Weighted Best Local Approximation
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Received 12 March 2015; Accepted (in revised version) 23 September 2016

Abstract. In this survey the notion of a balanced best multipoint local approximation is fully exposed since they were treated in the $L^p$ spaces and recent results in Orlicz spaces. The notion of balanced point, introduced by Chui et al. in 1984 are extensively used.

Key Words: Best Local approximation, multipoint approximation, balanced neighborhood.

AMS Subject Classifications: 41A10, 41A65

1 Introduction

The notion of a best multipoint local approximations of a function is fully treated in [2] where the $L^p$ norm is used. Later, other approaches to best multipoint local approximations with $L^p$ norms appeared in [7] and [8]. And finally, for Orlicz norms, we mention [3,5,9,12,13] and for a general family of norms [6,10] and [14]. However, in [2], Chui et al. introduced the concept of balanced points in $L^p$ which includes different importance in each point.

More precisely, a rather general view of the problem is as follows. Let $f : \mathbb{R} \to \mathbb{R}$ be a function in a normed space $X$ with norm $\| \cdot \|$. Let $\Pi^m$ denote the set of polynomial in $\mathbb{R}$ of degree less or equal than $m$ and suppose $\Pi^m \subseteq X$. Consider $n$ points $x_1, \cdots, x_n$ in $\mathbb{R}$ and a net of small Lebesgue measurable neighborhoods $V_i^\delta$ around each point $x_i$ such that the Lebesgue measure $|V_i^\delta|$ goes to 0 as $\delta \to 0$ for $i = 1, \cdots, n$. We select the best approximation to $f$ near the points $x_1, \cdots, x_n$ by polynomial in $\Pi^m$. Formally, for each $n$-tuple of neighborhoods $V_1, \cdots, V_n$ we consider the polynomial $g_V \in \Pi^m$ which minimizes

$$\|(f-h)X_V\|$$

for all $h \in \Pi^m$, where $V = \bigcup_{i=1}^n V_i$ and $V_i = V_i^\delta$. It is well known that a best $\| \cdot \|$-approximation $g_V$ always exists since $\Pi^m$ has finite dimension. If any net $g_V$ converges to a unique

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element $g \in \Pi^m$, as $\delta \to 0$, then $g$ is said to be a best local approximation to $f$ at the points $x_1, \ldots, x_n$. We will mention in this survey all the works which consider that the velocity of convergence $|V_i| \to 0$, as $|V| \to 0$, can be different at each point $x_i$. According to [2], this problem has been treated considering the concept of balanced neighborhoods in local approximation and it reflects the different importance of the points $x_1, \ldots, x_n$. We need to deal with the necessary definition of balanced neighborhoods in each context.

As we pointed out above, Chui et al. study in [2] this problem when the space $X$ is the usual $L^p$ space, with the norm $\|f\|_p = \left( \int_B |f(x)|^p dx \right)^{1/p}$, where $B$ is a measurable set. They get results for balanced and non balanced neighborhoods. At last they generalize the results to the case of $\mathbb{R}^k$ instead of $\mathbb{R}$. On the other hand, in [4], the authors get balanced results in $L^p$ using other technique. We will discuss the $L^p$ problem with more details in Section 3.

In [11] and [12] the authors study the problem in Orlicz spaces, it means,

$$X = L^\phi(B) := \left\{ f : \int_B \phi(\alpha |f(x)|) dx < \infty \text{ for some } \alpha > 0 \right\},$$

where $\phi$ is a convex function, non negative, defined on $\mathbb{R}_0^+$ and $B$ is a Lebesgue measurable set. In these two works, the authors studied the best local approximation problem with the Luxemburg norm

$$\|f\|_\phi = \|f\|_{L^\phi(B)} = \inf \left\{ \lambda > 0 : \int_B \phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}$$

(1.2)

and get results for balanced and non balanced neighborhoods. Furthermore, we can consider a different Luxemburg norms in Orlicz Spaces, that is

$$\|f\|_{\phi,B} = \inf \left\{ \lambda > 0 : \int_B \phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq |B| \right\}$$

(1.3)

and it can generate different best approximation functions $g_V$ than those obtained with the standard Luxemburg norm given in (1.2). In [13] the authors study the balanced neighborhoods problem with this norm using a different technique than that used in [11] and [12].

Moreover, in [9] the authors study the balanced problem in Orlicz spaces $L^\phi$ when the error (1.1) does not come from a norm, but considering

$$\int_V \phi(|f(x) - g_V(x)|) dx = \min_{h \in \Pi^m} \int_V \phi(|f(x) - h(x)|) dx.$$

The last three problems in $L^p$ are equivalent, but in Orlicz spaces they are different problems and have different concepts of balanced neighborhoods. In Section 4 we will present the three problems in Orlicz spaces in detail.
2 Notations

We now introduce some notation. Let $B \subset \mathbb{R}$ be a bounded open set and set $|B|$ for its Lebesgue measure. Denote by $\mathcal{M}$ the system of all equivalence classes of Lebesgue measurable real valued functions defined on $B$.

Let $x_1, \cdots, x_n$ be $n$ distinct points in $B$. Consider a net of measurable sets $\{V_i\}_{|V| > 0}$ such that $V = \bigcup_{i=1}^{m} V_i$, where $V_i$ is a neighborhood of the point $x_i$ and

$$\sup_{1 \leq i \leq n} \sup_{y \in V_i} |x_i - y| \to 0,$$

as $|V| \to 0$. It is easy to see that $V_i = x_i + |V_i| A_i$, $i \leq i \leq n$, where $A_i$ is a measurable set with measure 1. Henceforward, we assume the sets $A_i$ are uniformly bounded.

For each $p$, $1 \leq p \leq \infty$, we consider the space $L^p = L^p(B)$ and the following norms

$$\|h\|_p = \|h\|_{L^p(B)} = \left( \int_B |h(x)|^p dx \right)^{1/p}$$

and

$$\|h\|_{p,B} = \left( \frac{1}{|B|} \int_B |h(x)|^p dx \right)^{1/p},$$

for $h \in \mathcal{M}$ and $p < \infty$. Sometimes we write $\|h\|_{L^p(W)}$ instead of $\|h\|_{L^p(B)}$, where $\chi_B$ denote the characteristic function of the set $W \subset B$.

Let $\Phi$ be the set of convex functions $\phi : [0, \infty) \to [0, \infty)$, with $\phi(x) > 0$ for $x > 0$ and $\phi(0) = 0$. For $\phi \in \Phi$ define the Orlicz space

$$L^\phi(B) = \left\{ f \in \mathcal{M} : \int_B \phi(\alpha |f(x)|) dx < \infty \text{ for some } \alpha > 0 \right\}.$$

This space can be endowed with the Luxemburg norm $\|f\|_\phi$ defined in (1.2) as well as with the norm $\|f\|_{\phi,B}$ defined in (1.3). If $\phi(x) = x^p$, the last norm coincides with the norm $\|h\|_{p,B}$. Sometimes we write $\|f\|_{L^\phi(W)}$ instead of $\|f\|_{L^\phi(B)}$. The space $L^\phi$ with both norms is a Banach space and we refer to [1] for a detailed study of Orlicz spaces.

