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Weighted Best Local Approximation

Segio Favier* and Claudia Ridolfi

Instituto de Matemática Aplicada San Luis, Conicet and Universidad Nacional de San Luis, Ejército de los Andes 950, San Luis, Argentina.

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Abstract. In this survey the notion of a balanced best multipoint local approximation is fully exposed since they were treated in the L^p spaces and recent results in Orlicz spaces. The notion of balanced point, introduced by Chui et al. in 1984 are extensively used.

Key Words: Best Local approximation, multipoint approximation, balanced neighborhood.

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1 Introduction

The notion of a best multipoint local approximations of a function is fully treated in [2] where the L^p norm is used. Later, other approaches to best multipoint local approximations with L^p norms appeared in [7] and [8]. And finally, for Orlicz norms, we mention [3,5,9,12,13] and for a general family of norms [6,10] and [14]. However, in [2], Chui et al. introduced the concept of balanced points in L^p which includes different importance in each point.

More precisely, a rather general view of the problem is as follows. Let $f:\mathbb{R}\to\mathbb{R}$ be a function in a normed space X with norm $\|\cdot\|$. Let Π^m denote the set of polynomial in $\mathbb{R}$ of degree less or equal than m and suppose $\Pi^m\subseteq X$. Consider n points $x_1,\cdots,x_n$ in $\mathbb{R}$ and a net of small Lebesgue measurable neighborhoods V_i^δ around each point x_i such that the Lebesgue measure $|V_i^\delta|$ goes to 0 as $\delta\to 0$ for $i=1,\cdots,n$. We select the best approximation to f near the points $x_1,\cdots,x_n$ by polynomial in Π^m. Formally, for each n-tuple of neighborhoods $V_1,\cdots,V_n$ we consider the polynomial $g_V\in\Pi^m$ which minimizes

$$\|(f-h)\mathcal{X}_V\| \tag{1.1}$$

for all $h\in\Pi^m$, where $V=\cup_{i=1}^n V_i$ and $V_i=V_i^\delta$. It is well known that a best $\|\cdot\|$-approximation g_V always exists since Π^m has finite dimension. If any net g_V converges to a unique

*Corresponding author. *Email addresses:* `sfavier@unsl.edu.ar` (S. Favier), `ridolfi@unsl.edu.ar` (C. Ridolfi)

element $g\in\Pi^m$, as $\delta\to 0$, then g is said to be a best local approximation to f at the points $x_1,\cdots,x_n$. We will mention in this survey all the works which consider that the velocity of convergence $|V_i|\to 0$, as $|V|\to 0$, can be different at each point x_i. According to [2], this problem has been treated considering the concept of balanced neighborhoods in local approximation and it reflects the different importance of the points $x_1,\cdots,x_n$. We need to deal with the necessary definition of balanced neighborhoods in each context.

As we pointed out above, Chui et al. study in [2] this problem when the space X is the usual L^p space, with the norm $\|f\|_p=\left(\int_B|f(x)|^p dx\right)^{1/p}$, where B is a measurable set. They get results for balanced and non balanced neighborhoods. At last they generalize the results to the case of $\mathbb{R}^k$ instead of $\mathbb{R}$. On the other hand, in [4], the authors get balanced results in L^p using other technique. We will discuss the L^p problem with more details in Section 3.

In [11] and [12] the authors study the problem in Orlicz spaces, it means,

$$X=L^\phi(B):=\Big\{f:\ \int_B\phi(\alpha|f(x)|)dx<\infty\ \text{ for some }\ \alpha>0\Big\},$$

where ϕ is a convex function, non negative, defined on $\mathbb{R}_0^+$ and B is a Lebesgue measurable set. In these two works, the authors studied the best local approximation problem with the Luxemburg norm

$$\|f\|_\phi=\|f\|_{L^\phi(B)}=\inf\Big\{\lambda>0:\ \int_B\phi\left(\frac{|f(x)|}{\lambda}\right)dx\le 1\Big\} \tag{1.2}$$

and get results for balanced and non balanced neighborhoods. Furthermore, we can consider a different Luxemburg norms in Orlicz Spaces, that is

$$\|f\|_{\phi,B}=\inf\Big\{\lambda>0:\ \int_B\phi\left(\frac{|f(x)|}{\lambda}\right)dx\le|B|\Big\} \tag{1.3}$$

and it can generate different best approximation functions g_V than those obtained with the standard Luxemburg norm given in (1.2). In [13] the authors study the balanced neighborhoods problem with this norm using a different technique than that used in [11] and [12].

Moreover, in [9] the authors study the balanced problem in Orlicz spaces L^ϕ when the error (1.1) does not come from a norm, but considering

$$\int_V\phi(|f(x)-g_V(x)|)\,dx=\min_{h\in\pi^n}\int_V\phi(|f(x)-h(x)|)\,dx.$$

The last three problems in L^p are equivalent, but in Orlicz spaces they are different problems and have different concepts of balanced neighborhoods. In Section 4 we will present the three problems in Orlicz spaces in detail.

2 Notations

We now introduce some notation. Let $B\subset\mathbb{R}$ be a bounded open set and set $|B|$ for its Lebesgue measure. Denote by $\mathcal{M}$ the system of all equivalence classes of Lebesgue measurable real valued functions defined on B.

Let $x_1,\cdots,x_n$ be n distinct points in B. Consider a net of measurable sets $\{V\}_{|V|>0}$ such that $V=\cup_{i=1}^n V_i$, where V_i is a neighborhood of the point x_i and

$$\sup_{1\le i\le n}\sup_{y\in V_i}|x_i-y|\to 0,$$

as $|V|\to 0$. It is easy to see that $V_i=x_i+|V_i|A_i$, $i\le i\le n$, where A_i is a measurable set with measure 1. Henceforward, we assume the sets A_i are uniformly bounded.

For each p, $1\le p\le\infty$, we consider the space $L^p=L^p(B)$ and the following norms

$$\|h\|_p=\|h\|_{L^p(B)}=\left(\int_B|h(x)|^p dx\right)^{1/p}$$

and

$$\|h\|_{p,B}=\left(\frac{1}{|B|}\int_B|h(x)|^p dx\right)^{1/p},$$

for $h\in\mathcal{M}$ and $p<\infty$. Sometimes we write $\|h\|_{L^p(B)}$ instead $\|h\chi_B\|_p$, where χ_B denote the characteristic function of the set $W\subset B$.

