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A Note on the Operator Equation Generalizing the Notion of Slant Hankel Operators

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Abstract. The operator equation $\lambda M_{\bar{z}} X = X M_{z^k}$, for $k \geq 2$, $\lambda \in \mathbb{C}$, is completely solved. Further, some algebraic and spectral properties of the solutions of the equation are discussed.

Key Words: Hankel operators, slant Hankel operators, generalized slant Toeplitz operators, generalized slant Toeplitz operators, spectrum of an operator.

AMS Subject Classifications: 47B35

1 Introduction

Let $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{T}$ denote the set of integers, set of complex numbers and the unit circle, respectively. Let $L^2(\mathbb{T})$ (simply written as L^2) denote the classical Hilbert space with standard orthonormal basis $\{e_n : n \in \mathbb{Z}\}$, where $e_n(z) = z^n$ for each $z \in \mathbb{T}$. The symbol H^2 denotes the space generated by $\{e_n : n \geq 0\}$. The symbol L^∞ is used to denote the space of all essentially bounded measurable functions on $\mathbb{T}$ and $H^\infty = L^\infty \cap H^2$. The theory of Hankel operators, which is a beautiful area of mathematical analysis, admits of vast applications. In 1861, Hankel [12] began the study of finite matrices whose entries depend only on the sum of the coordinates and such objects are called Hankel matrices. In 1881, Kronecker [14] obtained first theorem about infinite Hankel matrices that characterizes Hankel matrices of finite rank.

The development of the theory of Hankel operators led to different generalizations of the original concept, like, slant Hankel operators, λ-Hankel operators and (λ,μ)-Hankel operators (see [2, 5] and [8]). A lot of progress has taken place in the study of Hankel operators on Bergman spaces on the disk, Dirichlet type spaces, Bergman and Hardy spaces on the unit ball in C^n or on symmetric domains, etc. [16].

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Hankel operators on the space L^2 are characterized by the operator equation $M_{\bar{z}}X = XM_z$, whereas on the Hardy space H^2 these are characterized by the operator equation $U^*X = XU$, where U is the forward unilateral shift operator on the Hardy space H^2. We refer to [10, 11, 16] and the references therein for the basic study of Hankel operators on these spaces. Motivated by the approach initiated by Barría and Halmos [2], various equations, like, $M_{\bar{z}}X = XM_{z^2}$ (solutions of which are named as slant Hankel operators [2]), $U^*X - XU = \lambda X$, $\lambda \in \mathbb{C}$ (solutions of which are named as λ-Hankel operators [5]) etc. are attained by mathematicians. In this row, generalized slant Hankel operators [3] have also been obtained which are nothing but the solution of the operator equation $M_{\bar{z}}X = XM_{z^k}$, for $k \geq 2$ and are named as k^{th}-order slant Hankel operators. Work of Avendaño [5] dragged our attention to the operator equation $\lambda M_{\bar{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$. Clearly, for $\lambda = 1$, this equation characterizes the k^{th}-order slant Hankel operators and if further $k = 2$ then it is nothing but the equation characterizing slant Hankel operators.

From the work of Nehari [15], it is known that each Hankel operator is induced by an essentially bounded measurable symbol $\phi \in L^\infty$ and is denoted as H_ϕ. Not much is known about spectral properties of Hankel operators in terms of the inducing symbol. Power [17] described the essential spectrum of H_ϕ for piecewise continuous functions $\phi \in L^\infty$. In this paper, we completely solve the operator equation $\lambda M_{\bar{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$. We describe some of the spectral properties of the solutions of the equation $\lambda M_{\bar{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$. We achieve the containment of a closed disc in the spectrum of each non-zero operator satisfying the equation $\lambda M_{\bar{z}}X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$.

2 Operator equation: $\lambda M_{\bar{z}}X = XM_{z^k}$ for $k \geq 2$, $\lambda \in \mathbb{C}$

In last two decades various operator equations generalizing the notion of Hankel operators have been discussed, for the details and importance of which we suggest the references [2, 3] and [4]. The purpose here is to call attention to the operator equation $\lambda M_{\bar{z}}X = XM_{z^k}$, for an integer $k \geq 2$ and $\lambda \in \mathbb{C}$. Throughout the paper, k is assumed to be an integer greater than or equal to 2. We begin with the following result.