We recall that a function $\phi \in \Phi$ satisfies the $\Delta_2$-condition if there exists a constant $k > 0$ such that $\phi(2x) \leq k \phi(x)$, for $x \geq 0$. We also say that $\phi \in \Phi$ satisfies the $\Delta'$ condition if there exists a constant $C > 0$ such that $\phi(x_1 x_2) \leq C \phi(x_1) \phi(y)$ for $x, y \geq 0$. Note that it is easy to see that $\Delta'$ condition implies $\Delta_2$ condition.

Let $f \in PC^m(B)$, where $PC^m(B)$ is the class of functions in $L^\phi(B)$ with $m - 1$ continuous derivatives and with bounded piecewise continuous $m^{th}$ derivative on $B$.

3 Balanced and non balanced problem in space $L^p$

In this section we present the results given in [2] and [4], both in $L^p$ spaces, $p \geq 1$. In [2] Chui et al. introduce the balanced concept as follows. For each $a \in \mathbb{R}$ and $k$, $1 \leq k \leq n$, we
denote

\[ \mathcal{V}_k(\alpha) := |V_k|^\alpha, \]
and assume the following condition which allows us to compare \( \mathcal{V}_k(\alpha) \) with each other as functions of \( \alpha \).

For any nonnegative real numbers \( \alpha \) and \( \beta \) and any pair \( j, k, 1 \leq j, k \leq n, \)

\[ \text{either } \mathcal{V}_k(\alpha) = o(\mathcal{V}_j(\beta)) \text{ or } \mathcal{V}_j(\beta) = o(\mathcal{V}_k(\alpha)), \quad \text{as } |V| \to 0. \] (3.1)

Given a collection of neighborhoods \( \{V_i\}_{i=1}^n \) and a set of \( n \) nonnegative real numbers \( a_1, \ldots, a_n \) in \( \mathbb{R} \), we say that \( V_j(a_j) \) is maximal if for all \( k \), \( \mathcal{V}_k(a_k) = O(V_j(a_j)) \). When it happens we write \( \mathcal{V}_j(a_j) = \max \{ \mathcal{V}_k(a_k) \} \).

In the balanced case, the neighborhoods can have different measure, but it is not at random, there is a relationship between the measure of the sets \( |V_k| \) and the amount of information of \( f \) over the points \( i_k \).

**Definition 3.1.** A \( n \)-tuple of nonnegative integers \( (i_k) \) is balanced if for each \( j \) such that \( i_j > 0 \), \( \max \{ \mathcal{V}_k(i_k + 1/p) \} = o(\mathcal{V}_j(i_j - 1 + 1/p)) \). In this case, we say that \( m+1 = \sum_{k=1}^n i_k \) is a balanced integer and the neighborhoods \( V_k \) are balanced.

It is easy to see that to each balanced integer \( m+1 \) there corresponds exactly one balanced \( n \)-tuple \( (i_k) \) such that \( \sum_{k=1}^n i_k = m+1 \).

**Example 3.1.** If \( L^p = L^2 \) and the neighborhoods are \( V_1 = V_1(e) = x_1 + e[-\frac{1}{2}, \frac{1}{2}] \) and \( V_2 = V_2(e) = x_2 + e^{1/2}[-\frac{1}{2}, \frac{1}{2}] \), then \( (0,0), (0,1), (1,1), (1,2) \) and \( (1,3) \) are balanced \( n \)-tuples, while \( (2,2), (1,0), (2,0) \) and \( (2,1) \) are non balanced \( n \)-tuples.

There exists a simple way to find all the balanced \( n \)-tuples.

**Algorithm 3.1.** It begins with the balanced \( n \)-tuple \( (i_k^{(0)}) := (0) \) corresponding to the balanced integer 0. Let \( (i_k^{(l)}) \) be a balanced \( n \)-tuple. Let \( C = C((i_k^{(l)})) := \{ j : \mathcal{V}_j(i_j^{(l)} + 1/p) = \max \{ \mathcal{V}_k(i_k^{(l)} + 1/p) \} \} \). To build the next \( n \)-tuple, \( (i_k^{(l+1)}) \), put \( i_k^{(l+1)} = i_k^{(l)} + 1 \) for \( k \in C((i_k^{(l)})) \) and \( i_k^{(l+1)} = i_k^{(l)} \) for \( k \notin C \).

In [2], the authors prove that this algorithm generates exactly all the balanced \( n \)-tuples.

The following Lemma gives an order of the error produced in the approximation (1.1) with the norm \( \| \cdot \|_p \). Also, this Lemma exposes how to define a maximal element.

**Lemma 3.1.** Let \( (i_k) \) be an ordered \( n \)-tuple of nonnegative integers. Suppose \( h \in PC^{(l)}(B) \), where \( l = \max \{ i_k \} \) and \( h^{(l)}(x_k) = 0, 0 \leq j \leq i_k - 1, 1 \leq k \leq n \). Then

\[ \| h \|_{L^p(V)} = O(\max \{ \mathcal{V}_k(i_k + 1/p) \}). \]

We now present the Lemma 3 stated in [2], which will be used in the sequel. This lemma have importance in the proof of the main Theorems.
Lemma 3.2. Let $1 \leq p \leq \infty$ and let $\Lambda$ be a family of uniformly bounded measurable subsets of the real line with measure 1. Then there exists a constant $M$ (depending on $m$ and $p$) such that for all the polynomials $P \in \Pi^m$ and all $A \in \Lambda$,

$$|c_k| \leq M \|P\|_{L^p(A)}, \quad 0 \leq k \leq m,$$

where $P(x) = \sum_{k=0}^{m} c_k x^k$.

Now we present the first main result given in [2], which solved the problem of best local approximation, for balanced neighborhoods. In the proof they used the above lemmas.

Theorem 3.1. If $m+1$ is a balanced integer with balanced $n$-tuple $(i_k)$ and $f \in PC^l(B)$, $l = \max\{i_k\}$, then the best local approximation to $f$ from $\Pi^m$ is the unique $g \in \Pi^m$ defined by the $m+1$ interpolation conditions $f^{(j)}(x_k) = g^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$.

Given a balanced integer $m+1$, as a consequence of the Algorithm, there exists a following and previous balanced integer and for example, the following balanced integer will be $m + \text{Car}(C)$, where $\text{Car}(C)$ is the cardinality of the set $C$. So we have the following definition.

