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Some L^γ Inequalities for the Polar Derivative of a Polynomial

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Abstract. In this paper, we consider an operator D_α which maps a polynomial $P(z)$ in to $D_\alpha P(z) := np(z) + (\alpha - z)P'(z)$, where $\alpha \in \mathbb{C}$ and obtain some L^γ inequalities for lucanary polynomials having zeros in $|z| \le k \le 1$. Our results yields several generalizations and refinements of many known results and also provide an alternative proof of a result due to Dewan et al. [7], which is independent of Laguerre's theorem.

Key Words: Polar derivative, polynomials, L^γ-inequalities in the complex domain, Laguerre's theorem.

AMS Subject Classifications: 30A10, 30C10, 30C15

1 Introduction

Let P_n be the class of polynomials

$$P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

of degree n. For $P \in P_n$, define

$$\|P\|_\gamma := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^\gamma \right\}^{\frac{1}{\gamma}}, \quad \gamma > 0,$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|, \quad m := \min_{|z|=k} |P(z)| \quad \text{and} \quad m_1 := \min_{|z|=1} |P(z)|.$$

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For fixed μ, $1\le\mu\le n$, let $P_{n,\mu}$, denote the class of polynomials

$$P(z)=a_nz^n+\sum_{v=\mu}^{n}a_{n-v}z^{n-v}$$

of degree n having all zeros in $|z|\le k$, $k\le 1$.

If $P\in P_n$, then according to the following well-known Bernstein's inequality (for reference see [5]), we have

$$\|P'\|_\infty\le n\|P\|_\infty. \tag{1.1}$$

Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin.

For the class of polynomials $P\in P_n$ having all zeros in $|z|\le 1$, Turán [14] proved that

$$\|P'\|_\infty\ge\frac{n}{2}\|P\|_\infty. \tag{1.2}$$

Inequality (1.2) was refined by Aziz and Dawood [1] and they proved under the same hypothesis that

$$\|P'\|_\infty\ge\frac{n}{2}\Big\{\|P\|_\infty+m_1\Big\}. \tag{1.3}$$

Both the inequalities (1.2) and (1.3) are best possible and become equality for polynomials $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$. As an extension of (1.2), it was shown by Malik [12], that if $P\in P_{n,1}$, then

$$\|P'\|_\infty\ge\frac{n}{1+k}\|P\|_\infty, \tag{1.4}$$

where as the corresponding extension of (1.3) and a refinement of (1.4) was given by Govil [9] who under the same hypothesis proved that

$$\|P'\|_\infty\ge\frac{n}{1+k}\Big\{\|P\|_\infty+\frac{m}{k^{n-1}}\Big\}. \tag{1.5}$$

In the literature, there already exist some refinements and generalizations of all the above inequalities, for example see Aziz and Shah [4], Dewan, Mir and Yadav [8], Govil, Rahman and Schemeisser [10], Dewan, Singh and Lal [6], etc.

Aziz and Shah [4] (see also Dewan, Mir and Yadav [8]) generalized inequality (1.5) and proved that, if $P\in P_{n,\mu}$, then

$$\|P'\|_\infty\ge\frac{n}{1+k^\mu}\Big\{\|P\|_\infty+\frac{m}{k^{n-\mu}}\Big\}. \tag{1.6}$$

For $\mu=1$, inequality (1.6) reduces to inequality (1.5).

For a complex number α and for $P\in P_n$, let

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $P(z)$ with respect to α (see [13]). It generalizes the ordinary derivative in the following sense

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\}=P'(z).$$

Aziz and Rather [3] extended (1.4) to the polar derivative of a polynomial and proved that if $P\in P_{n,1}$, then for every complex number α with $|\alpha|\geq k$,

$$\|D_\alpha P\|_\infty\geq n\Big(\frac{|\alpha|-k}{1+k}\Big)\|P\|_\infty. \tag{1.7}$$