Let Φ be the set of convex functions $\phi:[0,\infty)\longrightarrow[0,\infty)$, with $\phi(x)>0$ for $x>0$ and $\phi(0)=0$. For $\phi\in\Phi$ define the Orlicz space

$$L^\phi(B)=\left\{f\in\mathcal{M}:\int_B\phi(\alpha|f(x)|)\,dx<\infty\text{ for some }\alpha>0\right\}.$$

This space can be endowed with the Luxemburg norm $\|f\|_\phi$ defined in (1.2) as well as with the norm $\|f\|_{\phi,B}$ defined in (1.3). If $\phi(x)=x^p$, the last norm coincides with the norm $\|h\|_{p,B}$. Sometimes we write $\|f\|_{L^\phi(W)}$ instead of $\|f\chi_W\|_\phi$. The space L^ϕ with both norms is a Banach space and we refer to [1] for a detailed study of Orlicz spaces.

We recall that a function $\phi\in\Phi$ satisfies the Δ_2-condition if there exists a constant $k>0$ such that $\phi(2x)\le k\phi(x)$, for $x\ge 0$. We also say that $\phi\in\Phi$ satisfies the Δ' condition if there exists a constant $C>0$ such that $\phi(xy)\le C\phi(x)\phi(y)$ for $x,y\ge 0$. Note that it is easy to see that Δ' condition implies Δ_2 condition.

Let $f\in PC^m(B)$, where $PC^m(B)$ is the class of functions in $L^\phi(B)$ with $m-1$ continuous derivatives and with bounded piecewise continuous m^{th} derivative on B.

3 Balanced and non balanced problem in space L^p

In this section we present the results given in [2] and [4], both in L^p spaces, $p\ge 1$. In [2] Chui et al. introduce the balanced concept as follows. For each $\alpha\in\mathbb{R}$ and k, $1\le k\le n$, we denote

$$\mathcal{V}_k(\alpha):=|V_k|^\alpha,$$

and assume the following condition which allows us to compare $\mathcal{V}_k(\alpha)$ with each other as functions of α.

For any nonnegative real numbers α and β and any pair j, k, $1\le j,k\le n$,

$$\text{either } \mathcal{V}_k(\alpha)=\mathcal{O}(\mathcal{V}_j(\beta)) \text{ or } \mathcal{V}_j(\beta)=o(\mathcal{V}_k(\alpha)), \text{ as } |V|\to 0. \tag{3.1}$$

Given a collection of neighborhoods $\{V_i\}_1^n$ and a set of n non negative real numbers $\alpha_1,\cdots,\alpha_n$ in $\mathbb{R}$, we say that $\mathcal{V}_j(\alpha_j)$ is maximal if for all k, $\mathcal{V}_k(\alpha_k)=\mathcal{O}(\mathcal{V}_j(\alpha_j))$. When it happens we write $\mathcal{V}_j(\alpha_j)=\max\{\mathcal{V}_k(\alpha_k)\}$.

In the balanced case, the neighborhoods can have different measure, but it is not at random, there is a relationship between the measure of the sets $|V_k|$ and the amount of information of f over the points i_k.

Definition 3.1. A n-tuple of non negative integers (i_k) is balanced if for each j such that $i_j>0$, $\max\{\mathcal{V}_k(i_k+1/p)\}=o(\mathcal{V}_k(i_j-1+1/p))$. In this case, we say that $m+1=\sum_{k=1}^n i_k$ is a balanced integer and the neighborhoods V_k are balanced.

It is easy to see that to each balanced integer $m+1$ there corresponds exactly one balanced n-tuple (i_k) such that $\sum_{1=1}^n i_k=m+1$.

Example 3.1. If $L^p=L^2$ and the neighborhoods are $V_1=V_1(\epsilon)=x_1+\epsilon[-\frac{1}{2},\frac{1}{2}]$ and $V_2=V_2(\epsilon)=x_2+\epsilon^{1/2}[-\frac{1}{2},\frac{1}{2}]$, then (0,0), (0,1), (1,1), (1,2) and (1,3) are balanced n-tuples, while (2,2), (1,0), (2,0) and (2,1) are non balanced n-tuples.

There exists a simple way to find all the balanced n-tuples.

Algorithm 3.1. It begins with the balanced n-tuple $(i_k^{(0)}):=(0)$ corresponding to the balanced integer 0. Let $(i_k^{(l)})$ be a balanced n-tuple. Let $C=C((i_k^{(l)})):=\{j:\ \mathcal{V}_j(i_j^{(l)}+1/p)=\max\{\mathcal{V}_k(i_k^{(l)}+1/p)\}\}$. To build the next n-tuple, $(i_k^{(l+1)})$, put $i_k^{(l+1)}=i_k^{(l)}+1$ for $k\in C((i_k^{(l)}))$ and $i_k^{(l+1)}=i_k^{(l)}$ for $k\notin C$.

In [2], the authors prove that this algorithm generates exactly all the balanced n-tuples.

The following Lemma gives an order of the error produced in the approximation (1.1) with the norm $\|\cdot\|_p$. Also, this Lemma exposes how to define a maximal element.

Lemma 3.1. *Let (i_k) be an ordered n-tuple of nonnegative integers. Suppose $h\in PC^{(l)}(B)$, where $l=\max\{i_k\}$ and $h^{(j)}(x_k)=0$, $0\le j\le i_k-1$, $1\le k\le n$. Then*

$$\|h\|_{L^p(V)}=\mathcal{O}(\max\{\mathcal{V}_k(i_k+1/p)\}).$$

We now present the Lemma 3 stated in [2], which will be used in the sequel. This lemma have importance in the proof of the main Theorems.

Lemma 3.2. *Let $1\le p\le\infty$ and let Λ be a family of uniformly bounded measurable subsets of the real line with measure* 1*. Then there exists a constant M (depending on m and p) such that for all the polynomials $P\in\Pi^m$ and all $A\in\Lambda$,*

$$|c_k|\le M\|P\|_{L_p(A)},\quad 0\le k\le m,$$

where $P(x)=\sum_{k=0}^m c_k x^k$.

Now we present the first main result given in [2], which solved the problem of best local approximation, for balanced neighborhoods. In the proof they used the above lemmas.

Theorem 3.1. *If $m+1$ is a balanced integer with balanced n-tuple (i_k) and $f\in PC^l(B)$, $l=\max\{i_k\}$, then the best local approximation to f from Π^m is the unique $g\in\Pi^m$ defined by the $m+1$ interpolation conditions $f^{(j)}(x_k)=g^{(j)}(x_k)$, $0\le j\le i_k-1$, $1\le k\le n$.*

Given a balanced integer $m+1$, as a consequence of the Algorithm, there exists a following and previous balanced integer and for example, the following balanced integer will be $m+Car(C)$, where $Car(C)$ is the cardinality of the set C. So we have the following definition.