Definition 3.2. Given the neighborhoods $V_1, \cdots, V_n$, $p$ and an integer $m+1$ we define:

- $\underline{m+1}$ the largest balanced integer less than or equal to $m+1$.
- $(\underline{i}_k)$ the balanced $n$-tuple satisfying $\sum \underline{i}_k = m+1$.
- $\underline{m+1}$ the smallest balanced integer greater than or equal to $m+1$.
- $(\overline{i}_k)$ the balanced $n$-tuple satisfying $\sum \overline{i}_k = m+1$.

Remark 3.1. Given a non balanced integer $m+1$, set $C = C((\underline{i}_k)) = \{j : V_j(\underline{i}_k + 1/p) = \max\{V_{j_k}(\underline{i}_k + 1/p)\}\}$. Then, as the algorithm generates exactly all the balanced integers, the next balanced integer is $m+1$, with $\underline{i}_k = \overline{i}_k$ for $k \notin C$ and $\overline{i}_k = \underline{i}_k + 1$ for $k \in C$.

We establish the following auxiliary lemma from [2] which it is used to prove one of the main results.

Lemma 3.3. Given $(|V_1|, \cdots, |V_n|)$ and $m+1$, define $l = \max\{\overline{i}_k\}$. If $f \in PC^l(B)$ and for each $V$,

$$\|f - g_V\|_{L^p(V)} = \min_{h \in \Pi^m} \|f - h\|_{L^p(V)},$$

then $g_V$ is bounded on $B$ uniformly for all $|V| > 0$ and for each $k$, $1 \leq k \leq n$

$$\frac{(f - g_V)^{(j)}(x_k)V_k(j+1/p)}{E} = O(1), \quad j = 1, \cdots, \overline{i}_k - 1,$$

where $E = \max\{V_j(\underline{i}_k + 1/p)\}$. 


From Lemma 3.3, if there exists a best local approximation \( g \), then it satisfy the equations \( f^{(j)}(x_k) = g^{(j)}(x_k), 0 \leq j \leq \ell_k - 1, 1 \leq k \leq n \), since

\[
(f - g)^{(j)}(x_k) = \mathcal{O} \left( \frac{E}{V_k(j+1/p)} \right) = \mathcal{O} \left( \frac{E}{V_k(\ell_k-1+1/p)} \right) = o(1)
\]

for the values \( j \) and \( k \) above because \( m+1 \) is a balanced integer. These are \( m+1 \) constraints and there are \( m+1-m+1 \) degrees of freedom. The remaining \( m+1-m+1 \) degrees of freedom must then be chosen to minimize the local \( L^p \) error around the \( m+1-m+1 \) points. The calculations required to do this are more difficult than the previous ones and the following strong assumption is needed to prove the best local approximation existence.

The \( n \)-tuple of neighborhoods \( (V_1, \cdots, V_n) \) satisfy

\[
\frac{V_k(\ell_k+1/p)}{E} = e_k + o(1), \quad 1 \leq k \leq n,
\]

where \( e_k \) is a fixed constant.

**Remark 3.2.** Given \( m+1 \), set \( C = \{ k : 1 \leq k \leq n \text{ and } V_k(\ell_k+1/p) = \max \{ V_1(\ell_1+1/p) \} \} \). From the algorithm \( e_k = 0 \) for \( k \notin C \) and \( e_k \neq 0 \) for \( k \in C \).

Now we present the main Theorem from [2].

**Theorem 3.2.** Suppose \( m+1 \) is not balanced. Assume that each \( A_k \) is either an interval for each \( V_k \) or is independent of the net \( \{ V_k \} \). Assume that the measure \( (|V_1|, \cdots, |V_n|) \) satisfies for each \( k \),

\[
\frac{V_k(\ell_k+1/p)}{\max \{ V_i(\ell_i+1/p) \}} = e_k + o(1)
\]

with \( e_k \) a constant independent of the net \( \{ V_k \} \). Let \( I_{A_k}(i,p) \) denote the minimum \( L^p \) norm over the measurable set \( A \) of an \( i \)th degree polynomial with unit leading coefficient. If \( f \in PC^l(B) \), where \( l = \max \{ T_k \} \) and \( 1 < p \leq \infty \), then the best local approximation to \( f \) from \( \pi^m \) is the unique solution of the constrained \( L^p \) minimization problem

\[
\min_{h \in \pi^m} \| (e_k I_{A_k}(\ell_k,p)(f - g)^{(j)}(x_k))_{j=1}^p \|_{L^p}
\]

subject to

\[
(f - g)^{(j)}(x_k) = 0,
0 \leq j \leq \ell_k - 1, \quad 1 \leq k \leq n,
\]

where, if \( A_k \) is an interval, we can replace \( I_{A_k}(\ell_k,p) \) by \( I_{[0,1]}(\ell_k,p) \).

If \( p = 1 \) the \( L^p \) minimization may not have a unique solution; if it does, however, it is the best local approximation.

By the other hand, in [4] the balanced result (Theorem 3.1) is proved with other technique. The authors prove a Polya-type inequality for polynomials in \( L^p \) spaces and it has an application to best local approximation. The Polya-type inequality is the following.
**Theorem 3.3.** Let $0 < p \leq \infty$, and $m,n \in \mathbb{N}$. Let $i_k$, $1 \leq k \leq n$, be $n$ positive integers such that $i_1 + \cdots + i_n = m + 1$. Then there exists a constant $K$ depending on $p$, $i_k$, for $1 \leq k \leq n$, such that

$$ |c_j| \leq \min_{1 \leq k \leq n} \frac{K}{|V_k|^k-1+1/p} \|P\|_{L^p(V)}, \quad 0 \leq j \leq m, $$

for all $P(x) = \sum_{j=0}^{m} c_j x^j \in \pi^m$, $V = \bigcup_{k=1}^{n} V_k$, with $|V_k| > 0$, $1 \leq k \leq n$.

In [2] the authors prove that if $(i_1, \cdots, i_n)$ is a balanced $n$-tuple and $f$ is a function sufficiently differentiable in a neighborhood of the $n$-points $x_1, \cdots, x_n$, the best local approximation is the classical Hermite polynomial on the points $x_1, \cdots, x_n$, fixed from the interpolation conditions of the function $f$ in $x_k$ up to order $i_k - 1$, $i \leq k \leq n$. In [4], the authors get a similar result for more general functions $f$. They introduce the following class of Lebesgue measurable functions.

**Definition 3.3.** Given $p > 0$ and $m+1 = i_1 + \cdots + i_n$, a function $f$ belongs to the class $\mathcal{H}_{m,p}$ $(i_1, \cdots, i_n)$ if $f \in L^p(B)$ and there exists a polynomial $H \in \pi^m$ satisfying

$$ \|f - H\|_{V,p} = o(|V_k|^{i_k-1+1/p}), \quad 1 \leq k \leq n, \quad \text{as } |V| \to 0. \quad (3.2) $$

These classes are similar to those introduced in [4] and [14], for $n = 1$. As a consequence of Theorem 3.3 it follows that the polynomial $H$ is unique if $f \in \mathcal{H}_{m,p}$ $(i_1, \cdots, i_n)$. It is called the **generalized Hermite polynomial** of $f$ on $x_1, \cdots, x_n$ with respect to the $n$-tuple $(i_1, \cdots, i_n)$. Moreover,

**Theorem 3.4.** Let $f \in \mathcal{H}_{m,p}$ $(i_1, \cdots, i_n)$. Then the best local approximation to $f$ from $\pi^m$, say $H$, is the generalized Hermite polynomial of $f$ on $x_1, \cdots, x_n$ with respect to the $n$-tuple $(i_1, \cdots, i_n)$.