Theorem 2.1. *The only solution of the operator equation $\lambda M_{\bar{z}}X = XM_{z^k}$, $|\lambda| \neq 1$ is the zero operator.*

Proof. Suppose that X satisfies $\lambda M_{\bar{z}}X = XM_{z^k}$. First, consider the case $|\lambda| < 1$. Define a map $\tau : \mathfrak{B}(L^2) \to \mathfrak{B}(L^2)$ as $\tau(X) = \lambda M_{\bar{z}}XM_{\bar{z}^k}$. Then $\|\tau\| \leq |\lambda| < 1$ and $(I - \tau)$ is invertible. Now $(I - \tau)X = 0$, which implies that $X = 0$.

Now consider the case $|\lambda| > 1$. This time we define the mapping τ as $\tau(X) = M_z XM_{z^k}$. Now $\|\tau\| \leq 1$ so $(\lambda I - \tau)$ is invertible and this provides that $X = 0$. This completes the proof. $\square$

Now in view of the last result, we are left to solve the operator equation $\lambda M_{\bar{z}}X=XM_{z^k}$ for $|\lambda|=1$. We consider the operators W_k and J (the flip operator) on L^2 defined as

$$W_k e_n=\begin{cases} e_m, & \text{if } n=km, \quad m\in\mathbb{Z},\\ 0, & \text{otherwise,}\end{cases}$$

and $Je_n=e_{-n}$ for each $n\in\mathbb{Z}$. Then the facts $W_kJ=JW_k$, $M_{\phi(z)}W_k=W_kM_{\phi(z^k)}$ and $M_{\phi(z)}J=JM_{\phi(\bar{z})}$, where $\phi\in L^\infty$, are well known about these operators. It is interesting to know that each k^{th}-order slant Hankel operator on L^2 is of the form W_kJM_ϕ for some $\phi\in L^\infty$ (see [3]). Using these facts, we claim the following.

Theorem 2.2. *Let $\lambda\in\mathbb{C}$ be such that $|\lambda|=1$. The operator equation $\lambda M_{\bar{z}}X=XM_{z^k}$ admits of non-zero solutions and each non-zero solution is of the form $X=D_\lambda W_kJM_\phi$ for some $\phi\in L^\infty$, where D_λ is the composition operator on L^2 induced by $z\mapsto\bar{\lambda}z$, i.e., $D_\lambda f(z)=f(\bar{\lambda}z)$ for all $f\in L^2$.*

Proof. Suppose X is an operator of the form $D_\lambda W_kJM_\phi$ for some $\phi\in L^\infty$, where symbol D_λ is used in the sense it is defined in the statement. Now it can be seen that $\lambda M_{\bar{z}}D_\lambda=D_\lambda M_{\bar{z}}$ and $M_{\bar{z}}(W_kJM_\phi)=(W_kJM_\phi)M_{z^k}$, which provides that $\lambda M_{\bar{z}}X=XM_{z^k}$.

Conversely, suppose that X is an operator satisfying $\lambda M_{\bar{z}}X=XM_{z^k}$. Pre-multiplying by $D_{\bar{\lambda}}$, we get that $M_{\bar{z}}D_{\bar{\lambda}}X=D_{\bar{\lambda}}XM_{z^k}$. Therefore, $D_{\bar{\lambda}}X$ is a k^{th}-order slant Hankel operator on L^2 and hence we get the result. □

Now onward, we are focussed to study the behavior of the solutions of the equation $\lambda M_{\bar{z}}X=XM_{z^k}$, $|\lambda|=1$. In fact, here onward, the term solution is always used in reference to the solution of this equation only. The notion of Toeplitz operators, Hankel operators, slant Toeplitz operators and slant Hankel operators are characterized in terms of matrices (see [2, 7, 13, 17]) and in the same direction we would like to have a look at the matrix characterization to the solutions of the equation $\lambda M_{\bar{z}}X=XM_{z^k}$, $|\lambda|=1$. For $\phi=\sum_{n\in\mathbb{Z}}a_ne_n$ in L^∞ and $\lambda\in\mathbb{C}$ with $|\lambda|=1$, the solution $X=D_\lambda W_kJM_\phi$ satisfies

$$\langle Xe_j,e_i\rangle=\langle D_\lambda W_kJM_\phi e_j,e_i\rangle=\bar{\lambda}^i a_{-ki-j}$$

for each $i,j\in\mathbb{Z}$. So the matrix representation of the solution X is

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \lambda a_{k+1} & \lambda a_k & \lambda a_{k-1} & \cdots & \lambda a_0 & \cdots\\ \cdots & a_1 & a_0 & a_{-1} & \cdots & a_{-k} & \cdots\\ \cdots & \lambda^{-1}a_{-k+1} & \lambda^{-1}a_{-k} & \lambda^{-1}a_{-k+1} & \cdots & \lambda^{-1}a_{-2k} & \cdots\\ \cdots & \lambda^{-2}a_{-2k+1} & \lambda^{-2}a_{-2k} & \lambda^{-2}a_{-2k-1} & \cdots & \lambda^{-2}a_{-3k} & \cdots\\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

We now have the following characterization to the solutions in terms of matrices.