Recently, Dewan et al. [7] generalized as well as refined inequality (1.7) by proving that if $P\in P_{n,\mu}$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq s_\mu$,

$$\|D_\alpha P\|_\infty\geq n\Big(\frac{|\alpha|-s_\mu}{1+k^\mu}\Big)\|P\|_\infty, \tag{1.8}$$

where

$$s_\mu=\frac{n|a_n|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1}+\mu|a_{n-\mu}|}. \tag{1.9}$$

In the same paper, Dewan et al. [7] extended (1.6) to the polar derivative and proved that if $P\in P_{n,\mu}$, then for $\alpha\in\mathbb{C}$ with $|\alpha|\geq k^\mu$, we have

$$\|D_\alpha P\|_\infty\geq n\left(\frac{|\alpha|-A_\mu}{1+k^\mu}\right)\|P\|_\infty+\frac{mn}{k^n}\left(\frac{|\alpha|k^\mu+A_\mu}{1+k^\mu}\right), \tag{1.10}$$

where

$$A_\mu=\frac{n\Big(|a_n|-\frac{m}{k^n}\Big)k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n\Big(|a_n|-\frac{m}{k^n}\Big)k^{\mu-1}+\mu|a_{n-\mu}|}. \tag{1.11}$$

If we divide both sides of (1.11) by $|\alpha|$ and let $|\alpha|\to\infty$, we recover (1.6).

The main aim of this paper is to provide an L^γ analogue of (1.10) and to present a proof of it independent of Laguerre's theorem. Firstly, we shall present the following extension of inequality (1.8).

Theorem 1.1. *If $P\in P_{n,\mu}$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\geq s_\mu$ and for every $\gamma>0$, we have*

$$n\Big(|\alpha|-s_\mu\Big)\left\|\frac{P}{D_\alpha P}\right\|_\gamma\leq\|1+k^\mu z\|_\gamma, \tag{1.12}$$

where s_μ is as defined in (1.9).

Remark 1.1. Since for every $\alpha\in\mathcal{C}$, $\left|D_\alpha P(e^{i\theta})\right|\le\|D_\alpha P\|_\infty$, $0\le\theta<2\pi$, the following result easily follows from Theorem 1.1.

Corollary 1.1. If $P\in P_{n,\mu}$, then for every $\alpha\in\mathcal{C}$ with $|\alpha|\ge s_\mu$ and for every $\gamma>0$, we have

$$n\Big(|\alpha|-s_\mu\Big)\|P\|_\gamma\le\|1+k^\mu z\|_\gamma\|D_\alpha P\|_\infty. \tag{1.13}$$

If we let $\gamma\to\infty$ in (1.13) and note that $\|1+k^\mu z\|_\gamma\to(1+k^\mu)$, we get (1.8). Also, if we divide both sides of (1.13) by $|\alpha|$ and then let $|\alpha|\to\infty$, we get a result of Aziz and Rather [3].

By Lemma 2.2, we have

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right|\le k^\mu,$$

which further implies $s_\mu\le k^\mu$. Therefore Theorem 1.1 holds for every $\alpha\in\mathcal{C}$ with $|\alpha|\ge k^\mu$ as well. We immediately get the following useful consequence from Theorem 1.1.

Corollary 1.2. If $P\in P_{n,\mu}$, then for every $\alpha\in\mathcal{C}$ with $|\alpha|\ge k^\mu$ and for every $\gamma>0$, we have

$$n\Big(|\alpha|-k^\mu\Big)\left\|\frac{P}{D_\alpha P}\right\|_\gamma\le\|1+k^\mu z\|_\gamma. \tag{1.14}$$

Next, we shall prove the following more general result which as a special case provides a proof of inequality (1.10) independent of Laguerre's theorem.

Theorem 1.2. *If $P\in P_{n,\mu}$, then for every $\alpha,\beta\in\mathcal{C}$ with $|\alpha|\ge k^\mu$, $|\beta|<1$ and for each $\gamma>0$, we have*

$$n\Big(|\alpha|-A_\mu\Big)\left\|\frac{P-\frac{m\beta z^n}{k^n}}{D_\alpha P-\frac{\alpha\beta mnz^{n-1}}{k^n}}\right\|_\gamma\le\|1+k^\mu z\|_\gamma, \tag{1.15}$$

where A_μ is defined by formula (1.11).