In particular, under certain differentiability conditions of the function $f$, from Lemma 3.1 the polynomial $H$, which interpolates the data $f^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, satisfies

$$ \|f - H\|_{L^p(V)} = o \left( \sum_{k=1}^{n} |V_k|^{i_k+1/p} \right) = o \left( \max \{|V_k|^{i_k+1/p}\} \right). $$

If in addition, $(i_1, \cdots, i_n)$ is a balanced $n$-tuple, then $f \in \mathcal{H}_{m,p}$ $(i_1, \cdots, i_n)$ and $H$ fulfills (3.2). Therefore, it is obtained as a consequence of Theorem 3.3 the analogous result for balanced $k$-tuples, proved in Theorem 1 of [2]. However in [4] the authors do not assume any condition of classic differentiability over the function $f$.

Here we have exposed two techniques to get existence of multipoint local approximation with balanced neighborhoods. They can be generalized to Orlicz spaces as we show in the next section.
4 Balanced and non balanced problems in Orlicz spaces $L^\phi$

In this section we present the best local approximation polynomials using balanced neighborhoods in Orlicz spaces $L^\phi$. According with the norm that we consider to minimize the error (1.1), we obtain three different problems which we include in the following subsection.

4.1 Best local approximation in $L^\phi$ with norm $\|\cdot\|_\phi$

In this subsection we will expose the existence of best multipoint local $\|\cdot\|_\phi$-approximation to a function $f$ from $\Pi^n$ for a suitable integer $n$, it means, for the balanced and non balanced cases. This problem is considered in an arbitrary Orlicz space $L^\phi$ with the Luxemburg norm $\|\cdot\|_\phi$. We refer to [1] for a detailed treatment of Orlicz spaces. For this purpose, we introduce the concept of $\|\cdot\|_\phi$-balanced integer in this context. The following results follow the pattern given in [2] for $L^p$ spaces and they appeared in [11] and [12].

Now we assume in this article that $\phi \in \Phi$ and it satisfies the $\Delta_2$-condition and recall that $V = \bigcup_{k=1}^n V_k$ is a net of union of neighborhoods of the points $x_1, \cdots, x_n$ and denote by $g_V$ a best $\|\cdot\|_\phi$-approximation to $f$ from $\Pi^m$ on $V$, it means,

$$\|(f - g_V)X_V\|_\phi = \min_{h \in \Pi^m} \|(f - h)X_V\|_\phi.$$ 

For each $\alpha > 0$ and $1 \leq k \leq n$, we denote

$$v_k(\alpha) := \frac{|V_k|^\alpha}{\phi^{-1} \left( \frac{1}{|V_k|^\alpha} \right)},$$

and instead of (3.1) we assume in this context that for any nonnegative integers $\alpha$ and $\beta$, and any pair $j, k$, $1 \leq j, k \leq n$, either

$$v_k(\alpha) = O(v_j(\beta)) \quad \text{or} \quad v_j(\beta) = o(v_k(\alpha)). \quad (4.1)$$

Let $(i_k)$ be an ordered $n$-tuple of nonnegative integers. We say that $v_j(i_j)$ is maximal if $v_k(i_k) = O(v_j(i_j))$ for all $1 \leq k \leq n$. We denote it by

$$v_j(i_j) = \max \{v_k(i_k)\}.$$

**Definition 4.1.** An $n$-tuple $(i_k)$ of nonnegative integers is said to be $\|\cdot\|_\phi$-balanced if for each $i_j > 0$,

$$\frac{1}{v_j(i_j - 1)} \max \{v_k(i_k)\} = o(1).$$

If $(i_k)$ is $\|\cdot\|_\phi$-balanced, we say that $\sum_{k=1}^n i_k$ is a $\|\cdot\|_\phi$-balanced integer.

Next we set, from [11], an example of $\|\cdot\|_\phi$-balanced integers.
Example 4.1. Define \( \phi(x) = \frac{x^2}{\ln(e+x)}, \ x \geq 0. \) It can be seen that \( \phi \) satisfies the \( \Delta_2 \)-condition (see [1, pp. 30]). Given \( x_1, x_2 \) and let the neighborhoods satisfy \( |V_2| = |V_1|^2 \). Thus these neighborhoods satisfy the conditions (4.1) and every integer is \( \|\cdot\|_\phi \)-balanced.

In [11] an algorithm is presented and it generates all the \( \|\cdot\|_{\phi} \)-balanced integer as in \( L^p \).

Algorithm 4.1. Begin with the \( \|\cdot\|_{\phi} \)-balanced \( n \)-tuple \( (i_k^{(0)}) \) corresponding to the \( \|\cdot\|_{\phi} \)-balanced integer 0. Then, given \( (i_k^{(s)}) \) for \( s \geq 0 \), set \( C = \{ l : v_l(i_k^{(s)}) = \max \{ v_l(i_k^{(s)}) \} \} \). We build the next \( \|\cdot\|_{\phi} \)-balanced \( n \)-tuple \( (i_k^{(m+1)}) \) taking \( i_k^{(s+1)} = i_k^{(s)} + 1 \), for \( k \in \mathbb{C} \) and \( i_k^{(s+1)} = i_k^{(s)}, \) for \( k \notin C \).

Remark 4.1. It is proved in [11] that to each \( \|\cdot\|_{\phi} \)-balanced integer there corresponds exactly one \( \|\cdot\|_{\phi} \)-balanced \( n \)-tuple. Also an integer \( m+1 \) is \( \|\cdot\|_{\phi} \)-balanced if only if \( m+1 = \sum_{k=1}^{n} i_k \) for some \( (i_k) \) generated by this algorithm.

Now, we cite from [11] the following auxiliary lemmas and the first main result. Instead of Lemma 3.1, in [11] the authors prove the following auxiliary result.

Lemma 4.1. Let \( (i_k) \) be an increasing ordered \( n \)-tuple of nonnegative integers. Suppose \( h \in PC^n(X) \), where \( l = \max \{ i_k \} \) and \( h^{(j)}(x_k) = 0, \ 0 \leq j \leq i_k - 1, 1 \leq k \leq n. \) Then

\[
\|h\|_{L^p(V)} = o(\max \{ v_k(i_k) \}).
\]

Instead of Lemma 3.2 we have

Lemma 4.2. Let \( \Lambda \) be a family of uniformly bounded measurable subsets of the real line with measure 1 and let \( 0 < r < 1 \), then there exists a constant \( s > 0 \) such that

\[
\|P\|_{-1} \left( \left[ \frac{\|P\|_{\infty,A}}{s} \right] \right) \cap A \geq r, \quad (4.2)
\]

for all \( A \in \Lambda \) and for all \( P \in \Pi^m \).

