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# On the Relation of Shadowing and Expansivity in Nonautonomous Discrete Systems

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**Abstract.** In this paper we study shadowing property for sequences of mappings on compact metric spaces, i.e., nonautonomous discrete dynamical systems. We investigate the relations of various expansivity properties with shadowing and $h$-shadowing property.

**Key Words**: Shadowing, $h$-shadowing, locally expanding, uniformly weak expanding, locally weak expanding.

**AMS Subject Classifications**: 54H20, 37B55, 39A05

## 1 Introduction

Let $(X,d)$ be a compact metric space, and $f$ be a continuous map on $X$. We consider the associated autonomous difference equation of the following form:

$$x_{i+1}=f(x_i). \tag{1.1}$$

A finite or infinite sequence $\{x_0,x_1,\cdots\}$ of points in $X$ is called a $\delta$-pseudo-orbit ($\delta>0$) of (1.1) if $d(f(x_{i-1}),x_i)<\delta$ for all $i\geq 1$. We say that Eq. (1.1), (or $f$) has usual shadowing property if for every $\varepsilon>0$, there exists $\delta>0$ such that for every $\delta$-pseudo-orbit $\{x_0,x_1,\cdots\}$, there exists $y\in X$ with $d(f^i(y),x_i)<\varepsilon$ for all $i\geq 0$. The notion of pseudo-orbits appeared in several branches of dynamical systems theory, and various types of the shadowing property were presented and investigated extensively, see [5,6,11,12].

In this paper we study shadowing property of nonautonomous discrete systems. We consider the compact metric space $X$ and a sequence $f_{1,\infty}=\{f_i\}_{i=1}^{\infty}$ in which each $f_i$ :

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$X \to X$ is continuous. We call the pair $(X, f_{1,\infty})$ a nonautonomous discrete system (on $X$). For further simplicity we use only $f_{1,\infty}$ in the sequel. The associated nonautonomous difference equation has the following form:

$$x_{i+1} = f_i(x_i). \tag{1.2}$$

For every $n \geq i \geq 1$, we write $f_i^n = f_n \circ f_{n-1} \circ \cdots \circ f_i$.

Orbit of a nonautonomous system $f_{1,\infty}$ in a point $x$ is the following sequence:

$$O(x) = \{x, f_1(x), f_2 \circ f_1(x), \cdots, f_n \circ \cdots \circ f_1(x), \cdots\}.$$

On the other hand a pseudo-orbit of the system is as follows:

**Definition 1.1.** A finite or infinite sequence $\{x_0, x_1, \cdots\}$ of points in $X$ is called a $\delta$-pseudo-orbit ($\delta > 0$) of (1.2), if $d(f_i(x_{i-1}), x_i) < \delta$ for all $i \geq 1$.

In the nonautonomous case the standard definition of shadowing has the following form, see [12]:

**Definition 1.2.** We say that $f_{1,\infty}$ has shadowing property if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\delta$-pseudo-orbit $\{x_0, x_1, \cdots\}$, there exists $y \in X$ with $d(y, x_0) < \varepsilon$ and $d(f_1^i(y), x_i) < \varepsilon$, for all $i \geq 1$.

In this paper we investigate the relation of various expansivity such as positively expansive, locally expanding, weakly locally expanding, $\cdots$, with shadowing and $h$-shadowing property.

## 2 Shadowing and expansivity

First we prove the following simple lemma.

**Lemma 2.1.** *The sequence $f_{1,\infty}$ has shadowing property if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every finite $\delta$-pseudo-orbit is $\varepsilon$-shadowed.*

*Proof.* Let $\varepsilon > 0$ and $\delta > 0$ be such that every finite $\delta$-pseudo-orbit, $\frac{\varepsilon}{2}$-shadowed. Let $\{x_i\}_{i=1}^{\infty}$ be a $\delta$-pseudo-orbit. For every $n \geq 1$, $\{x_0, x_1, \cdots, x_n\}$, $\frac{\varepsilon}{2}$-shadowed by $y_n \epsilon X$ and there is a subsequence $\{y_{n_k}\}_{k\geq 0}$ and a point $y \epsilon X$ such that $y_{n_k} \to y$ as $k \to \infty$. Now for each $i \geq 1$, there is a $n_k > i$ such that $d(f_1^i(y_{n_k}), f_1^i(y)) < \frac{\varepsilon}{2}$. Therefore

$$d(f_1^i(y), x_i) \leq d(f_1^i(y), f_1^i(y_{n_k})) + d(f_1^i(y_{n_k}), x_i) < \varepsilon$$

and hence $f_{1,\infty}$ has the shadowing property. □

There are several variants of shadowing property, we define a stronger form which is called $h$-shadowing, see [2, 9].