Theorem 2.3. *A necessary and sufficient condition for an operator X on L^2 to be a solution of the equation $\lambda M_{\overline{z}}X = XM_{z^k}$, $|\lambda|=1$ is that its matrix $[a_{ij}]$ with respect to the standard orthonormal basis $\{e_n : n\in\mathbb{Z}\}$ satisfies*

$$a_{i-1,j+k} = \lambda a_{i,j}$$

for every $i,j\in\mathbb{Z}$.

Theorem 2.2 can be restated in the following form.

Theorem 2.4. *Let $\lambda\in\mathbb{C}$ be such that $|\lambda|=1$. An operator X on L^2 is a solution of the operator equation $\lambda M_{\overline{z}}X = XM_{z^k}$ if and only if it is of the form $X = D_\lambda W_k J M_\phi$ for some $\phi\in L^\infty$.*

As each solution is induced by an element of L^∞ and also depend on the choice of λ, we'll denote the solution X of the form $X = D_\lambda W_k J M_\phi$ by $X_{\phi,\lambda}$. It is clear that for $\phi\in L^\infty$, $\|X_{\phi,\lambda}\| = \|D_\lambda W_k J M_\phi\| \le \|\phi\|_\infty$. We see the following.

Theorem 2.5. *For $\phi\in L^\infty$ and $|\lambda|=1$, $\|X_{\phi,\lambda}\| = \sqrt{\|\xi_\lambda\|_\infty}$, where ξ_λ in L^∞ is given by*

$$\xi_\lambda(z) = \sum_{n\in\mathbb{Z}} \langle \psi, e_n\rangle \overline{\lambda}^n z^n.$$

Proof. Proof follows as

$$X_{\phi,\lambda}X^*_{\phi,\lambda} = D_\lambda W_k J M_\phi (D_\lambda W_k J M_\phi)^* = D_\lambda W_k J M_\phi M_{\overline{\phi}} J^* W_k^* D_{\overline{\lambda}} = D_\lambda M_\psi D_{\overline{\lambda}} = M_{\xi_\lambda},$$

where ψ and ξ_λ are elements of L^∞ given by

$$\psi = W_k(|\phi|^2) \quad \text{and} \quad \xi_\lambda(z) = \sum_{n\in\mathbb{Z}} \langle \psi, e_n\rangle \overline{\lambda}^n z^n.$$

Thus, we complete the proof. □

Without any extra efforts, along the lines of techniques used in [4], it is easy to check the following observation about the set S_λ of all the solutions of the equation $\lambda M_{\overline{z}}X = XM_{z^k}$, $|\lambda|=1$.

Theorem 2.6. *We have the following:*

1. *The mapping $\phi\mapsto X_{\phi,\lambda}$ is one-one from L^∞ into S_λ.*

2. *S_λ is weakly and hence strongly closed.*

3. *S_λ is a norm closed subspace of $\mathfrak{B}(L^2)$, the algebra of all bounded operators on L^2.*

4. *S_λ is not self adjoint.*

Proof. We just give proof for (4). It follows using Theorem 2.3 and the fact that for $\phi = \sum_{n\in\mathbb{Z}} a_n e_n \in L^\infty$, the adjoint of a non-zero $X_{\phi,\lambda}$ is $X^*_{\phi,\lambda} = M_{\overline{\phi}} J^* W_k^* D_{\overline{\lambda}}$ and hence

$$\langle X^*_{\phi,\lambda} e_j, e_i\rangle = \langle e_j, D_\lambda W_k J M_\phi e_i\rangle = \Big\langle e_j, \sum_{m\in\mathbb{Z}} \overline{\lambda}^m a_{-km-i} e_m \Big\rangle = \lambda^j \overline{a}_{-kj-i}$$

for each $i,j\in\mathbb{Z}$. $\square$

In order to see whether S_λ form an algebra or not, we first claim the following about a solution $X_{\phi,\lambda}$, $\phi\in L^\infty$.