Proposition 4.1. Given an integer \( m+1 \), consider the Definition 3.2 for the Luxemburg norm, then

a) If \( i_j + 1 = i_j \), then \( \max \{ v_k(i_k) \} = o(v_j(i_j - 1)) \);

b) If \( i_j = i_j \), then \( \max \{ v_k(i_k) \} = o(v_j(i_j - 1)) \);

c) If \( m+1 < m+1 \), then \( \max \{ v_k(i_k) \} = o(\max \{ v_k(i_k) \}) \).

We now present the first important result from [11] concerning to the behavior of a net \( \{ g(V) \}_{V \geq 0} \) of best \( \|\cdot\|_{\phi} \)-approximations from \( \Pi^m \), as \( |V| \to 0 \).
Theorem 4.1. Let $m+1$ be a positive integer and $l = \max \{ \lceil k \rceil \}$. If $f \in PC^l(X)$ and $\{ g_V \}_{V \geq 0}$ is a net of best $\| \cdot \|_\phi$-approximations of $f$ from $\pi^m$ on $V$, then $\{ g_V \}_{V \geq 0}$ is uniformly bounded on $X$.

Using the same technique it is obtained

Lemma 4.3. Given an integer $m+1$, set $l = \max \{ \lceil k \rceil \}$. If $f \in PC^l(X)$ and $\{ g_V \}_{V \geq 0}$ is a net of best $\| \cdot \|_\phi$-approximations of $f$ from $\pi^m$ on $V$, then

$$|(f - g_V)^{(i)}(x_k)v_k(j)| = O(\max \{ v_k(\lceil k \rceil) \}),$$

(4.3)

$0 \leq j \leq \lceil k \rceil - 1, 1 \leq k \leq n$.

Thus, using the $\| \cdot \|_\phi$-balanced definition, it follows the main result of [11].

Theorem 4.2. Let $(i_k)$ be a $\| \cdot \|_\phi$-balanced $n$-tuple and let $0 < m+1 = \sum_i k_i$. If $l = \max \{ i_k \}$, $f \in PC^l(X)$, then the best local $\| \cdot \|_\phi$-approximation to $f$ from $\Pi^m$ is the unique $g \in \Pi^m$ defined by the $m+1$ interpolation conditions

$$f^{(i)}(x_k) = g^{(i)}(x_k),$$

$0 \leq j \leq i_k - 1, 1 \leq k \leq n$.

Now, we cite the following results from [12], which are a continuity of the above analysis.

Set $E = \max \{ v_l(\lceil i \rceil) \}$ and

$$c_{i,k} = c_{i,k}(V) := (f - g_V)^{(i)}(x_k) \frac{v_k(j)}{E},$$

(4.4)

for $k = 1,2,\ldots,n$ and $j = 0,1,\ldots,\lceil k \rceil - 1$. As a consequence of Lemma 4.3 we have

$$c_{i,k} = O(1), \quad k = 1,2,\ldots,n, \quad j = 0,1,\ldots,\lceil k \rceil - 1.$$  

(4.5)

Since

$$g^{(i)}_V(x_s) = f^{(i)}(x_s) - c_{i,s} \frac{E}{v_s(i)} \quad \text{and} \quad \frac{E}{v_s(i)} = o(1)$$

(4.6)

for $i = 0,1,\ldots,\lceil k \rceil - 1$, from (4.5) we obtain

$$g^{(i)}_V(x_s) = f^{(i)}(x_s) + o(1), \quad s = 1,2,\ldots,n, \quad i = 0,1,\ldots,\lceil k \rceil - 1.$$

Consider the following basis for $\Pi^m$, say $\{ u_{i,k} \} \cup \{ w_r \}$, with $k = 1,2,\ldots,n$, $j = 0,1,\ldots,\lceil k \rceil - 1$, and $r = 1,2,\ldots,(m+1) - m+1$, which satisfies

$$u_{i,k}^{(i)}(x_k) = \delta_{(i,k),(j,k')}, \quad \text{and} \quad w_r^{(i)}(x_k) = 0, \quad k = 1,2,\ldots,n, \quad \text{and} \quad r = 1,2,\ldots,(m+1) - m+1.$$
where \( \delta_{(j,k),(j',k')} \) is the Kronecker delta. Observe that if \( g \in \pi^n \), then
\[
g(x) = \sum_{k=1}^{n} \sum_{j=0}^{k-1} a_{j,k} u_{j,k}(x) + \sum_{r=1}^{(m+1)-m+1} b_r w_r(x),
\]
where \( g^{(j)}(x_k) = a_{j,k}, \ k = 1,2,\ldots,n \) and \( j = 0,1,\ldots,j_k-1 \).

Thus, if there exists a best local approximation \( g \), since (4.6) it will satisfy the equations
\[
f^{(j)}(x_k) = g^{(j)}(x_k), \ 0 \leq j \leq j_k-1, \ 1 \leq k \leq n.
\]
The remaining \( m+1-m+1 \) degrees of freedom must then be chosen so as to minimize the local \( L^p \) error around the \( m+1-m+1 \) points. It required a delicate analysis and it appears in [12]. There are many auxiliary lemmas here to prove the main result, which solve the best local approximation problem when \( (m+1) \) is not a balanced integer. As an example of the auxiliary lemmas we expose the following (see [12]).

Given a non balanced integer \( m+1 \), set
\[
C = C((j_k)) = \{ j : v_j(j_k+1/p) = \max \{ v_k(j_k+1/p) \} \}.
\]

**Lemma 4.4.** There holds
\[
\Gamma := \left\| \sum_{\lambda \in C} \left( \sum_{s=1}^{n} \sum_{i=0}^{k-1} \frac{c_{i,s}}{\nu_s(i)} \nu_{s}(i) (x_k) \right) \frac{(x-x_k)_{j_k}}{j_k!} x_{j_k}(x) \right\|_{L^p(V)} = o(1).
\]

**Lemma 4.5.** Let \( \phi \in \Phi \) satisfying the \( \Delta_2 \)-condition. If for each \( x \geq 0 \) there exists
\[
\lim_{\delta \to \infty} \frac{\phi(\delta x)}{\phi(x)} =: \psi(x),
\]
then \( \psi(x) = x^p \) for some \( p \geq 1 \).