**Definition 2.1.** The sequence $f_{1,\infty}$ has $h$-shadowing property if for every $\varepsilon>0$ there exists $\delta>0$ such that for every $\delta$-pseudo-orbit $\{x_0,x_1,\cdots,x_n\}\subseteq X$ there is $y\in X$ with $d(y,x_0)<\varepsilon$ and,

$$d(f_1^i(y),x_i)<\varepsilon \text{ for all } 1\leq i<n \quad \text{and} \quad f_1^n(y)=x_n.$$

In the case of an autonomous difference equation various notions of expansivity such as positively expansive, locally expanding, $\cdots$, have been introduced and their properties studied extensively, see [1,8,14]. We consider a nonautonomous form of expansivity and a modified form of equicontinuouity.

**Definition 2.2.** We say that the sequence $f_{1,\infty}$ is positively expansive, with expansive constant $e>0$, if $x\neq y$, then for every $N\in\mathbb{N}$ there is $n\geq N$ such that $d(f_N^n(x),f_N^n(y))>e$.

**Theorem 2.1.** *Suppose that the sequence $f_{1,\infty}$ is positively expansive and has shadowing property then it has h-shadowing property.*

*Proof.* Let $e>0$ be the expansive constant, $\varepsilon<e$ and $\delta>0$ is provided by the shadowing of $f_{1,\infty}$. Suppose that $\{x_0,x_1,\cdots,x_m\}$ is a $\delta$-pseudo-orbit. The following sequence:

$$\{x_0,x_1,\cdots,x_m,f_{m+1}(x_m),f_{m+1}^{m+2}(x_m),f_{m+1}^{m+3}(x_m),\cdots\}$$

is an infinite $\delta$-pseudo-orbit. Now since $f_{1,\infty}$ has the shadowing property, there is $y\in X$ such that for each $j\geq 1$, $d(f_1^{m+j}(y),f_{m+1}^{m+j}(x_m)<\varepsilon$, which yields $f_1^m(y)=x_m$. Hence the sequence $f_{1,\infty}$ has $h$-shadowing property. □

**Definition 2.3.** The sequence $f_{1,\infty}$ called inverse equicontinuous if for every $x\in X$ and for every $\varepsilon>0$ there exists $\delta(x)>0$ such that:

$$B_{\delta(x)}(f_i(x))\subseteq f_i(B_\varepsilon(x)) \quad \text{for all } i,$$

in which $B_\varepsilon(x)$ is the open ball with radius $\varepsilon$ and center $x$.

**Proposition 2.1.** Suppose that $f_i:X\to X$ is one to one and surjective, for all $i$. Then the sequence $f_{1,\infty}$ is inverse equicontinuous if and only if the sequence $\{f_i^{-1}\}_{i=1}^{\infty}$ is equicontinuous.

*Proof.* The proof is trivial. □

**Definition 2.4.** We say that $f_{1,\infty}$ is weakly expanding small distances if there exists $\gamma>0$ such that for every $x,y\in X$ and every $i$,

$$d(x,y)<\gamma\Longrightarrow d(f_i(x),f_i(y))>d(x,y).$$

**Definition 2.5.** We say that $f_{1,\infty}$ is locally expanding if there exists $\lambda>1$ such that for every $x\in X$, $i\geq 1$ and $\varepsilon>0$, $B_{\lambda\varepsilon}(f_i(x))\subseteq f_i(B_\varepsilon(x))$.

**Definition 2.6.** We say that $f_{1,\infty}$ is weakly locally expanding if there exists $\gamma > 0$ such that for every $x \in X$, $i \geq 1$ and $\varepsilon < \gamma$, $B_\varepsilon(f_i(x)) \subseteq f_i(B_\varepsilon(x))$.

