\pi})$.

For $xH\in G/H$ and $\zeta,\xi\in\mathcal{H}_{\pi}$, we have

$$|\pi^H_{\zeta,\xi}(xH)|=|\langle\pi(x)T_H^{\pi}\zeta,\xi\rangle|\leq\|\pi(x)T_H^{\pi}\zeta\|\|\xi\|\leq\|T_H^{\pi}\zeta\|\|\xi\|\leq\|\zeta\|\|\xi\|.$$

Also we can write

$$\pi^H_{\zeta,\xi}(xH)=\langle\pi(x)T_H^{\pi}\zeta,\xi\rangle=\pi_{T_H^{\pi}\zeta,\xi}(x). \tag{5.2}$$

Invoking definition of the linear map T_H and also T_H^{π} we have

$$\begin{aligned}T_H(\pi_{\zeta,\xi})(xH)&=\int_H\pi_{\zeta,\xi}(xh)dh=\int_H\langle\pi(xh)\zeta,\xi\rangle dh\\&=\int_H\langle\pi(x)\pi(h)\zeta,\xi\rangle dh=\langle\pi(x)T_H^{\pi}\zeta,\xi\rangle,\end{aligned}$$

which implies that

$$T_H(\pi_{\zeta,\xi})=\pi^H_{\zeta,\xi}. \tag{5.3}$$

Theorem 5.1. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure and $(\pi,\mathcal{H}_{\pi})$ be a continuous unitary representation of G such that $T_H^{\pi}\neq 0$. Then*

1. *The subspace $\mathcal{E}_\pi(G/H)$ depends on the unitary equivalence class of π.*

2. *The subspace $\mathcal{E}_\pi(G/H)$ is a closed left invariant subspace of $L^1(G/H,\mu)$.*

Proof. (1) Let $(\sigma,\mathcal{H}_\sigma)$ be a continuous unitary representation of G such that $[\pi]=[\sigma]$. Let $S:\mathcal{H}_\pi\to\mathcal{H}_\sigma$ be the unitary operator which satisfies $\sigma(x)S=S\pi(x)$ for all $x\in G$. Then $ST_H^\pi=T_H^\sigma S$ and also $T_H^\sigma\neq 0$. Thus for $x\in G$ and $\zeta,\xi\in\mathcal{H}_\pi$ we can write

$$\begin{aligned}\pi^H_{\zeta,\xi}(xH)&=\langle\pi(x)T_H^\pi\zeta,\xi\rangle_{\mathcal{H}_\pi}=\langle S^{-1}\sigma(x)ST_H^\pi\zeta,\xi\rangle_{\mathcal{H}_\pi}\\&=\langle\sigma(x)ST_H^\pi\zeta,S\xi\rangle_{\mathcal{H}_\sigma}=\langle\sigma(x)T_H^\sigma S\zeta,S\xi\rangle_{\mathcal{H}_\sigma}=\sigma^H_{S\zeta,S\xi}(xH),\end{aligned}$$

which implies that $\mathcal{E}_\pi(G/H)=\mathcal{E}_\sigma(G/H)$.
(2) It is straightforward. □

If ζ,ξ belongs to an orthonormal basis $\{e_i\}$ for $\mathcal{H}_\pi$, H-matrix elements of $[\pi]$ with respect to an orthonormal basis $\{e_j\}$ changes in the form

$$\pi^H_{ij}(xH)=\pi^H_{e_j,e_i}(xH)=\langle\pi(x)T_H^\pi e_j,e_i\rangle\quad\text{for}\ \ xH\in G/H. \tag{5.4}$$

The linear span of the H-matrix elements of a continuous unitary representation $(\pi,\mathcal{H}_\pi)$ satisfying $T_H^\pi\neq 0$, is denoted by $\mathcal{E}_\pi(G/H)$ which is a subspace of $\mathcal{C}(G/H)$.