**Lemma 4.6.** For every \( k \in C \), set
\[
P_{k,V}(y) := \sum_{j=0}^{j_k-1} \frac{c_{j,k}(V)}{j!} y^j + c_{j,k}(\delta) y^{j_k}, \ k \in C,
\]
such that \( \lim_{|V| \to 0} c_{j,k}(V) = d_{j,k} \) and \( \lim_{|V| \to 0} c_{j,k}(\delta) = m_k \). If
\[
\lim_{\delta \to \infty} \frac{\phi(\delta x)}{\phi(x)} =: \psi(x)
\]
extists for \( x \geq 0 \) and \( \lim_{|V| \to 0} a_k(V) = \infty \) for each \( k \in C \), then
\[
\lim_{|V| \to 0} \left[ \inf \left\{ \lambda > 0 : \sum_{k \in C} \int_{A_k} \phi(a_k(V) |P_{k,V}(y)|) \frac{dy}{\phi(a_k(V))} \leq 1 \right\} \right]
\]
\[
= \inf \left\{ \lambda > 0 : \sum_{k \in C} \int_{A_k} \psi \left( \frac{|P_k(y)|}{\lambda} \right) dy \leq 1 \right\},
\]
where

\[ P_k(y) = \sum_{j=0}^{k-1} \frac{d_j}{j!} y^j + m_k y^k. \]

We now can give, under certain conditions, the existence of the best local \( \| \cdot \|_\phi \)-approximation.

**Theorem 4.3.** Let \( \phi \in \Phi \) satisfying the \( \Delta_2 \)-condition. Assume that there exists \( \lim_{a \to \infty} \frac{\phi(ax)}{\phi(a)} \) for all \( x \geq 0 \) and therefore this limit is \( x^p \) for some \( p \geq 1 \). Let \( m+1 \) be a non \( \| \cdot \|_\phi \)-balanced integer and \( l = \max_{1 \leq k \leq n} \{ \tilde{t}_k \} \). For each \( k \in C \) suppose

\[ \lim_{\delta \to 0} \frac{v_k(\tilde{t}_k)}{E} = e_k > 0. \quad (4.7) \]

If \( f \in PC^l(X) \) then, for \( |V| \to 0 \), the limit of any convergent subsequence of \( \{ g_V \} \), a net of best \( \| \cdot \|_\phi \)-approximations of \( f \) from \( \Pi^m \), is a solution of the following minimization problem in \( \mathbb{R}^{m+1-m+1} \).

\[
\begin{aligned}
\min_{h \in \Pi^m} & \left\| \left< e_k J_{A_k}(\tilde{t}_k,p)(f-h)(\tilde{t}_k) (x_k)/\tilde{t}_k! \right>_{l_k} \right\|, \\
\text{with the constraints} & \ (f-h)(j) (x_k) = 0, \ k = 1,2,\ldots,n, \text{and } j = 0,1,\ldots,i_k-1,
\end{aligned}
\]

where, for \( k \in C \), \( J_{A_k}(\tilde{t}_k,p) \) is the minimum \( L_p \) norm over \( A_k \) of an \( \tilde{t}_k \)-degree polynomial with unit leading coefficient. In particular, if (4.8) has a unique solution \( g \), then \( g = \lim_{|V| \to 0} g_V \) and therefore this is a best local \( \| \cdot \|_\phi \)-approximation to \( f \) from \( \Pi^m \) on \( \{ x_1,\ldots,x_n \} \).

The following example shows that \( \lim_{|V| \to 0} g_V \) may not exist if \( \phi \) does not satisfy the assumption that \( \lim_{a \to \infty} \frac{\phi(ax)}{\phi(a)} \) exists for all \( x \geq 0 \). The proof is in [12].

**Example 4.2.** Let \( x_1 = 0 \), \( x_2 = 1 \), \( A_1 = A_2 = [-\frac{1}{2},\frac{1}{2}] \), \( |V_1| = 2\delta \), \( |V_2| = \delta \), for \( 0 < \delta < \frac{1}{3} \) and let \( \Pi^m = \Pi^0 \) be the subspace formed by the constant functions in \( L_\phi \). Define

\[
\phi(x) = \begin{cases} 
 x, & \text{if } x \in [0,1], \\
 2x-1, & \text{if } x \in [1,2], \\
 23^\eta x - 3^2\eta, & \text{if } x \in [23^{\eta-1},3^\eta], \ \eta \in \mathbb{N},
\end{cases}
\]

and \( f(x) = 0 \) if \( x \in [-\frac{1}{2},\frac{1}{2}] \), \( f(x) = 1 \) if \( x \in \left[ \frac{5}{6},\frac{7}{6} \right] \).

### 4.2 Best local approximation with the norm \( \| \cdot \|_{\phi,B} \)

In this section we expose the analysis given in [13] to prove the existence of the best local approximation to a function \( f \), with balanced neighborhoods, when the error (1.1) is the following. Denote \( g_V \in \Pi^m \) such that

\[
\| f - g_V \|_{\phi,V} = \min_{h \in \Pi^m} \| f - h \|_{\phi,V}.
\]
These best approximation can be different to that given with the Luxemburg norm $\| \cdot \|_{L^\phi(V)}$.

The analysis in [13] follows the pattern used in [4] for $L^p$ spaces. We begin with the following auxiliary lemmas and properties.

If $\phi$ satisfies the $\Delta'$-condition, it is easy to see that there exists a constant $K > 0$ such that

$$\phi^{-1}(x)\phi^{-1}(y) \leq K\phi^{-1}(xy) \quad \text{for all } x, y \geq 0. \quad (4.10)$$

We assume in this section that $\phi \in \Phi$ and it satisfies the $\Delta'$-condition.

**Proposition 4.2.** The family of all seminorms $\| \cdot \|_{\phi, V}$ with $|V| > 0$, has the following properties:

(a) $\|X_V\|_{\phi, V} = \frac{1}{\phi^{-1}(1)}$.

(b) If $f, g \in L^\phi(X)$ satisfy $|f| \leq |g|$ on $V$, then $\|f\|_{\phi, V} \leq \|g\|_{\phi, V}$. The inequality is strict if $|f| < |g|$ on some subset of $V$ with positive measure.

(c) There exists a constant $M > 0$ such that

$$\|f\|_{\phi, G} \leq \frac{M}{\phi^{-1}\left(\frac{|G|}{|D|}\right)} \|f\|_{\phi, D}, \quad f \in L^\phi(X), \quad (4.11)$$

for all pair of measurable sets $G, D$, with $G \subset D$ and $|G| > 0$.

**Lemma 4.7.** There exists a constant $M > 0$ such that

$$\left|P(a)\right| \leq \frac{M}{e^{|j|}} \|P\|_{\phi, [a-\epsilon, a+\epsilon]}$$

for all $P \in \Pi^m$, $[a-\epsilon, a+\epsilon] \subset B$ and $0 \leq j \leq m$.

**Lemma 4.8.** Let $C \subset B$ be an interval, $E \subset C$, $|E| > 0$. For all $P \in \Pi^m$, there exists an interval $F := F(E, P) \subset C$ such that

(a) $|F| \geq \frac{|E|}{m}$,

(b) $\|P\|_{\phi, F} \leq 2m \|P\|_{\phi, E}$.