Remark 1.2. Since

$$\left|D_\alpha P(e^{i\theta})-\frac{\alpha\beta mne^{i(n-1)\theta}}{k^n}\right|\le\left\|D_\alpha P-\frac{\alpha\beta mnz^{n-1}}{k^n}\right\|_\infty,\quad 0\le\theta<2\pi,$$

we get from inequality (1.15) that

$$n\Big(|\alpha|-A_\mu\Big)\left\|P-\frac{m\beta z^n}{k^n}\right\|_\gamma\le\|1+k^\mu z\|_\gamma\left\|D_\alpha P-\frac{\alpha\beta mnz^{n-1}}{k^n}\right\|_\infty. \tag{1.16}$$

If we let $\gamma\to\infty$ in (1.16) and note that $\|1+k^\mu z\|_\gamma\to(1+k^\mu)$, we get

$$\left\|D_\alpha P-\frac{\alpha\beta mnz^{n-1}}{k^n}\right\|_\infty\ge n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)\left\|P-\frac{m\beta z^n}{k^n}\right\|_\infty. \tag{1.17}$$

Let z_0 be on $|z|=1$ such that $|P(z_0)|=\max_{|z|=1}|P(z)|$, then from (1.17), we get

$$\begin{aligned}\Big|\Big\{D_\alpha P(z)\Big\}_{z=z_0}-\frac{\alpha\beta mnz_0^{n-1}}{k^n}\Big|&\geq n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)\Big|P(z_0)-\frac{m\beta z_0^n}{k^n}\Big|\\&\geq n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)\Big\{|P(z_0)|-\frac{m|\beta|}{k^n}\Big\}.\end{aligned}\tag{1.18}$$

Since the polynomial $P(z)-\frac{m\beta z^n}{k^n}$ has all zeros in $|z|<k$, $k\leq 1$, where $|\beta|<1$, therefore by the Guass-Lucas theorem, the polynomial $P'(z)-\frac{mn\beta z^{n-1}}{k^n}$ also has all its zeros in $|z|<k$, $k\leq 1$ and hence

$$|P'(z)|\geq\frac{mn|z|^{n-1}}{k^n}\quad\text{for}\quad |z|\geq k.\tag{1.19}$$

Because if (1.19) is not true, then there is a point $z=z_0$ with $|z_0|\geq k$ such that

$$|P'(z_0)|<\frac{mn|z_0|^{n-1}}{k^n}.$$

If we take $\beta=\frac{k^nP'(z_0)}{mnz_0^{n-1}}$, so that $|\beta|<1$, then with this choice of β, we have

$$P'(z_0)-\frac{mn\beta z_0^{n-1}}{k^n}=0,$$

where $|z_0|\geq k$, which contradicts the fact that all the zeros of $P'(z)-\frac{mn\beta z^{n-1}}{k^n}$ lie in $|z|<k$, $k\leq 1$.

Also for $|z|=1$,

$$\begin{aligned}|D_\alpha P(z)|&=|nP(z)+(\alpha-z)P'(z)|\\&\geq|\alpha||P'(z)|-|nP(z)-zP'(z)|\\&=|\alpha||P'(z)|-|Q'(z)|.\end{aligned}$$

Combining this inequality with Lemma 2.3, we get for $|z|=1$ and $|\alpha|\geq k^\mu$,

$$|D_\alpha P(z)|\geq(|\alpha|-k^\mu)|P'(z)|+\frac{mn}{k^{n-\mu}}.\tag{1.20}$$

Inequality (1.20) in conjunction with (1.19) gives for $|z|=1$ and $|\alpha|\geq k^\mu$,

$$|D_\alpha P(z)|\geq\frac{|\alpha|mn}{k^n}.\tag{1.21}$$

If in (1.18), we choose the argument of β such that

$$\Big|\Big\{D_\alpha P(z)\Big\}_{z=z_0}-\frac{\alpha\beta mnz_0^{n-1}}{k^n}\Big|=\Big|\Big\{D_\alpha P(z)\Big\}_{z=z_0}\Big|-\frac{mn|\beta||\alpha||z_0|^{n-1}}{k^n},$$

which easily follows from (1.21), we obtain

$$\Big|\Big\{D_\alpha P(z)\Big\}_{z=z_0}\Big|-\frac{mn|\beta||\alpha||z_0|^{n-1}}{k^n}\geq n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)|P(z_0)|-n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)\frac{m|\beta|}{k^n}. \tag{1.22}$$