Definition 3.2. Given the neighborhoods $V_1,\cdots,V_n$, p and an integer $m+1$ we define:

- $\underline{m+1}$ the largest balanced integer less than or equal to $m+1$.
- $(\underline{i}_k)$ the balanced n-tuple satisfying $\sum\underline{i}_k=\underline{m+1}$.
- $\overline{m+1}$ the smallest balanced integer greater than or equal to $m+1$.
- $(\bar{i}_k)$ the balanced n-tuple satisfying $\sum\bar{i}_k=\overline{m+1}$.

Remark 3.1. Given a non balanced integer $m+1$, set $C=C((\underline{i}_k))=\{j:\mathcal{V}_j(\underline{i}_j+1/p)=\max\{\mathcal{V}_k(\underline{i}_k+1/p)\}\}$. Then, as the algorithm generates exactly all the balanced integers, the next balanced integer is $\overline{m+1}$, with $\underline{i}_k=\bar{i}_k$ for $k\notin C$ and $\bar{i}_k=\underline{i}_k+1$ for $k\in C$.

We establish the following auxiliary lemma from [2] which it is used to prove one of the main results.

Lemma 3.3. *Given $(|V_1|,\cdots,|V_n|)$ and $m+1$, define $l=\max\{\bar{i}_k\}$. If $f\in PC^l(B)$ and for each V,*

$$\|f-g_V\|_{L^p(V)}=\min_{h\in\pi^m}\|f-h\|_{L^p(V)},$$

then g_V is bounded on B uniformly for all $|V|>0$ and for each k, $1\le k\le n$

$$\frac{(f-g_V)^{(j)}(x_k)\mathcal{V}_k(j+1/p)}{E}=\mathcal{O}(1),\quad j=1,\cdots,\bar{i}_k-1,$$

where $E=\max\{\mathcal{V}_j(\underline{i}_j+1/p)\}$.

From Lemma 3.3, if there exists a best local approximation g, then it satisfy the equations $f^{(j)}(x_k)=g^{(j)}(x_k)$, $0\le j\le \underline{i}_k-1$, $1\le k\le n$, since

$$(f-g)^{(j)}(x_k)=\mathcal{O}\Big(\frac{E}{\mathcal{V}_k(j+1/p)}\Big)=\mathcal{O}\Big(\frac{E}{\mathcal{V}_k(\underline{i}_k-1+1/p)}\Big)=o(1)$$

for the values j and k above because $\underline{m+1}$ is a balanced integer. These are $\underline{m+1}$ constraints and there are $m+1-\underline{m+1}$ *degrees of freedom*. The remaining $m+1-\underline{m+1}$ degrees of freedom must then be chosen to minimize the local L^p error around the $\overline{m+1-\underline{m+1}}$ points. The calculations required to do this are more difficult than the previous ones and the following strong assumption is needed to prove the best local approximation existence.

The n-tuple of neighborhoods $(V_1,\cdots,V_n)$ satisfy

$$\frac{\mathcal{V}_k(\underline{i}_k+1/p)}{E}=e_k+o(1),\quad 1\le k\le n,$$

where e_k is a fixed constant.

Remark 3.2. Given $m+1$, set $C=\{k\colon 1\le k\le n \text{ and } \mathcal{V}_k(\underline{i}_k+1/p)=\max\{\mathcal{V}_l(\underline{i}_l+1/p)\}\}$. From the algorithm $e_k=0$ for $k\notin C$ and $e_k\neq 0$ for $k\in C$.

Now we present the main Theorem from [2].

Theorem 3.2. *Suppose $m+1$ is not balanced. Assume that each A_k is either an interval for each V_k or is independent of the net $\{V_k\}$. Assume that the measure $(|V_1|,\cdots,|V_n|)$ satisfies for each k,*

$$\frac{\mathcal{V}_k(\underline{i}_k+1/p)}{\max\{\mathcal{V}_j(\underline{i}_j+1/p)\}}=e_k+o(1)$$

with e_k a constant independent of the net $\{V_k\}$. Let $J_A(i,p)$ denote the minimum L^p norm over the measurable set A of an i^{th} degree polynomial with unit leading coefficient. If $f\in PC^l(B)$, where $l=\max\{\bar{i}_k\}$ and $1<p\le\infty$, then the best local approximation to f from π^m is the unique solution of the constrained l_p minimization problem

$$\min_{h\in\pi^m}\|(e_kJ_{A_k}(\underline{i}_k,p)(f-g)^{(\underline{i}_k)}(x_k))_{k=1}^n\|_{l_p}$$
$$\text{subject to}\begin{cases}(f-g)^{(j)}(x_k)=0,\\ 0\le j\le\underline{i}_k-1,\quad 1\le k\le n,\end{cases}$$

where, if A_k is an interval, we can replace $J_{A_k}(\underline{i}_k,p)$ by $J_{[0,1]}(\underline{i}_k,p)$.

If $p=1$ the l_p minimization may not have a unique solution; if it does, however, it is the best local approximation.

By the other hand, in [4] the balanced result (Theorem 3.1) is proved with other technique. The authors prove a Polya-type inequality for polynomials in L^p spaces and it has an application to best local approximation. The Polya-type inequality is the following.

Theorem 3.3. *Let $0<p\leq\infty$, and $m,n\in\mathbb{N}$. Let i_k, $1\leq k\leq n$, be n positive integers such that $i_1+\cdots+i_n=m+1$. Then there exists a constant K depending on p, i_k, for $1\leq k\leq n$, such that*

$$|c_j|\leq\frac{K}{\min\limits_{1\leq k\leq n}|V_k|^{i_k-1+1/p}}\|P\|_{L^p(V)},\quad 0\leq j\leq m,$$

for all $P(x)=\sum_{j=0}^m c_jx^j\in\pi^m$, $V=\cup_{k=1}^n V_k$, with $|V_k|>0$, $1\leq k\leq n$.

In [2] the authors prove that if $(i_1,\cdots,i_n)$ is a balanced n-tuple and f is a function sufficiently differentiable in a neighborhood of the n-points $x_1,\cdots,x_n$, the best local approximation is the classical Hermite polynomial on the points $x_1,\cdots,x_n$, fixed from the interpolation conditions of the function f in x_k up to order i_k-1, $i\leq k\leq n$. In [4], the authors get a similar result for more general functions f. They introduce the following class of Lebesgue measurable functions.