Now, we present the main result concerning to Pólya inequality in $L^\phi$.

**Theorem 4.4.** Let $\phi \in \Phi$ and $n, m \in \mathbb{N}$. Let $i_k$, $1 \leq k \leq n$, be $n$ positive integers such that $\sum_{k=1}^n i_k = m+1$. Let $E_k$, $1 \leq k \leq n$, be disjoint pairwise compact intervals in $\mathbb{R}$, with $0 < |E_k| \leq 1$. Then there exists a positive constant $M$ depending on $\phi, i_k$ and $E_k$, $1 \leq k \leq n$, such that

$$|c_j| \leq \frac{M}{\min_{1 \leq k \leq n} \left\{ \frac{|V \cap E_k|}{|V|} \phi^{-1}\left(\frac{|V \cap E_k|}{|V|}\right) \right\}} \|P\|_{\phi, V}, \quad 0 \leq j \leq m, \quad (4.12)$$

for all $P(x) = \sum_{j=0}^m c_j x^j, V \subset \bigcup_{k=1}^n E_k$ with $|V \cap E_k| > 0, 1 \leq k \leq n$. 
Now, we will introduce the concept of balanced integer in that context. For each \( \alpha \in \mathbb{R} \) and \( k, 1 \leq k \leq n \), we denote
\[
A_k(\alpha) := \left| V_k \right|^a \phi^{-1}\left( \frac{|V|}{|V_k|} \right).
\]
The following condition allows us that \( A_k(\alpha) \) can be compared with each other as functions of \( \alpha \) when \( |V| \to 0 \).

For any nonnegative integers \( \alpha \) and \( \beta \) and any pair \( j, k, 1 \leq j, k \leq n \),
\[
either A_k(\alpha) = \mathcal{O}(A_j(\beta)) \quad \text{or} \quad A_j(\beta) = o(A_k(\alpha)), \quad \text{as } |V| \to 0. \quad (4.13)
\]

Let \( (i_k) \) be an ordered \( n \)-tuple of nonnegative integers. We say that \( A_j(i_j) \) is a maximal element of \( (A_k(i_k)) \) if \( A_k(i_k) = \mathcal{O}(A_j(i_j)) \) for all \( 1 \leq k \leq n \). We denote it by
\[
A_j(i_j) = \max\{A_k(i_k)\}.
\]

Observe that
\[
\sum_{k=1}^n A_k(i_k) = \mathcal{O}(\max\{A_k(i_k)\}).
\]

**Definition 4.2.** An \( n \)-tuple \( <i_k> \) of nonnegative integers is balanced if
\[
\sum_{k=1}^n A_k(i_k) = \mathcal{O}\left( \min_{1 \leq k \leq n} \left\{ \left| V_k \right|^k \phi^{-1}\left( \frac{|V|}{|V_k|} \right) \right\} \right).
\]

In this case, we say that \( \sum_{k=1}^n i_k \) is a balanced integer and \( (V_k) \) are balanced neighborhoods.

To each balanced integer there corresponds exactly one balanced \( n \)-tuple. Moreover, there are an algorithm which gives all balanced \( n \)-tuples which it is proved in [13].

Given \( (i_k) \), set
\[
C = C((i_k)) := \{ j : A_j(i_j) = \max\{A_k(i_k)\} \}.
\]

**Algorithm 4.2.** Let \( v_q \) be a balanced integer and let \( (i_k^{(v_q)}) \) be the corresponding balanced \( n \)-tuple. To build the next \( n \)-tuple, \( (i_k^{(v_q+1)}) \), put \( i_k^{(v_q+1)} = i_k^{(v_q)} + 1 \) for \( k \in C((i_k^{(v_q)})) \) and \( i_k^{(v_q+1)} = i_k^{(v_q)} \) for \( k \notin C((i_k^{(v_q)})) \).

The algorithm generates \( n \)-tuples candidates to be balanced. We can observe it with the following example.

**Example 4.3.** Define \( \phi(x) = x^2(1 + |\ln x|), \ x > 0 \) and \( \phi(0) = 0 \). Consider two points \( x_1, x_2 \) with \( |V_1| = \delta^{4/3}, \ |V_2| = \delta^{1/3} \) and \( A_1 = A_2 = [0, 1] \). The 2-tuple \((0,1)\) is balanced. Here, the set \( C((0,1)) = \{0\} \), however \((1,1)\) is not a balanced 2-tuple.
Lemma 4.9. Let \((i_k)\) be an ordered \(n\)-tuple of nonnegative integers. Suppose \(h \in PC^l(X)\), where \(l = \max\{i_k\}\) and \(h^{(i)}(x_k) = 0, 0 \leq j \leq i_k - 1, 1 \leq k \leq n\). Then
\[
\|h\|_{\phi,V} = O\left(\max\{A_k(i_k)\}\right).
\]

If a polynomial \(P \in \Pi^m, m+1 = \sum_{k=1}^n i_k\), satisfies \(P^{(i)}(x_k) = f^{(i)}(x_k), 1 \leq j \leq i_k - 1, 1 \leq k \leq n\), we call it the Hermite interpolating polynomial of the function \(f\) on \(\{x_1, \ldots, x_n\}\).

Now, we are in condition to prove the main result in this Section.

Theorem 4.5. Let \((i_k)\) be a balanced \(n\)-tuple and \(m+1 = \sum_{k=1}^n i_k\). If \(l = \max\{i_k\}\) and \(f \in PC^l(X)\), then the best local approximation to \(f\) from \(\Pi^m\) on \(\{x_1, \ldots, x_n\}\) is the Hermite interpolating polynomial of \(f\) on \(\{x_1, \ldots, x_n\}\).

Proof. Let \(H \in \Pi^m\) be the Hermite interpolating polynomial and let \(\{g_V\}\) be a net of best approximations of \(f\) from \(\Pi^m\) respect to \(\|\cdot\|_{\phi,V}\). From Lemma 4.9,
\[
\|g_V - H\|_{\phi,V} = O\left(\max\{A_k(i_k)\}\right).
\]

Using Theorem 4.4 and the equivalence of the norms in \(\Pi^m\), we get
\[
\|g_V - H\| \leq \min_{1 \leq k \leq n} \left\{ \left| \frac{V_k}{V_k^{i_k}} \right| \right\} \|g_V - H\|_{\phi,V}.
\]

So, the definition of balanced \(n\)-tuple implies \(g_V \to H\), as \(|V| \to 0\). \(\square\)

4.3 Best local \(\phi\)-approximation

In this section we present the analysis of the problem given in [9]. Here the authors study the existence of the best local approximation, with balanced neighborhoods, when the error (1.1) is the following
\[
\int_V \phi(|f(x) - g_V(x)|) \, dx = \min_{h \in \Pi^m} \int_V \phi(|f(x) - h(x)|) \, dx.
\]

The technique used in [9] follows the pattern used in [2]. Assume that \(\phi \in \Phi\) satisfies the \(\Delta^\ell\)-condition.