Lemma 2.1. *$D_\lambda W_k J X_{\phi,\lambda}$ is a solution if and only if $X_{\phi,\lambda}$ is zero.*

Proof. Let $\phi = \sum_{n\in\mathbb{Z}} a_n e_n \in L^\infty$ and let $D_\lambda W_k J X_{\phi,\lambda}$ be a solution. Then on applying Theorem 2.3, we have

$$\lambda\langle D_\lambda W_k J X_{\phi,\lambda} e_j, e_i\rangle = \langle D_\lambda W_k J X_{\phi,\lambda} e_{j+k}, e_{i-1}\rangle$$

for each $i,j\in\mathbb{Z}$. This gives

$$\Big\langle \sum_{n\in\mathbb{Z}} \overline{\lambda}^n a_{-kn-j} e_n, e_{-ki}\Big\rangle = \Big\langle \sum_{n\in\mathbb{Z}} \overline{\lambda}^n a_{-kn-j-k} e_n, e_{-ki+k}\Big\rangle,$$

which implies that $a_{k(ki)-j} = \lambda^{-k} a_{k^2(i-1)-(j+k)}$ for each $i,j\in\mathbb{Z}$. Hence, if $i=0$ then we get $a_{-j} = \lambda^{-k} a_{-k^2-j-k}$ for each $j\in\mathbb{Z}$. This gives that for integer t, $a_t = \lambda^{-kn} a_{-n(k^2+k)+t} \to 0$ as $n\to 0$. As a matter of fact $\phi=0$ and $X_{\phi,\lambda}=0$. Converse is obvious. $\square$

For $\phi,\psi\in L^\infty$, the product of two solutions $X_{\phi,\lambda}$ and $X_{\psi,\lambda}$ satisfies

$$\begin{aligned} X_{\phi,\lambda} X_{\psi,\lambda} &= D_\lambda W_k J M_\phi D_\lambda W_k J M_\psi = D_\lambda W_k J M_\phi D_\lambda J W_k M_\psi \\ &= D_\lambda W_k J D_\lambda M_{\phi(\lambda\overline{z})} J W_k M_\psi = D_\lambda W_k J D_\lambda J W_k M_{\phi(\lambda\overline{z}^k)} M_\psi \\ &= D_\lambda W_k X_{\xi,\lambda}, \end{aligned}$$

where $\xi(z) = \phi(\lambda\overline{z}^k)\psi(z)$. This observation along with Lemma 2.1 help to conclude that S_λ does not form an algebra.

If we multiply the solution $X_{\phi,\lambda}$ with a Laurent (multiplication) operator A $(=M_\psi)$, where $\phi,\psi\in L^\infty$ then $X_{\phi,\lambda}A = D_\lambda W_k J M_\phi M_\psi = D_\lambda W_k J M_{\phi\psi}$ and $AX_{\phi,\lambda} = M_\psi D_\lambda W_k J M_\phi = D_\lambda M_{\psi(\lambda z)} W_k J M_\phi = D_\lambda J = M_{\psi(\lambda\overline{z})} W_k M_\phi = D_\lambda J W_k M_{\psi(\lambda\overline{z}^k)\phi(z)}$. These observations land at the conclusion that a solution $X_{\phi,\lambda}$ commutes with a Laurent operator M_ψ if and only if $\psi\phi = \psi(\lambda\overline{z}^k)\phi$. It is interesting to attain the following about these products.

Theorem 2.7. *The product of a solution and a Laurent operator is always a solution.*

Proof. For any $\phi\in L^\infty$, a straight forward computation shows that for any Laurent operator M, $\lambda M_{\overline{z}}(X_{\phi,\lambda}A) = (\lambda M_{\overline{z}} X_{\phi,\lambda})A = (X_{\phi,\lambda} M_{z^k})A = (X_{\phi,\lambda}A)M_{z^k}$. Similarly, $\lambda M_{\overline{z}}(X_{\phi,\lambda}A) = (\lambda M_{\overline{z}}A)X_{\phi,\lambda} = A(\lambda M_{\overline{z}} X_{\phi,\lambda}) = A(X_{\phi,\lambda} M_{z^k}) = (X_{\phi,\lambda}A)M_{z^k}$. Hence the result. $\square$

It is shown in [3] that the only compact k^{th}-order slant Hankel operator is the zero operator. When we use this with the fact that the composition operator D_λ is unitary, we get the following.