**Lemma 2.2.** *Suppose that the sequence $f_{1,\infty}$ is inverse equicontinuous and weakly expanding small distance then it has weakly locally expanding property.*

*Proof.* Let $\gamma > 0$ be a constant as in the definition of weakly expanding small distances. Since $f_{1,\infty}$ is inverse equicontinuous, for each $x \in X$ there exists $\lambda(x) > 0$ such that:

$$B_{\lambda(x)}(f_i(x)) \subseteq f_i(B_{\frac{\gamma}{2}}(x)) \quad \text{for all } i \geq 1. \tag{2.1}$$

We denote $B_{\frac{\gamma}{2}}(x) \cap f_i^{-1}(B_{\frac{\lambda(x)}{2}}(f_i(x))$ by $U_i$, then for $x \in U_i$ there exists $\eta = \eta(x) < \frac{\lambda(x)}{2}$ such that $B_\eta(x) \subseteq U_i$. Now we have $f_i(U_i) = B_{\frac{\lambda(x)}{2}}(f_i(x))$, in fact if $y \in B_{\frac{\lambda(x)}{2}}(f_i(x))$ then using (2.1) we have $y = f_i(t)$ with $t \in B_{\frac{\gamma}{2}}(x)$ and $f_i(t) = y \in B_{\frac{\lambda(x)}{2}}(f_i(x))$ implies $t \in f_i^{-1}(B_{\frac{\lambda(x)}{2}}(f_i(x))$, hence $B_{\frac{\lambda(x)}{2}}(f_i(x)) \subseteq f_i(U_i)$. The other side is trivial.

Let $z \in B_\eta(x)$, $\rho < \eta$ such that $B_\rho(z) \subseteq B_\eta(x)$ then $f_i(z) \in f_i(U_i) = B_{\frac{\lambda(x)}{2}}(f_i(x))$, thus

$$B_\rho(f_i(z)) \subseteq B_{\lambda(x)}(f_i(x)) \subseteq f_i(B_{\frac{\gamma}{2}}(x)). \tag{2.2}$$

We denote $B_{\frac{\gamma}{2}}(x) \cap f_i^{-1}(B_\rho(f_i(z))$ by $V_i$, as in the proof of the similar result for $U_i$, we see that $f_i(V_i) = B_\rho(f_i(z))$. We claim $V_i \subseteq B_\rho(z)$. Suppose that $V_i \nsubseteq B_\rho(z)$, then there is $y \in V_i - B_\rho(z)$. $z \in B_\eta(x) \subseteq U_i \subseteq B_{\frac{\gamma}{2}}(x)$ so $\rho < d(y,z) \leq d(y,x) + d(x,z) < \gamma$. Now from this relation and the fact that $f_{1,\infty}$ is weakly expanding small distance we have $d(f_i(y), f_i(z)) \geq d(y,z) > \rho$ which is in contradiction with $y \in V_i$. Therefore we have $V_i \subseteq B_\rho(z)$ which yields $B_\rho(f_i(z)) = f_i(V_i) \subseteq f_i(B_\rho(z))$. Now $X$ is compact, and there is $x_1, x_2, \cdots, x_n$ in $X$ such that $X \subseteq \cup_{i=1}^n B_{\frac{\eta(x_i)}{2}}(x_i)$. Define $r = \min \frac{\eta(x_i)}{2}$ and consider $x \in X$ and $\rho < r$, so there is $1 \leq i \leq n$ such that $x \in B_{\frac{\eta(x_i)}{2}}(x_i)$ which implies $B_\rho(x) \subseteq B_{\eta(x_i)}(x_i)$ and therefore $B_\rho(f_i(x)) \subseteq f_i(B_\rho(x))$. □

**Definition 2.7.** We say that $f_{1,\infty}$ is uniformly expanding if there exist $\lambda > 1$ and $\gamma > 0$ such that for every $x, y \in X$ and $i \geq 1$:

$$d(f_i(x), f_i(y)) < \gamma \Rightarrow d(f_i(x), f_i(y)) > \lambda d(x,y).$$

**Definition 2.8.** We say that $f_{1,\infty}$ is weakly uniformly expanding if there exists $\gamma > 0$ such that for every $x, y \in X$ and $i \geq 1$:

$$d(f_i(x), f_i(y)) < \gamma \Rightarrow d(f_i(x), f_i(y)) > d(x,y).$$

**Proposition 2.2.** If $f_{1,\infty}$ is weakly uniformly expanding and for all $i \geq 1$, $f_i$ is surjective, then $f_{1,\infty}$ is weakly locally expanding.