Definition 3.3. Given $p>0$ and $m+1=i_1+\cdots+i_n$, a function f belongs to the class $\mathcal{H}_{m,p}$ $(i_1,\cdots,i_n)$ if $f\in L^p(B)$ and there exists a polynomial $H\in\pi^m$ satisfying

$$\|f-H\|_{V,p}=o(|V_k|^{i_k-1+1/p}),\quad 1\leq k\leq n,\quad \text{as}\ \ |V|\to 0. \tag{3.2}$$

These classes are similar to those introduced in [4] and [14], for $n=1$. As a consequence of Theorem 3.3 it follows that the polynomial H is unique if $f\in\mathcal{H}_{m,p}$ $(i_1,\cdots,i_n)$. It is called the *generalized Hermite polynomial* of f on $x_1,\cdots,x_n$ with respect to the n-tuple $(i_1,\cdots,i_n)$. Moreover,

Theorem 3.4. *Let $f\in\mathcal{H}_{m,p}$ $(i_1,\cdots,i_n)$. Then the best local approximation to f from π^m, say H, is the generalized Hermite polynomial of f on $x_1,\cdots,x_n$ with respect to the n-tuple $(i_1,\cdots,i_n)$.*

In particular, under certain differentiability conditions of the function f, from Lemma 3.1 the polynomial H, which interpolates the data $f^{(j)}(x_k)$, $0\leq j\leq i_k-1$, $1\leq k\leq n$, satisfies

$$\|f-H\|_{L^p(V)}=\mathcal{O}\Big(\sum_{k=1}^n|V_k|^{i_k+1/p}\Big)=\mathcal{O}(\max\{|V_k|^{i_k+1/p}\}).$$

If in addition, $(i_1,\cdots,i_n)$ is a balanced n-tuple, then $f\in\mathcal{H}_{m,p}$ $(i_1,\cdots,i_n)$ and H fulfills (3.2). Therefore, it is obtained as a consequence of Theorem 3.3 the analogous result for balanced k-tuples, proved in Theorem 1 of [2]. However in [4] the authors do not assume any condition of classic differentiability over the function f.

Here we have exposed two techniques to get existence of multipoint local approximation with balanced neighborhoods. They can be generalized to Orlicz spaces as we show in the next section.

4 Balanced and non balanced problems in Orlicz spaces L^ϕ

In this section we present the best local approximation polynomials using balanced neighborhoods in Orlicz spaces L^ϕ. According with the norm that we consider to minimize the error (1.1), we obtain three different problems which we include in the following subsection.

4.1 Best local approximation in L^ϕ with norm $\|\cdot\|_\phi$

In this subsection we will expose the existence of best multipoint local $\|\cdot\|$-approximation to a function f from Π^n for a suitable integer n, it means, for the balanced and non balanced cases. This problem is considered in an arbitrary Orlicz space L^ϕ with the Luxemburg norm $\|\cdot\|_\phi$. We refer to [1] for a detailed treatment of Orlicz spaces. For this purpose, we introduce the concept of $\|\cdot\|$-balanced integer in this context. The following results follow the pattern given in [2] for L^p spaces and they appeared in [11] and [12].

Now we assume in this article that $\phi\in\Phi$ and it satisfies the Δ_2-condition and recall that $V=\bigcup_{k=1}^n V_k$ is a net of union of neighborhoods of the points $x_1,\cdots,x_n$ and denote by g_V a best $\|\cdot\|_\phi-$approximation to f from Π^m on V, it means,

$$\|(f-g_V)\mathcal{X}_V\|_\phi=\min_{h\in\pi^m}\|(f-h)\mathcal{X}_V\|_\phi.$$

For each $\alpha>0$ and $1\le k\le n$, we denote

$$v_k(\alpha):=\frac{|V_k|^\alpha}{\phi^{-1}\left(\frac{1}{|V_k|}\right)},$$

and instead of (3.1) we assume in this context that for any nonnegative integers α and β, and any pair j, k, $1\le j,k\le n$, either

$$v_k(\alpha)=\mathcal{O}(v_j(\beta))\quad\text{or}\quad v_j(\beta)=o(v_k(\alpha)). \tag{4.1}$$

Let (i_k) be an ordered n-tuple of nonnegative integers. We say that $v_j(i_j)$ is *maximal* if $v_k(i_k)=\mathcal{O}(v_j(i_j))$ for all $1\le k\le n$. We denote it by

$$v_j(i_j)=\max\{v_k(i_k)\}.$$

Definition 4.1. An n-tuple (i_k) of nonnegative integers is said to be $\|\cdot\|_\phi$-balanced if for each $i_j>0$,

$$\frac{1}{v_j(i_j-1)}\max\{v_k(i_k)\}=o(1).$$

If (i_k) is $\|\cdot\|_\phi$-balanced, we say that $\sum_{k=1}^n i_k$ is a $\|\cdot\|_\phi$-balanced integer.

Next we set, from [11], an example of $\|\cdot\|_\phi$-balanced integers.

Example 4.1. Define $\phi(x)=\frac{x^2}{\ln(e+x)}$, $x\geq 0$. It can be seen that ϕ satisfies the Δ_2-condition (see [1, pp. 30]). Given x_1, x_2 and let the neighborhoods satisfy $|V_2|=|V_1|^2$. Thus these neighborhoods satisfy the conditions (4.1) and every integer is $\|\cdot\|_\phi$-balanced.

In [11] an algorithm is presented and it generates all the $\|\cdot\|_\phi$-balanced integer as in L^p.

Algorithm 4.1. Begin with the $\|\cdot\|_\phi$-balanced n-tuple $\langle i_k^{(0)}\rangle=\langle 0\rangle$ corresponding to the $\|\cdot\|_\phi$-balanced integer 0. Then, given $(i_k^{(s)})$ for $s\geq 0$, set $C=\{l:\ v_l(i_l^{(s)})=\max\{v_k(i_k^{(s)})\}\}$. We build the next $\|\cdot\|_\phi$-balanced n-tuple $(i_k^{(m+1)})$ taking $i_k^{(s+1)}=i_k^{(s)}+1$, for $k\in C$ and $i_k^{(s+1)}=i_k^{(s)}$, for $k\notin C$.

Remark 4.1. It is proved in [11] that to each $\|\cdot\|_\phi$-balanced integer there corresponds exactly one $\|\cdot\|_\phi$-balanced n-tuple. Also an integer $m+1$ is $\|\cdot\|_\phi$-balanced if only if $m+1=\sum_{k=1}^n i_k$ for some (i_k) generated by this algorithm.

Now, we cite from [11] the following auxiliary lemmas and the first main result. Instead of Lemma 3.1, in [11] the authors prove the following auxiliary result.