Theorem 2.8. *The solution $X_{\phi,\lambda}$ is compact if and only if it is zero operator.*

Now we discuss the isometric behavior of the solutions and find a result similar to the result obtained for generalized λ-slant Toeplitz operators in [9]. If $\phi\in L^\infty$ is unimodular (i.e., $|\phi|=1$) then simple computation shows that $X_{\phi,\lambda}$ is co-isometry (i.e., $X^*_{\phi,\lambda}$ is isometry). However, by the same technique as used in [9], we get the following.

Theorem 2.9. *For $\phi\in L^\infty$, $X_{\phi,\lambda}$ is co-isometry if and only if*

$$\Big|\phi\Big(\frac{\theta}{k}\Big)\Big|^2+\Big|\phi\Big(\frac{\theta+2\pi}{k}\Big)\Big|^2+\cdots+\Big|\phi\Big(\frac{\theta+(k-1)\pi}{k}\Big)\Big|^2=k$$

for a.e. $\theta\in[0,2\pi]$.

There is a dearth of hyponormal solutions and we find that the only solution $X_{\phi,\lambda}$ which is hyponormal is the zero operator.

Theorem 2.10. *For $\phi\in L^\infty$ and $|\lambda|=1$, the solution $X_{\phi,\lambda}$ is hyponormal if and only if $X_{\phi,\lambda}=0$.*

Proof. Suppose $\phi=\sum_{n\in\mathbb{Z}}a_ne_n\in L^\infty$ and $X_{\phi,\lambda}$ is hyponormal. Then for all $f\in L^2$,

$$\|Xf\|\geq\|X^*f\|.$$

This, in particular, for $f=e_0$ gives

$$\sum_{n\in\mathbb{Z}}|a_{-kn}|^2\geq\sum_{n\in\mathbb{Z}}|\overline{a}_{-n}|^2,$$

which implies that $a_{-kn-m}=0$ for $m=1,2,\cdots,k-1$ and for all $n\in\mathbb{Z}$. Now on substituting $f=e_1$ in the inequality, we find

$$\sum_{n\in\mathbb{Z}}|a_{-kn-1}|^2\geq\sum_{n\in\mathbb{Z}}|\overline{a}_{-k-n}|^2,$$

which yields that $a_{-kn}=0$ for all $n\in\mathbb{Z}$. Thus $\phi=0$ and so $X_{\phi,\lambda}=0$. This completes the proof as converse is trivial. $\square$

3 Spectral behavior of solutions of the equation $\lambda M_{\overline{z}}X = XM_{z^k}$

In this section, our aim is to investigate information about the spectral behavior of solutions of the equation $\lambda M_{\overline{z}}X = XM_{z^k}$, $|\lambda|=1$. We also prove that the spectrum of the solution contains a closed disc for an invertible symbol in L^∞, which is a well known result in case of k^{th}-order slant Toeplitz operators [1]. For an operator A on a Hilbert space, the symbols $\sigma(A)$ and $\sigma_p(A)$ are used to denote the spectrum and the point spectrum of A respectively. The result here are just stated as can be obtained without any extra efforts by adopting the methods used to obtain the same for k^{th}-order slant Toeplitz operators in [1].

For $\phi \in L^\infty$, we write

$$\varphi = \sum_{n\in\mathbb{Z}} \langle \phi, e_n\rangle \lambda^n e_{-n}.$$

Theorem 3.1. *If ϕ is invertible in L^∞, then $\sigma_p(X_{\varphi,\lambda}) = \sigma_p(X_{\phi(z^k),\lambda})$.*

Theorem 3.2. *For $\phi \in L^\infty$, $\sigma(X_{\varphi,\lambda}) = \sigma(X_{\phi(z^k),\lambda})$.*

Our next result shows the containment of a closed disc in the spectrum of a solution of the operator equation $\lambda M_{\overline{z}}X = XM_{z^k}$.

Theorem 3.3. *For any invertible ϕ in L^∞, $\sigma(X_{\varphi,\lambda})$ contains a closed disc, where X is a solution of the equation $\lambda M_{\overline{z}}X = XM_{z^k}$.*