Since z_0 lies on $|z|=1$ and $|P(z_0)|=\max_{|z|=1}|P(z)|$, inequality (1.22) is equivalent to

$$\Big|\Big\{D_\alpha P(z)\Big\}_{z=z_0}\Big|\geq n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)\max_{|z|=1}|P(z)|-n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)\frac{m|\beta|}{k^n}+\frac{mn|\beta||\alpha|}{k^n}. \tag{1.23}$$

Now, if in (1.23) we make $|\beta|\to 1$, we get

$$\max_{|z|=1}|D_\alpha P(z)|\geq n\Big(\frac{|\alpha|-A_\mu}{1+k^\mu}\Big)\max_{|z|=1}|P(z)|+\frac{mn}{k^n}\Big(\frac{|\alpha|k^\mu+A_\mu}{1+k^\mu}\Big),$$

which is (1.10) and this proves the required claim.

2 Lemmas

We need the following lemmas to prove the theorems.

Lemma 2.1. *If $P\in P_{n,\mu}$, then on $|z|=1$,*

$$|Q'(z)|\leq k^\mu|P'(z)|, \tag{2.1}$$

where here and throughout this paper $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$.

The above lemma is due to Aziz and Shah [4]. The following lemma is due to Aziz and Rather [2].

Lemma 2.2. *If $P\in P_{n,\mu}$, then on $|z|=1$,*

$$|Q'(z)|\leq s_\mu|P'(z)| \tag{2.2}$$

and

$$\frac{\mu}{n}\Big|\frac{a_{n-\mu}}{a_n}\Big|\leq k^\mu, \tag{2.3}$$

where s_μ is defined by the formula (1.9).

Lemma 2.3. *If $P\in P_{n,\mu}$, then on $|z|=1$,*

$$|Q'(z)|\leq k^\mu|P'(z)|-\frac{nm}{k^{n-\mu}}. \tag{2.4}$$

Lemma 2.4. *If $P\in P_n$ with all its zeros in $|z|\le k, k>0$, then $|Q(z)|\ge \frac{m}{k^n}$ for $|z|\le \frac{1}{k}$ and in particular*

$$|a_n|>\frac{m}{k^n}. \tag{2.5}$$

Lemma 2.5. *If $P\in P_{n,\mu}$, then*

$$A_\mu \le k^\mu, \tag{2.6}$$

where A_μ is defined by the formula (1.1).

Lemma 2.6. *The function*

$$S_\mu(x)=\frac{nxk^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{nxk^{\mu-1}+\mu|a_{n-\mu}|}, \tag{2.7}$$

where $k\le 1$ and $\mu\ge 1$, is a non-increasing function of x.

The above Lemmas 2.3-2.6 are due to Dewan et al. [7].

3 Proof of theorems

Proof of Theorem 1.1. If

$$Q(z)=z^n\overline{P\Big(\frac{1}{\bar z}\Big)},$$

then

$$P(z)=z^n\overline{Q\Big(\frac{1}{\bar z}\Big)}$$

and it can be easily verified that for $|z|=1$,

$$|Q'(z)|=|nP(z)-zP'(z)| \tag{3.1}$$

and

$$|P'(z)|=|nQ(z)-zQ'(z)|. \tag{3.2}$$

As $P(z)$ has all its zeros in $|z|\le k$, therefore, by using Lemma 2.1 and (3.2), we have for $|z|=1$,

$$|Q'(z)|\le k^\mu|nQ(z)-zQ'(z)|. \tag{3.3}$$

Now for every complex number α with $|\alpha|\ge s_\mu$, we have

$$|D_\alpha P(z)|=|nP(z)+(\alpha-z)P'(z)|\ge|\alpha||P'(z)|-|nP(z)-zP'(z)|,$$

which on using (3.1) and Lemma 2.2 gives for $|z|=1$,

$$|D_\alpha P(z)|\geq|\alpha||P'(z)|-|Q'(z)|\geq(|\alpha|-s_\mu)|P'(z)|. \tag{3.4}$$