Lemma 4.1. *Let (i_k) be an increasing ordered n-tuple of nonnegative integers. Suppose $h\in PC^l(X)$, where $l=\max\{i_k\}$ and $h^{(j)}(x_k)=0$, $0\leq j\leq i_k-1$, $1\leq k\leq n$. Then*

$$\|h\|_{L^\phi(V)}=\mathcal{O}(\max\{v_k(i_k)\}).$$

Instead of Lemma 3.2 we have

Lemma 4.2. *Let Λ be a family of uniformly bounded measurable subsets of the real line with measure* 1 *and let $0<r<1$, then there exists a constant $s>0$ such that*

$$\left\||P|^{-1}\left(\left[\frac{\|P\|_{\infty,A}}{s},\|P\|_{\infty,A}\right]\right)\cap A\right|\geq r, \tag{4.2}$$

for all $A\in\Lambda$ and for all $P\in\Pi^m$.

Proposition 4.1. Given an integer $m+1$, consider the Definition 3.2 for the Luxemburg norm, then

a) If $\underline{i}_j+1=\bar{i}_j$, then $\max\{v_k(\underline{i}_k)\}=\mathcal{O}(v_j(\bar{i}_j-1))$;

b) If $\underline{i}_j=\bar{i}_j$, then $\max\{v_k(\underline{i}_k)\}=o(v_j(\bar{i}_j-1))$;

c) If $\underline{m+1}<\overline{m+1}$, then $\max\big\{v_k(\bar{i}_k)\big\}=o(\max\{v_k(\underline{i}_k)\})$.

We now present the first important result from [11] concerning to the behavior of a net $\{g_V\}_{|V|>0}$ of best $\|\cdot\|_\phi$-approximations from Π^m, as $|V|\to 0$.

Theorem 4.1. *Let $m+1$ be a positive integer and $l=\max\{\bar{i}_k\}$. If $f\in PC^l(X)$ and $\{g_V\}_{|V|>0}$ is a net of best $\|\cdot\|_\phi$-approximations of f from π^m on V, then $\{g_V\}_{|V|>0}$ is uniformly bounded on X.*

Using the same technique it is obtained

Lemma 4.3. *Given an integer $\underline{m+1}$, set $l=\max\{\underline{i}_k\}$. If $f\in PC^l(X)$ and $\{g_V\}_{|V|>0}$ is a net of best $\|\cdot\|_\phi$-approximations of f from π^m on V, then*

$$|(f-g_V)^{(j)}(x_k)v_k(j)|=\mathcal{O}(\max\{v_k(\underline{i}_k)\}), \tag{4.3}$$

$0\le j\le \underline{i}_k-1$, $1\le k\le n$.

Thus, using the $\|\cdot\|_\phi$-balanced definition, it follows the main result of [11].

Theorem 4.2. *Let (i_k) be a $\|\cdot\|_\phi$-balanced n-tuple and let $0<m+1=\sum i_k$. If $l=\max\{i_k\}$, $f\in PC^l(X)$, then the best local $\|\cdot\|_\phi$-approximation to f from Π^m is the unique $g\in\Pi^m$ defined by the $m+1$ interpolation conditions*

$$f^{(j)}(x_k)=g^{(j)}(x_k),$$

$0\le j\le i_k-1$, $1\le k\le n$.

Now, we cite the following results from [12], which are a continuity of the above analysis.

Set $E=\max\{v_l(\underline{i}_l)\}$ and

$$c_{j,k}=c_{j,k}(V):=(f-g_V)^{(j)}(x_k)\frac{v_k(j)}{E}, \tag{4.4}$$

for $k=1,2,\cdots,n$ and $j=0,1,\cdots,\underline{i}_k-1$. As a consequence of Lemma 4.3 we have

$$c_{j,k}=\mathcal{O}(1),\quad k=1,2,\cdots,n,\quad j=0,1,\cdots,\underline{i}_k-1. \tag{4.5}$$

Since

$$g_V^{(i)}(x_s)=f^{(i)}(x_s)-c_{i,s}\frac{E}{v_s(i)}\quad\text{and}\quad \frac{E}{v_s(i)}=o(1)$$

for $i=0,1,\cdots,\underline{i}_s-1$, from (4.5) we obtain

$$g_V^{(i)}(x_s)=f^{(i)}(x_s)+o(1),\quad s=1,2,\cdots,n,\quad i=0,1,\cdots,\underline{i}_s-1. \tag{4.6}$$

Consider the following basis for Π^m, say $\{u_{j,k}\}\cup\{w_r\}$, with $k=1,2,\cdots,n$, $j=0,1,\cdots,\underline{i}_k-1$, and $r=1,2,\cdots,(m+1)-\underline{m+1}$, which satisfies

$$u_{j,k}^{(j')}(x_{k'})=\delta_{(j,k),(j',k')}\quad\text{and}\quad w_r^{(j')}(x_{k'})=0,\quad k'=1,2,\cdots,n,\quad\text{and}\quad j'=0,1,\cdots,\underline{i}_{k'}-1,$$

where $\delta_{(j,k),(j',k')}$ is the Kronecker delta. Observe that if $g\in\pi^m$, then

$$g(x)=\sum_{k=1}^{n}\sum_{j=0}^{\underline{i}_k-1}a_{j,k}u_{j,k}(x)+\sum_{r=1}^{(m+1)-\underline{m+1}}b_r w_r(x),$$

where $g^{(j)}(x_k)=a_{j,k}$, $k=1,2,\cdots,n$ and $j=0,1,\cdots,\underline{i}_k-1$.

Thus, if there exists a best local approximation g, since (4.6) it will satisfy the equations $f^{(j)}(x_k)=g^{(j)}(x_k)$, $0\le j\le \underline{i}_k-1$, $1\le k\le n$. The remaining $m+1-\underline{m+1}$ degrees of freedom must then be chosen so as to minimize the local L^p error around the $\overline{m+1}-\underline{m+1}$ points. It required a delicate analysis and it appears in [12]. There are many auxiliary lemmas here to prove the main result, which solve the best local approximation problem when $(m+1)$ is not a balanced integer. As an example of the auxiliary lemmas we expose the following (see [12]).

Given a non balanced integer $m+1$, set

$$C=C((\underline{i}_k))=\{j: v_j(\underline{i}_j+1/p)=\max\{v_k(\underline{i}_k+1/p)\}\}.$$

Lemma 4.4. *There holds*

$$\Gamma:=\left\|\sum_{k\in C}\left(\sum_{s=1}^{n}\sum_{i=0}^{\underline{i}_s-1}\frac{c_{i,s}}{v_s(i)}u_{i,s}^{(\underline{i}_k)}(x_k)\right)\frac{(x-x_k)^{\underline{i}_k}}{\underline{i}_k!}\mathcal{X}_{V_k}(x)\right\|_{L^{\phi}(V)}=o(1).$$

Lemma 4.5. *Let $\phi\in\Phi$ satisfying the Δ_2-condition. If for each $x\ge 0$ there exists*

$$\lim_{\alpha\to\infty}\frac{\phi(\alpha x)}{\phi(\alpha)}=:\psi(x),$$

then $\psi(x)=x^p$ for some $p\ge 1$.