Given a net of neighborhoods \(\{V\}\), denote for each \(1 \leq k \leq n\) and \(\beta \in \mathbb{R}\)
\[
c_k(\beta) := \phi(\|V_k^\beta\| |V_k|).
\]

Assume for any \(\alpha, \beta \geq 0\) and any \(j, k\) such that \(1 \leq j, k \leq n\), that either
\[
c_j(\beta) = O(c_k(\alpha)) \quad \text{or} \quad c_k(\alpha) = O(c_j(\beta))
\]
or both. Then, \(c_j(\alpha_j)\) is the maximal of the \(n\)-tuple \((c_k(a_k))\), with \(a_k \in \mathbb{R}\), if for all \(k, 1 \leq k \leq n, c_k(a_k) = O(c_j(\alpha_j))\). We denote it by
\[
\max\{c_k(a_k)\}.
\]
**Definition 4.3.** An $n$-tuple $(i_k)$ of nonnegative integers is said to be $\phi$-balanced if for each $j$ such that $i_j > 0$,

$$\phi \left( \frac{1}{|V_j|^b} \right) \max \left\{ \frac{c_k(i_k)}{|V_j|} \right\} = o(1).$$

If $(i_k)$ is $\phi$-balanced, then $\sum_{k=1}^{m} i_k$ is said to be a $\phi$-balanced integer.

The $n$-tuple $(V_k)$ is said to be $\phi$-balanced neighborhoods if the dimension $m+1$ of the space $\Pi^m$ is a $\phi$-balanced integer.

To each $\phi$-balanced integer there corresponds exactly one $\phi$-balanced $(i_k)$.

**Remark 4.2.** If $\phi(x) = x^p$, $1 \leq p < \infty$ the last definition of $\phi$-balanced is equivalent to those considered by Chui et al. in [2].

**Example 4.4.** Let $\phi(x) = x^3(1 + |\ln x|)$ with $\phi(0) = 0$ a convex function that satisfies the $\Delta'$ condition and $(|V_1|, |V_2|) = (\delta, e^{-1}/\delta)$, for $\delta > 0$; then each integer $m$ is a $\phi$-balanced integer.

Now we state an algorithm that generates all the $\phi$-balanced $n$-tuples.

**Algorithm 4.3.** Begin with the $\phi$-balanced $n$-tuple $(i^{(0)}_k) = (0)$ corresponding to the $\phi$-balanced integer 0. Given $(i^{(l)}_k)$, determine a maximal element of $(c_k(i^{(l)}_k))$, say $c_k(i^{(l)}_k) = \max \{c_k(i^{(l)}_k)\}$ and define $i^{(l+1)}_k = i^{(l)}_k$ for $k \neq k^*$ and $i^{(l+1)}_k = i^{(l)}_k + 1$ for $k = k^*$.

In [9], the authors proved the following lemma.

**Lemma 4.10.**

a) The above algorithm generates all $\phi$-balanced $(i_k)$.

b) If a $n$-tuple $(i^{(l)}_k)$ generated by the algorithm ($l \geq 1$) is $\phi$-balanced, then there is a unique maximal element of $(v_k(i^{(m-1)}_k))$.

As we see in the following example, the lemma gives a way to find candidates of $\phi$-balanced $n$-tuples.

**Example 4.5.** If $\phi(x) = x^3(|\ln x| + 1)$ with $\phi(0) = 0$ and $(|V_1|, |V_2|) = (\delta, \delta^4)$, for $\delta > 0$, then in the first step the algorithm generates the 2-tuple (1,0) and the corresponding maximal $\max \{c_k(i_k)\} = c_1(1)$ is unique. However the second 2-tuple generated by the algorithm is (2,0) and it is not $\phi$-balanced.

Now we expose the auxiliary lemmas given in [9].

**Lemma 4.11.** Let $i_1, \ldots, i_n$ be nonnegative integers. Suppose $h \in PC^l(X)$, where $l = \max\{i_k\}$ and that $h^{(j)}(x_k) = 0$, $1 \leq j \leq i_k - 1$, $1 \leq k \leq n$. Then

$$\int_X \phi(|h|)dx = O(\max\{c_k(i_k)\}).$$

As a corollary of Lemma 3.1, we mention the following result.
Proposition 4.3. Let \( \Lambda \) be a family of uniformly bounded measurable subsets of the real line with measure 1. Let \( P(x) = b_0 + b_1 x + \cdots + b_m x^m \) be an arbitrary polynomial of degree \( m \). Then there exists a constant \( M \) (depending on \( m \)) such that for all \( P(x) \) and all \( A \in \Lambda \),

\[
\phi(|b_k|) \leq M \int_A \phi(|P(x)|) \, dx,
\]

\( 0 \leq k \leq m \).

Instead of lemma 3.3, in [9] the authors prove the following two lemmas.

Lemma 4.12. Given \( C \), set a \( \phi \)-balanced \( n \)-tuple \( (i_k) \) such that \( m+1 = \sum_{k=1}^n i_k \) and define \( l = \max \{ i_k \} \). If \( f \in PC^l(X) \) and \( \{ g_V \} \) is a net of best \( \phi \)-approximations, then there exists \( M > 0 \) such that for all \( |V| > 0 \),

\[
\int_X \phi(|g_V|) \, dx \leq M.
\]

Lemma 4.13. Given \( V \) and a \( \phi \)-balanced \( n \)-tuple \( < i_k > \) such that \( m+1 = \sum_{k=1}^n i_k \), define \( l = \max \{ i_k \} \). If \( f \in PC^l(X) \) and \( \{ g_V \} \) is a net of best \( \phi \)-approximations, then for each \( k \)

\[
\phi(||f - g_V||^j(x_k)||V_k||^l) = O \left( \max \left\{ c_l(i_l) \left| \frac{c_l(i_l)}{|V_k|^l} \right| \right\} \right),
\]

\( 0 \leq j \leq i_k - 1 \).

Using Lemma 4.13 and the definition of \( \phi \)-balanced \( n \)-tuple it is obtained the main result in that context, which solve the best local approximation problem when the neighborhoods are \( \phi \)-balanced.

Theorem 4.6. If \( m+1 \) is a \( \phi \)-balanced integer with \( \phi \)-balanced \( (i_k) \) and \( f \in PC^l(X) \), \( (l = \max \{ i_k \}) \), then the best local \( \phi \)-approximation to \( f \) from \( \Pi^m \) is the unique \( g \in \Pi^m \) defined by the \( m+1 \) interpolation conditions

\[
f^{(j)}(x_k) = g^{(j)}(x_k),
\]

\( 0 \leq j \leq i_k - 1, 1 \leq k \leq n \).

Acknowledgments

The author would like to thank Professor M. Marano for his helpful comments about this survey.
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