Again since $P(z)$ has all its zeros in $|z|\leq k$, $k\leq 1$, it follows by the Guass-Lucas theorem that all the zeros of $P'(z)$ also lie in $|z|\leq k$, $k\leq 1$. This implies that the polynomial

$$z^{n-1}\overline{P'\Big(\frac{1}{\overline{z}}\Big)}=nQ(z)-zQ'(z)$$

has all its zeros in $|z|\geq\frac{1}{k}\geq 1$. Therefore, it follows from (3.3) that the function

$$W(z)=\frac{zQ'(z)}{k^\mu(nQ(z)-zQ'(z))}$$

is analytic for $|z|\leq 1$ and $|W(z)|\leq 1$ for $|z|\leq 1$. Furthermore, $W(0)=0$ and so the function $1+k^\mu W(z)$ is subordinate to the function $1+k^\mu z$ for $|z|\leq 1$. Hence by a well-known property of sub-ordination [11], we have for each $\gamma>0$,

$$\int_0^{2\pi}\Big|1+k^\mu W(e^{i\theta})\Big|^\gamma d\theta\leq\int_0^{2\pi}\Big|1+k^\mu e^{i\theta}\Big|^\gamma d\theta. \tag{3.5}$$

Now

$$1+k^\mu W(z)=\frac{nQ(z)}{nQ(z)-zQ'(z)},$$

which gives with the help of (3.2) that for $|z|=1$,

$$n|Q(z)|=|1+k^\mu W(z)||P'(z)|. \tag{3.6}$$

Since $|P(z)|=|Q(z)|$ for $|z|=1$, therefore from (3.6), we get

$$|P'(z)|=\frac{n|P(z)|}{|1+k^\mu W(z)|}\quad\text{for}\ \ |z|=1. \tag{3.7}$$

From (3.4) and (3.7), we deduce that for each $\gamma>0$ and $0\leq\theta<2\pi$,

$$n^\gamma(|\alpha|-s_\mu)^\gamma\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^\gamma d\theta\leq\int_0^{2\pi}\big|1+k^\mu W(e^{i\theta})\big|^\gamma d\theta.$$

The above inequality in conjunction with (3.5) gives

$$n^\gamma(|\alpha|-s_\mu)^\gamma\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^\gamma\leq\big|1+k^\mu e^{i\theta}\big|^\gamma.$$

Equivalently, we write

$$n\Big(|\alpha|-s_\mu\Big)\left\|\frac{P}{D_\alpha P}\right\|_\gamma \le \|1+k^\mu z\|_\gamma,$$

which proves Theorem 1.1 completely. □

Proof of Theorem 1.2. By hypothesis, the polynomial

$$P(z)=a_n z^n+\sum_{v=\mu}^{n} a_{n-v} z^{n-v}, \quad 1\le\mu\le n,$$

has all its zeros in $|z|\le k$, $k\le 1$. If $P(z)$ has a zero on $|z|=k$, then $m=0$ and the result follows from Theorem 1.1 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z|<k$, $k\le 1$, so that $m>0$.

Now $m\le|P(z)|$ for $|z|=k$, therefore, if β is any complex number with $|\beta|<1$, then

$$\Big|\frac{m\beta z^n}{k^n}\Big|<|P(z)| \quad \text{for } |z|=k.$$

Since all the zeros of $P(z)$ lie in $|z|<k$, it follows by Rouche's theorem that all the zeros of $P(z)-\frac{m\beta z^n}{k^n}$ also lie in $|z|<k$, $k\le 1$. Hence we can apply Theorem 1.1 to $P(z)-\frac{m\beta z^n}{k^n}$ and obtain for $|\alpha|\ge k^\mu\ge s'_\mu$ and $\gamma>0$,

$$n\Big(|\alpha|-s'_\mu\Big)\left\|\frac{P-\frac{m\beta z^n}{k^n}}{D_\alpha\Big(P-\frac{m\beta z^n}{k^n}\Big)}\right\|_\gamma \le \|1+k^\mu z\|_\gamma, \tag{3.8}$$

where

$$s'_\mu=\frac{n\Big|a_n-\frac{m\beta}{k^n}\Big|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n\Big|a_n-\frac{m\beta}{k^n}\Big|k^{\mu-1}+\mu|a_{n-\mu}|}. \tag{3.9}$$