Lemma 4.6. *For every $k\in C$, set*

$$P_{k,V}(y):=\sum_{j=0}^{\underline{i}_k-1}\frac{c_{j,k}(V)}{j!}y^j+c_k^*(\delta)y^{\underline{i}_k},\quad k\in C,$$

such that $\lim_{|V|\to 0}c_{j,k}(V)=d_{j,k}$ and $\lim_{|V|\to 0}c_k^*(V)=m_k$. If

$$\lim_{\alpha\to\infty}\frac{\phi(\alpha x)}{\phi(\alpha)}=:\psi(x)$$

exists for $x\ge 0$ and $\lim_{|V|\to 0}\alpha_k(V)=\infty$ for each $k\in C$, then

$$\lim_{|V|\to 0}\left[\inf\left\{\lambda>0:\sum_{k\in C}\int_{A_k}\frac{\phi\left(\alpha_k(V)\frac{|P_{k,V}(y)|}{\lambda}\right)}{\phi(\alpha_k(V))}dy\le 1\right\}\right]$$
$$=\inf\left\{\lambda>0:\sum_{k\in C}\int_{A_k}\psi\left(\frac{|P_k(y)|}{\lambda}\right)dy\le 1\right\},$$

where

$$P_k(y)=\sum_{j=0}^{\underline{i}_k-1}\frac{d_{j,k}}{j!}y^j+m_ky^{\underline{i}_k}.$$

We now can give, under certain conditions, the existence of the best local $\|\cdot\|_\phi$-approximation.

Theorem 4.3. *Let $\phi\in\Phi$ satisfying the Δ_2-condition. Assume that there exists $\lim_{\alpha\to\infty}\frac{\phi(\alpha x)}{\phi(\alpha)}$ for all $x\geq 0$ and therefore this limit is x^p for some $p\geq 1$. Let $m+1$ be a non $\|\cdot\|_\phi$-balanced integer and $l=\max_{1\leq k\leq n}\{\bar{i}_k\}$. For each $k\in C$ suppose*

$$\lim_{\delta\to 0}\frac{v_k(\underline{i}_k)}{E}=e_k>0. \tag{4.7}$$

If $f\in PC^l(X)$ then, for $|V|\to 0$, the limit of any convergent subsequence of $\{g_V\}$, a net of best $\|\cdot\|_\phi$-approximations of f from Π^m, is a solution of the following minimization problem in $\mathbb{R}^{\overline{m+1}-\underline{m+1}}$:

$$\begin{cases} \min_{h\in\Pi^m}\left\|\left\langle e_kJ_{A_k}(\underline{i}_k,p)(f-h)^{(\underline{i}_k)}(x_k)/\underline{i}_k!\right\rangle_{k\in K}\right\|_{l_p}, \\ \text{with the constraints } (f-h)^{(j)}(x_k)=0, \quad k=1,2,\cdots,n, \text{ and } j=0,1,\cdots,\underline{i}_k-1, \end{cases} \tag{4.8}$$

where, for $k\in C$, $J_{A_k}(\underline{i}_k,p)$ is the minimum L_p norm over A_k of an $\underline{i}_k$th degree polynomial with unit leading coefficient. In particular, if (4.8) *has a unique solution g, then $g=\lim_{|V|\to 0}g_V$ and therefore this is a best local $\|\cdot\|_\phi$-approximation to f from Π^m on $\{x_1,\cdots,x_n\}$.*

The following example shows that $\lim_{|V|\to 0}g_V$ may not exist if ϕ does not satisfy the assumption that $\lim_{\alpha\to\infty}\frac{\phi(\alpha x)}{\phi(\alpha)}$ exists for all $x\geq 0$. The proof is in [12].

Example 4.2. Let $x_1=0$, $x_2=1$, $A_1=A_2=\left[-\frac{1}{2},\frac{1}{2}\right]$, $|V_1|=2\delta$, $|V_2|=\delta$, for $0<\delta<\frac{1}{3}$ and let $\Pi^m=\Pi^0$ be the subspace formed by the constant functions in L^ϕ. Define

$$\phi(x)=\begin{cases} x, & \text{if } x\in[0,1], \\ 2x-1, & \text{if } x\in[1,2], \\ 23^\eta x-3^{2\eta}, & \text{if } x\in[23^{\eta-1},3^\eta], \quad \eta\in\mathbb{N}, \end{cases} \tag{4.9}$$

and $f(x)=0$ if $x\in\left[-\frac{1}{3},\frac{1}{3}\right]$, $f(x)=1$ if $x\in\left[\frac{5}{6},\frac{7}{6}\right]$.

4.2 Best local approximation with the norm $\|\cdot\|_{\phi,B}$

In this section we expose the analysis given in [13] to prove the existence of the best local approximation to a function f, with balanced neighborhoods, when the error (1.1) is the following. Denote $g_V\in\Pi^m$ such that

$$\|f-g_V\|_{\phi,V}=\min_{h\in\Pi^m}\|f-h\|_{\phi,V}.$$

These best approximation can be different to that given with the Luxemburg norm $\|\cdot\|_{L^\phi(V)}$.

The analysis in [13] follows the pattern used in [4] for L^p spaces. We begin with the following auxiliary lemmas and properties.

If ϕ satisfies the Δ'-condition, it is easy to see that there exists a constant $K>0$ such that

$$\phi^{-1}(x)\phi^{-1}(y)\le K\phi^{-1}(xy) \quad \text{for all } \ x,y\ge 0. \tag{4.10}$$

We assume in this section that $\phi\in\Phi$ and it satisfies the Δ'-condition.

Proposition 4.2. The family of all seminorms $\|\cdot\|_{\phi.V}$ with $|V|>0$, has the following properties:

(a) $\|\mathcal{X}_V\|_{\phi.V}=\frac{1}{\phi^{-1}(1)}$.

(b) If $f,g\in L^\phi(X)$ satisfy $|f|\le|g|$ on V, then $\|f\|_{\phi,V}\le\|g\|_{\phi,V}$. The inequality is strict if $|f|<|g|$ on some subset of V with positive measure.

(c) There exists a constant $M>0$ such that

$$\|f\|_{\phi,G}\le\frac{M}{\phi^{-1}\left(\frac{|G|}{|D|}\right)}\|f\|_{\phi,D}, \quad f\in L^\phi(X), \tag{4.11}$$

for all pair of measurable sets G, D, with $G\subset D$ and $|G|>0$.

Lemma 4.7. *There exists a constant $M>0$ such that*

$$\left|P^{(j)}(a)\right|\le\frac{M}{\epsilon^j}\|P\|_{\phi,[a-\epsilon,\ a+\epsilon]}$$

for all $P\in\Pi^m$, $[a-\epsilon,a+\epsilon]\subset B$ and $0\le j\le m$.