Definition 5.1. Let H be a closed subgroup of a compact group G and $[\pi]\in\widehat{G/H}$. An ordered orthonormal basis $\mathfrak{B}=\{e_\ell:1\leq\ell\leq d_\pi\}$ of the Hilbert space $\mathcal{H}_\pi$ is called H-admissible, if it is an extension of an orthonormal basis $\{e_\ell:1\leq\ell\leq d_{\pi,H}\}$ of the closed subspace $\mathcal{K}_\pi^H$, which equivalently means that $d_{\pi,H}$-first elements of $\mathfrak{B}$ be an orthogonal basis of $\mathcal{K}_\pi^H$.

Let $[\pi]\in\widehat{G/H}$ and $\mathfrak{B}_\pi=\{e_\ell:1\leq\ell\leq d_\pi\}$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$. Then, each $\pi_{i\ell}$ with $1\leq i\leq d_\pi$ and $1\leq\ell\leq d_{\pi,H}$, is a well-defined continuous function over G/H. Let $\mathcal{E}_\pi^\ell(G/H)$ be the subspace of $\mathcal{C}(G/H)$ spanned by the set $\mathfrak{B}_\pi^\ell:=\{\sqrt{d_\pi}\pi_{i\ell}:1\leq i\leq d_\pi\}$.

Proposition 5.1. *Let $[\pi]\in\widehat{G/H}$, $\mathfrak{B}_\pi$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$, and $1\leq\ell\neq\ell'\leq d_{\pi,H}$. Then*

1. $\dim\mathcal{E}_\pi^\ell(G/H)=d_\pi$ *and $\mathfrak{B}_\pi^\ell$ is an orthonormal basis for $\mathcal{E}_\pi^\ell(G/H)$.*

2. $\mathcal{E}_\pi^\ell(G/H)$ *is a closed left translation invariant subspace of $\mathcal{C}(G/H)$.*

3. $\mathcal{E}_\pi^{\ell'}(G/H)\perp\mathcal{E}_\pi^\ell(G/H)$.

Proof. (1) Let $1\leq i,i'\leq d_\pi$. Then by Theorem 27.19 of [11] we get

$$\langle\pi_{i\ell},\pi_{i'\ell}\rangle_{L^2(G/H,\mu)}=\langle\pi_{i\ell},\pi_{i'\ell}\rangle_{L^2(G)}=d_\pi^{-1}\delta_{ii'}.$$

Since $\dim\mathcal{E}_\pi^\ell(G/H)\leq d_\pi$ we achieve that $\mathfrak{B}_\pi^\ell$ is an orthonormal basis for $\mathcal{E}_\pi^\ell(G/H)$ and hence $\dim\mathcal{E}_\pi^\ell(G/H)=d_\pi$.

(2) It is straightforward.
(3) Let $1\leq i,i'\leq d_\pi$. Applying Theorem 27.19 of [11] we get

$$\langle \pi_{i\ell},\pi_{i'\ell'}\rangle_{L^2(G/H,\mu)}=\langle \pi_{i\ell},\pi_{i'\ell'}\rangle_{L^2(G)}=d_\pi^{-1}\delta_{ii'}\delta_{\ell\ell'},$$

which completes the proof. □

The following theorem shows that H-admissible bases lead to orthogonal decompositions of the subspace $\mathcal{E}_\pi(G/H)$.

Theorem 5.2. *Let H be a closed subgroup of a compact group G. Let $[\pi]\in\widehat{G/H}$ and $\mathfrak{B}_\pi=\{e_{\ell,\pi}:1\leq\ell\leq d_\pi\}$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$. Then $\mathfrak{B}_\pi(G/H):=\{\sqrt{d_\pi}\pi_{i\ell}:1\leq i\leq d_\pi,1\leq\ell\leq d_{\pi,H}\}$ is an orthonormal basis for the Hilbert space $\mathcal{E}_\pi(G/H)$ and hence it satisfies the following direct sum decomposition*

$$\mathcal{E}_\pi(G/H)=\bigoplus_{\ell=1}^{d_{\pi,H}}\mathcal{E}_\pi^\ell(G/H). \tag{5.5}$$