Proof. Let μ be any non-zero complex number. As ϕ is invertible in L^∞ so is ϕ^{-1}. Now suppose that $(X^*_{\overline{\phi}^{-1}(z^k),\lambda} - \mu I)$ is onto. Then for each $f \in L^2$, we have

$$\begin{aligned}(X^*_{\overline{\phi}^{-1}(z^k),\lambda} - \mu I)f &= M_{\phi^{-1}(z^k)}J^*W_k^*D_{\overline{\lambda}}f - \mu(P_kf \oplus (I-P_k)f)\\ &= \mu W_k^* M_{\phi^{-1}}J^*\Big(\frac{1}{\mu}D_{\overline{\lambda}} - JM_\phi W_k\Big)f \oplus \big(-\mu(I-P_k)f\big),\end{aligned}$$

where P_k is the projection on the closed span of $\{e_{kn} : n\in\mathbb{Z}\}$ in L^2. Now, pick $0 \neq g_0$ in $(I-P_k)(L^2)$. Being $(X^*_{\overline{\phi}^{-1}(z^k),\lambda} - \mu I)$ is onto, we find a $f\in L^2$ such that

$$g_0 = \mu W_k^* M_{\phi^{-1}}J^*(\mu^{-1}D_{\overline{\lambda}} - JM_\phi W_k)f \oplus (-\mu(I-P_k)f).$$

Since $g_0 \in (I-P_k)(L^2)$, we have $\mu W_k^* M_{\phi^{-1}}J^*(\mu^{-1}D_{\overline{\lambda}} - JM_\phi W_k)f = 0$. This, on using the facts that $\mu \neq 0$, W_k is co-isometry (i.e., $W_kW_k^* = I$) and $M_{\phi^{-1}}$ and J are invertible, yields that $(\mu^{-1}D_{\overline{\lambda}} - JM_\phi W_k)f = 0$. This shows that

$$0 = (\mu^{-1}I - D_\lambda JM_\phi W_k)f = (\mu^{-1}I - D_\lambda W_kJM_{\phi(z^k)})f = (\mu^{-1}I - X_{\phi(z^k),\lambda})f.$$

It implies that $\mu^{-1} \in \sigma_p(X_{\phi(z^k),\lambda})$. Now $(X^*_{\overline{\phi}^{-1}(z^k),\lambda} - \mu I)$ is onto (in fact invertible) for each $\mu \in \rho(X^*_{\overline{\phi}^{-1}(z^k),\lambda})$, the resolvent of $X^*_{\overline{\phi}^{-1}(z^k),\lambda}$, so on applying Theorem 3.1, we get that

$$\{\mu^{-1} : \mu \in \rho(X^*_{\overline{\phi}^{-1}(z^k),\lambda})\} \subseteq \sigma_p(X_{\phi(z^k),\lambda}) = \sigma_p(X_{\varphi,\lambda}) \subseteq \sigma_(X_{\varphi,\lambda}),$$

where $\varphi=\sum_{n\in\mathbb{Z}}\langle\phi,e_n\rangle\lambda^n e_{-n}$. As spectrum of any operator is compact it follows that $\sigma(X_{\varphi,\lambda})$ contains a disc of eigenvalues of $X_{\varphi,\lambda}$. $\square$

Remark 3.1. Radius of closed disc contained in $\sigma(X_{\varphi,\lambda})$ is $\frac{1}{r(X_{\overline{\varphi}^{-1},\lambda})}$, where $r(A)$ denotes the spectral radius of the operator A. For,

$$\begin{aligned}&\max\Big\{|\lambda^{-1}|:\lambda\in\rho(X^*_{\overline{\phi}^{-1}(z^k),\lambda})\Big\}\\ =&\frac{1}{\min\{|\lambda|:\lambda\in\rho(X^*_{\overline{\phi}^{-1}(z^k),\lambda})\}}\\ =&(r(X^*_{\overline{\phi}^{-1}(z^k),\lambda}))^{-1}\\ =&(r(X_{\overline{\phi}^{-1}(z^k),\lambda}))^{-1}\\ =&(r(X_{\overline{\varphi}^{-1},\lambda}))^{-1}.\end{aligned}$$

Thus the radius of closed disc contained in $\sigma(X_{\varphi,\lambda})$ is $\frac{1}{r(X_{\overline{\varphi}^{-1},\lambda})}$. Since spectral radius of an operator is the radius of smallest disc containing its spectrum so we get that $\frac{1}{r(X_{\overline{\varphi}^{-1},\lambda})}\le r(X_{\varphi,\lambda})$.

For unimodular $\phi\in L^\infty$, $\|X^n_{\varphi,\lambda}\|^2=\|X^n_{\varphi,\lambda}X^{*n}_{\varphi,\lambda}\|=\|I\|=1$, so that $r(X_{\varphi,\lambda})=1$ (using Gelfand formula for spectral radius). Hence, if $|\phi|=1$, then $\sigma(X_{\varphi,\lambda})=\overline{\mathbb{D}}$, the closed unit disc.

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