Lemma 4.8. *Let $C\subset B$ be an interval, $E\subset C$, $|E|>0$. For all $P\in\Pi^m$, there exists an interval $F:=F(E,P)\subset C$ such that*

a) $|F|\ge\frac{|E|}{2m}$,

b) $\|P\|_{\phi,F}\le 2m\|P\|_{\phi,E}$.

Now, we present the main result concerning to Pólya inequality in L^ϕ.

Theorem 4.4. *Let $\phi\in\Phi$ and $n,m\in\mathbb{N}$. Let i_k, $1\le k\le n$, be n positive integers such that $\sum_{k=1}^n i_k=m+1$. Let E_k, $1\le k\le n$, be disjoint pairwise compact intervals in $\mathbb{R}$, with $0<|E_k|\le 1$. Then there exists a positive constant M depending on ϕ, i_k and E_k, $1\le k\le n$, such that*

$$|c_j|\le\frac{M}{\min\limits_{1\le k\le n}\left\{|V\cap E_k|^{i_k-1}\phi^{-1}\left(\frac{|V\cap E_k|}{|V|}\right)\right\}}\|P\|_{\phi,V}, \quad 0\le j\le m, \tag{4.12}$$

for all $P(x)=\sum_{j=0}^m c_jx^j$, $V\subset\bigcup_{k=1}^n E_k$ with $|V\cap E_k|>0$, $1\le k\le n$.

Now, we will introduce the concept of balanced integer in that context. For each $\alpha\in\mathbb{R}$ and k, $1\le k\le n$, we denote

$$\mathcal{A}_k(\alpha):=\frac{|V_k|^\alpha}{\phi^{-1}\left(\frac{|V|}{|V_k|}\right)}.$$

The following condition allows us that $\mathcal{A}_k(\alpha)$ can be compared with each other as functions of α when $|V|\to 0$.

For any nonnegative integers α and β and any pair j, k, $1\le j,k\le n$,

$$\text{either } \mathcal{A}_k(\alpha)=\mathcal{O}(\mathcal{A}_j(\beta)) \ \text{ or } \ \mathcal{A}_j(\beta)=o(\mathcal{A}_k(\alpha)), \quad \text{as } |V|\to 0. \tag{4.13}$$

Let (i_k) be an ordered n-tuple of nonnegative integers. We say that $\mathcal{A}_j(i_j)$ is a maximal element of $(\mathcal{A}_k(i_k))$ if $\mathcal{A}_k(i_k)=\mathcal{O}(\mathcal{A}_j(i_j))$ for all $1\le k\le n$. We denote it by

$$\mathcal{A}_j(i_j)=\max\{\mathcal{A}_k(i_k)\}.$$

Observe that

$$\sum_{k=1}^n \mathcal{A}_k(i_k)=\mathcal{O}\left(\max\{\mathcal{A}_k(i_k)\}\right).$$

Definition 4.2. An n-tuple $<i_k>$ of nonnegative integers is balanced if

$$\sum_{k=1}^n \mathcal{A}_k(i_k)=o\left(\min_{1\le k\le n}\left\{|V_k|^{i_k-1}\phi^{-1}\left(\frac{|V_k|}{|V|}\right)\right\}\right).$$

In this case, we say that $\sum_{k=1}^n i_k$ is a balanced integer and (V_k) are balanced neighborhoods.

To each balanced integer there corresponds exactly one balanced n-tuple. Moreover, there are an algorithm which gives all balanced n-tuples which it is proved in [13].

Given (i_k), set

$$C=C((i_k)):=\{j:\mathcal{A}_j(i_j)=\max\{\mathcal{A}_k(i_k)\}\}.$$

Algorithm 4.2. Let v_q be a balanced integer and let $(i_k^{(v_q)})$ be the corresponding balanced n-tuple. To build the next n-tuple, $(i_k^{(v_q+1)})$, put $i_k^{(v_q+1)}=i_k^{(v_q)}+1$ for $k\in C((i_k^{(v_q)}))$ and $i_k^{(v_q+1)}=i_k^{(v_q)}$ for $k\notin C((i_k^{(v_q)}))$.

The algorithm generates n-tuples candidates to be balanced. We can observe it with the following example.

Example 4.3. Define $\phi(x)=x^3(1+|\ln x|)$, $x>0$ and $\phi(0)=0$. Consider two points x_1, x_2 with $|V_1|=\delta^{4/3}$, $|V_2|=\delta^{1/3}$ and $A_1=A_2=[0,1]$. The 2-tuple $(0,1)$ is balanced. Here, the set $C((0,1))=\{0\}$, however $(1,1)$ is not a balanced 2-tuple.

Lemma 4.9. *Let (i_k) be an ordered n-tuple of nonnegative integers. Suppose $h\in PC^l(X)$, where $l=\max\{i_k\}$ and $h^{(j)}(x_k)=0$, $0\le j\le i_k-1$, $1\le k\le n$. Then*

$$\|h\|_{\phi,V}=\mathcal{O}(\max\{\mathcal{A}_k(i_k)\}).$$

If a polynomial $P\in\Pi^m$, $m+1=\sum_{k=1}^n i_k$, satisfies $P^{(j)}(x_k)=f^{(j)}(x_k)$, $1\le j\le i_k-1$, $1\le k\le n$, we call it the Hermite interpolating polynomial of the function f on $\{x_1,\cdots,x_n\}$.

Now, we are in condition to prove the main result in this Section.

Theorem 4.5. *Let (i_k) be a balanced n-tuple and $m+1=\sum_{k=1}^n i_k$. If $l=\max\{i_k\}$ and $f\in PC^l(X)$, then the best local approximation to f from Π^m on $\{x_1,\cdots,x_k\}$ is the Hermite interpolating polynomial of f on $\{x_1,\cdots,x_n\}$.*

Proof. Let $H\in\Pi^m$ be the Hermite interpolating polynomial and let $\{g_V\}$ be a net of best approximations of f from Π^m respect to $\|\cdot\|_{\phi,V}$. From Lemma 4.9,

$$\|g_V-H\|_{\phi,V}=\mathcal{O}(\max\{\mathcal{A}_k(i_k)\}).$$

Using Theorem 4.4 and the equivalence of the norms in Π^m, we get

$$\|g_V-H\|_\infty\le\frac{K}{\min\limits_{1\le k\le n}\left\{|V_k|^{i_k-1}\phi^{-1}\left(\frac{|V_k|}{|V|}\right)\right\}}\|g_V-H\|_{\phi,V}.$$

So, the definition of balanced n-tuple implies $g_V\to H$, as $|V|\to 0$. □

4.3 Best local ϕ-approximation

In this section we present the analysis of the problem given in [9]. Here the authors study the existence of the best local approximation, with balanced neighborhoods, when the error (1.1) is the following

$$\int_V\phi(|f(x)-g_V(x)|)\,dx=\min_{h\in\Pi^n}\int_V\phi(|f(x)-h(x)|)\,dx.$$

The technique used in [9] follows the pattern used in [2]. Assume that $\phi\in\Phi$ satisfies the Δ'-condition.