Proof. It is straightforward to check that $\mathfrak{B}_\pi(G/H)$ spans the subspace $\mathcal{E}_\pi(G/H)$. Then Proposition 5.1 guarantees that $\mathfrak{B}_\pi(G/H)$ is an orthonormal set in $\mathcal{E}_\pi(G/H)$. Since $\dim\mathcal{E}_\pi(G/H)\leq d_{\pi,H}d_\pi$ we deduce that it is an orthonormal basis for $\mathcal{E}_\pi(G/H)$, which automatically implies the decomposition (5.5). □

Next proposition lists basic properties of H-matrix elements.

Proposition 5.2. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $(\pi,\mathcal{H}_\pi)$ be a continuous unitary representation of G. Then,*

1. *$T_H^\pi=0$ if and only if $\mathcal{E}_\pi(G)\subseteq\mathcal{J}^2(G,H)$.*
2. *If $T_H^\pi\neq 0$ then $T_H(\mathcal{E}_\pi(G))=\mathcal{E}_\pi(G/H)$ and $T_H^*(\mathcal{E}_\pi(G/H))\subseteq\mathcal{E}_\pi(G)$.*
3. *$\mathcal{E}_\pi(G)\subseteq\mathcal{J}^2(G,H)^\perp$ if and only if $\pi(h)=I$ for all $h\in H$.*

Then we can prove the following orthogonality relation concerning the functions in $\mathcal{E}(G/H)$.

Theorem 5.3. *Let H be a closed subgroup of a compact group G, μ be a normalized G-invariant measure on G/H and $[\pi]\neq[\sigma]\in\widehat{G/H}$. The closed subspaces $\mathcal{E}_\pi(G/H)$ and $\mathcal{E}_\sigma(G/H)$ are orthogonal to each other as subspaces of the Hilbert space $L^2(G/H,\mu)$.*

Proof. Let $\psi\in\mathcal{E}_\pi(G/H)$ and $\varphi\in\mathcal{E}_\sigma(G/H)$. Then we have $\psi_q\in\mathcal{E}_\pi(G)$ and also $\varphi_q\in\mathcal{E}_\sigma(G)$. Using Proposition 5.2, Corollary 3.2, and Theorem 27.15 of [11], we get

$$\langle\varphi,\psi\rangle_{L^2(G/H,\mu)}=\langle\varphi_q,\psi_q\rangle_{L^2(G)}=0,$$

which completes the proof. □

We can define

$$\mathcal{E}(G/H):=\text{the linear span of} \bigcup_{[\pi]\in\widehat{G/H}} \mathcal{E}_\pi(G/H). \tag{5.6}$$

Next theorem presents some analytic aspects of the function space $\mathcal{E}(G/H)$.

Theorem 5.4. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then,*

1. *The linear operator T_H maps $\mathcal{E}(G)$ onto $\mathcal{E}(G/H)$.*
2. *$\mathcal{E}(G/H)$ is $\|.\|_{L^2(G/H,\mu)}$-dense in $L^2(G/H,\mu)$.*
3. *$\mathcal{E}(G/H)$ is $\|.\|_{\sup}$-dense in $\mathcal{C}(G/H)$.*

Proof. (1) It is straightforward.

(2) Let $\phi\in L^2(G/H,\mu)$ and also $f\in L^2(G)$ with $T_H(f)=\phi$. Then by $\|\cdot\|_{L^2(G)}$-density of $\mathcal{E}(G)$ in $L^2(G)$ we can pick a sequence $\{f_n\}$ in $\mathcal{E}(G)$ such that $f=\|\cdot\|_{L^2(G)}-\lim_n f_n$. By Proposition 5.2 we have $\{T_H(f_n)\}\subseteq\mathcal{E}(G/H)$. Then continuity of the linear map $T_H: L^2(G)\to L^2(G/H,\mu)$ implies

$$\phi=T_H(f)=\|\cdot\|_{L^2(G/H,\mu)}-\lim_n T_H(f_n),$$

which completes the proof.