Since for every β with $|\beta|<1$, we have

$$\Big|a_n-\frac{m\beta}{k^n}\Big|\ge|a_n|-\frac{m|\beta|}{k^n}\ge|a_n|-\frac{m}{k^n} \tag{3.10}$$

and $|a_n|>\frac{m}{k^n}$ by Lemma 2.4. Now combining (3.9), (3.10) and Lemma 2.6, we have for every β with $|\beta|<1$,

$$\begin{aligned} s'_\mu&=\frac{n\Big|a_n-\frac{m\beta}{k^n}\Big|k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n\Big|a_n-\frac{m\beta}{k^n}\Big|k^{\mu-1}+\mu|a_{n-\mu}|}\\ &\le\frac{n\Big(|a_n|-\frac{m}{k^n}\Big)k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n\Big(|a_n|-\frac{m}{k^n}\Big)k^{\mu-1}+\mu|a_{n-\mu}|}\\ &=A_\mu. \end{aligned} \tag{3.11}$$

Further by Lemma 2.5, we have $A_\mu \le k^\mu$, it follows from (3.8) and (3.11)) that for every α with $|\alpha| \ge k^\mu$ and $\gamma > 0$,

$$n\Big(|\alpha| - A_\mu\Big) \left\| \frac{P - \frac{m\beta z^n}{k^n}}{D_\alpha P - \frac{mn\alpha\beta z^{n-1}}{k^n}} \right\|_\gamma \le \|1 + k^\mu z\|_\gamma, \tag{3.12}$$

which is inequality (1.15) and this completes the proof of Theorem 1.2. □

References

[1] A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, J. Approx. Theory, 54 (1988), 306–313.
[2] A. Aziz and N. A. Rather, Some Zygmund type inequalities for polynomials, J. Math. Anal. Appl., 289 (2004), 14–29.
[3] A. Aziz and N. A. Rather, A refinement of a theorem of Paul Turan concerning polynomials, Math. Ineq. Appl., 1 (1998), 231–238.
[4] A. Aziz and W. M. Shah, An integral mean estimate for polynomials, Indian J. Pure Appl. Math., 28 (1997), 1413–1419.
[5] S. Bernstein, Lecons Sur Les Proprietes Extremals et la Meilleure Approximation des Fonctions Analytiques d'une Fonctions Reelle, Gauthier-villars, Paris, 1926.
[6] K. K. Dewan, N. Singh and R. Lal, Inequalities for the polar derivative of a polynomial, Int. J. Pure. Appl. Math., 33 (2006), 109–117.
[7] K. K. Dewan, N. Singh and A. Mir, Extensions of some polynomial inequlities to the polar derivative, J. Math. Anal. Appl., 352 (2009), 807–815.
[8] K. K. Dewan, A. Mir and R. S. Yadav, Integral mean estimates for polynomials whose zeros are with in a circle, IJMMS, 4 (2001), 231–235.
[9] N. K. Govil, Some inequalities for derivative of polynomials, J. Approx. Theory, 66 (1991), 29–35.
[10] N. K. Govil, Q. I. Rahman and G. Schemeisser, On the derivative of a polynomial, Illinois J. Math., 23 (1979), 319–330.
[11] E. Hille, Ananlytic Function Theory, Vol II, Ginn and Company, New York, Toranto, 1962.
[12] M. A. Malik, On the derivative of a polynomial, J. London Math. Soc., 1 (1969), 57–60.
[13] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press Inc., New York, 2002.
[14] P. Turán, Über die Ableitung von Polynomen, Compositio Math., 7 (1939), 89–95.