Given a net of neighborhoods $\{V\}$, denote for each $1\le k\le n$ and $\beta\in\mathbb{R}$

$$c_k(\beta):=\phi(|V_k|^\alpha)\ |V_k|.$$

Assume for any $\alpha,\beta\ge 0$ and any j, k such that $1\le j,k\le n$, that either

$$c_j(\beta)=\mathcal{O}(c_k(\alpha))\quad\text{or}\quad c_k(\alpha)=\mathcal{O}(c_j(\beta))$$

or both. Then, $c_j(\alpha_j)$ is the maximal of the n-tuple $(c_k(\alpha_k))$, with $\alpha_k\in\mathbb{R}$, if for all k, $1\le k\le n$, $c_k(\alpha_k)=\mathcal{O}(c_j(\alpha_j))$. We denote it by

$$\max\{c_k(\alpha_k)\}.$$

Definition 4.3. An n-tuple (i_k) of nonnegative integers is said to be ϕ-balanced if for each j such that $i_j>0$,

$$\phi\left(\frac{1}{|V_j|^{i_j-1}}\right)\max\left\{\frac{c_k(i_k)}{|V_j|}\right\}=o(1).$$

If (i_k) is ϕ-balanced, then $\sum_{k=1}^{n} i_k$ is said to be a ϕ-balanced integer.

The n-tuple (V_k) is said to be ϕ-balanced neighborhoods if the dimension $m+1$ of the space Π^m is a ϕ-balanced integer.

To each ϕ-balanced integer there corresponds exactly one ϕ-balanced (i_k).

Remark 4.2. If $\phi(x)=x^p$, $1\le p<\infty$ the last definition of ϕ-balanced is equivalent to those considered by Chui et al. in [2].

Example 4.4. Let $\phi(x)=x^3(1+|\ln x|)$ with $\phi(0)=0$ a convex function that satisfies the Δ' condition and $(|V_1|,|V_2|)=(\delta,e^{-1/\delta})$, for $\delta>0$; then each integer m is a ϕ-balanced integer.

Now we state an algorithm that generates all the ϕ-balanced n-tuples.

Algorithm 4.3. Begin with the ϕ-balanced n-tuple $(i_k^{(0)})=(0)$ corresponding to the ϕ-balanced integer 0. Given $(i_k^{(l)})$, determine a maximal element of $(c_k(i_k^{(l)}))$, say $c_{k*}(i_{k*}^{(l)})=\max\{c_k(i_k^{(l)})\}$ and define $i_k^{(l+1)}=i_k^{(l)}$ for $k\neq k*$ and $i_k^{(l+1)}=i_k^{(l)}+1$ for $k=k*$.

In [9], the authors proved the following lemma.

Lemma 4.10. *a) The above algorithm generates all ϕ-balanced (i_k).*

b) If a n-tuple $(i_k^{(l)})$ generated by the algorithm ($l\ge 1$) is ϕ-balanced, then there is a unique maximal element of $(v_k(i_k^{(m-1)}))$.

As we see in the following example, the lemma gives a way to find candidates of ϕ-balanced n-tuples.

Example 4.5. If $\phi(x)=x^3(|\ln x|+1)$ with $\phi(0)=0$ and $(|V_1|,|V_2|)=(\delta,\delta^4)$, for $\delta>0$, then in the first step the algorithm generates the 2-tuple (1,0) and the corresponding maximal $\max\{c_k(i_k)\}=c_1(1)$ is unique. However the second 2-tuple generated by the algorithm is (2,0) and it is not ϕ-balanced.

Now we expose the auxiliary lemmas given in [9].

Lemma 4.11. *Let $i_1,\cdots,i_n$ be nonnegative integers. Suppose $h\in PC^l(X)$, where $l=\max\{i_k\}$ and that $h^{(j)}(x_k)=0$, $1\le j\le i_k-1$, $1\le k\le n$. Then*

$$\int_V \phi(|h|)dx=\mathcal{O}(\max\{c_k(i_k)\}).$$

As a corollary of Lemma 3.1, we mention the following result.

Proposition 4.3. Let Λ be a family of uniformly bounded measurable subsets of the real line with measure 1. Let $P(x)=b_0+b_1x+\cdots+b_mx^m$ be an arbitrary polynomial of degree m. Then there exists a constant M (depending on m) such that for all $P(x)$ and all $A\in\Lambda$,

$$\phi(|b_k|)\leq M\int_A \phi(|P(x)|)dx,$$

$0\leq k\leq m$.

Instead of lemma 3.3, in [9] the authors prove the following two lemmas.

Lemma 4.12. *Given C, set a ϕ-balanced n-tuple (i_k) such that $m+1=\sum_{k=1}^n i_k$ and define $l=\max\{i_k\}$. If $f\in PC^l(X)$ and $\{g_V\}$ is a net of best ϕ-approximations, then there exists $M>0$ such that for all $|V|>0$,*

$$\int_X \phi(|g_V|)\,dx\leq M.$$

Lemma 4.13. *Given V and a ϕ-balanced n-tuple $<i_k>$ such that $m+1=\sum_{k=1}^n i_k$, define $l=\max\{i_k\}$. If $f\in PC^l(X)$ and $\{g_V\}$ is a net of best ϕ-approximations, then for each k*

$$\phi(|(f-g_V)^{(j)}(x_k)||V_k|^j)=\mathcal{O}\left(\max\left\{\frac{c_l(i_l)}{|V_k|}\right\}\right),$$

$0\leq j\leq i_k-1$.

Using Lemma 4.13 and the definition of ϕ-balanced n-tuple it is obtained the main result in that context, which solve the best local approximation problem when the neighborhoods are ϕ-balanced.

Theorem 4.6. *If $m+1$ is a ϕ-balanced integer with ϕ-balanced (i_k) and $f\in PC^l(X)$, $(l=\max\{i_k\})$, then the best local ϕ-approximation to f from Π^m is the unique $g\in\Pi^m$ defined by the $m+1$ interpolation conditions*

$$f^{(j)}(x_k)=g^{(j)}(x_k),$$

$0\leq j\leq i_k-1$, $1\leq k\leq n$.

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