(3) Invoking uniformly boundedness of T_H, uniformly density of $\mathcal{E}(G)$ in $\mathcal{C}(G)$, and the same argument as used in (1), we get $\|\cdot\|_{\sup}$-density of $\mathcal{E}(G/H)$ in $\mathcal{C}(G/H)$. $\square$

The following theorem can be considered as an abstract extension of the Peter-Weyl Theorem for homogeneous spaces of compact groups.

Theorem 5.5. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The Hilbert space $L^2(G/H,\mu)$ satisfies the following orthogonality decomposition*

$$L^2(G/H,\mu)=\bigoplus_{[\pi]\in\widehat{G/H}} \mathcal{E}_\pi(G/H). \tag{5.7}$$

Proof. Using Peter-Weyl Theorem, Proposition 5.2, and since the bounded linear map $T_H: L^2(G)\to L^2(G/H,\mu)$ is surjective we achieve that each $\varphi\in L^2(G/H,\mu)$ has a decomposition to elements of $\mathcal{E}_\pi(G/H)$ with $[\pi]\in\widehat{G/H}$, namely

$$\varphi=\sum_{[\pi]\in\widehat{G/H}} c_\pi\varphi_\pi, \tag{5.8}$$

with $\varphi_\pi\in\mathcal{E}_\pi(G/H)$ for all $[\pi]\in\widehat{G/H}$. Since the subspaces $\mathcal{E}_\pi(G/H)$ with $[\pi]\in\widehat{G/H}$ are mutually orthogonal we conclude that decomposition (5.8) is unique for each φ, which guarantees (5.7). $\square$

We immediately deduce the following corollaries.

Corollary 5.1. Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. For each $[\pi]\in\widehat{G/H}$, let $\mathfrak{B}_\pi=\{e_{\ell,\pi}:1\le\ell\le d_\pi\}$ be an H-admissible basis for the representation space $\mathcal{H}_\pi$. Then we have the following statements.

1. The Hilbert space $L^2(G/H,\mu)$ satisfies the following direct sum decomposition
$$L^2(G/H,\mu)=\bigoplus_{[\pi]\in\widehat{G/H}}\bigoplus_{\ell=1}^{d_{\pi,H}}\mathcal{E}_\pi^\ell(G/H). \tag{5.9}$$

2. The set $\mathfrak{B}(G/H):=\{\pi_{i\ell}:1\le i\le d_\pi,\ 1\le\ell\le d_{\pi,H}\}$ constitutes an orthonormal basis for the Hilbert space $L^2(G/H,\mu)$.

3. Each $\varphi\in L^2(G/H,\mu)$ decomposes as the following:
$$\varphi=\sum_{[\pi]\in\widehat{G/H}}d_\pi\sum_{\ell=1}^{d_{\pi,H}}\sum_{i=1}^{d_\pi}\langle\varphi,\pi_{i\ell}\rangle_{L^2(G/H,\mu)}\pi_{i\ell}, \tag{5.10}$$
where the series is converges in $L^2(G/H,\mu)$.

Remark 5.1. Let H be a closed normal subgroup of a compact group G. Also, let μ be the normalized G-invariant measure over G/H associated to the Weil's formula. Then G/H is a compact group and the normalized G-invariant measure μ is a Haar measure of the quotient compact group G/H. By Theorem 4.1, we deduce that $\widehat{G/H}=H^\perp$, and for each $[\pi]\in\widehat{G/H}$ we get $T_H^\pi=I$ and $d_{\pi,H}=d_\pi$. Thus we obtain
$$L^2(G/H)=\bigoplus_{[\pi]\in H^\perp}\mathcal{E}_\pi(G/H),$$
which precisely coincides with the decomposition associated to applying the Peter-Weyl Theorem to the compact quotient group G/H.

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