*Proof.* Let $\gamma > 0$ be as in the weakly uniformly expanding definition. It is enough to prove that for each $\epsilon < \gamma$, $B_\varepsilon(f_i(x)) \subseteq f_i(B_\varepsilon(x))$. If $z \in B_\varepsilon(f_i(x))$ then there is $y \in X$ such that $f_i(y) = z$. Since $d(f_i(x), f_i(y)) < \epsilon$, we obtain $d(x,y) < d(f_i(x), f_i(y)) < \epsilon$. So $z \in f_i(B_\varepsilon(x))$. □

Now we investigate the relation of $h$-shadowing and the expansivity notions mentioned above.

**Theorem 2.2.** *Suppose that there is a continuous map $f$ such that $f_i \to f$ pointwise. If the sequence $f_{1,\infty}$ is inverse equicontinuous and weakly expanding small distances, and $f$ is weakly expanding small distances then it has h-shadowing property.*

*Proof.* There exists $\gamma > 0$ such that $d(x,y) < \gamma$ implies that

$$d(x,y) < d(f(x),f(y)) \quad \text{and} \quad d(x,y) < d(f_i(x),f_i(y)) \quad \text{for all} \ \ i \geq 1.$$

By the above lemma there is $r > 0$ such that for every $\rho < r$, and for every $i \geq 1$, we have $B_\rho(f_i(x)) \subseteq f_i(B_\rho(x))$. Let $\varepsilon > 0$, we set $0 < \varepsilon' < \min\{\gamma, r, \varepsilon\}$ and define:

$$\eta(\varepsilon') := \sup\{d(x,y) : d(f_i(x),f_i(y)) < \varepsilon',\ i \geq 1\}.$$

Hence $\eta(\varepsilon') \leq \varepsilon'$. We claim that $\eta(\varepsilon') < \varepsilon'$. Indeed, if $\eta(\varepsilon') = \varepsilon'$, there exist sequences $\{d(x_i,y_i)\}_{i=1}^{\infty}$ and $\{k(i)\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that $d(f_{k(i)}(x_i), f_{k(i)}(y_i)) < \varepsilon'$ and

$$\lim_{i \longrightarrow \infty} d(x_i,y_i) = \eta(\varepsilon') = \varepsilon'.$$

Since $X$ is compact, there is a subsequence $\{n_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that $x_{n_i} \to x_0$ and $y_{n_i} \to y_0$. Thus

$$\begin{aligned}\varepsilon' = \eta(\varepsilon') &= \lim_{i \longrightarrow \infty} d(x_{n_i}, y_{n_i}) = d(x_0,y_0) < d(f(x_0),f(y_0)) \\ &= \lim_{i \longrightarrow \infty} d(f_{k(n_i)}(x_{n_i}), f_{k(n_i)}(y_{n_i})) \leq \varepsilon',\end{aligned}$$

which is impossible. Now we consider $0 < \delta < \min\{r, \gamma, \varepsilon' - \eta(\varepsilon')\}$. Let $\{x_0, x_1, \cdots, x_n\}$ be a $\delta$-pseudo-orbit for $f_{1,\infty}$ then $d(f_n(x_{n-1}),x_n) < \delta$, which implies that there is $y_{n-1} \in B_\delta(x_{n-1})$ such that $f_n(y_{n-1}) = x_n$. Since $d(f_n(x_{n-1}),x_n) < \delta \leq \varepsilon'$, we have:

$$d(x_{n-1},y_{n-1}) < \eta(\varepsilon') \leq \varepsilon.$$

And $d(f_{n-1}(x_{n-2}),y_{n-1}) \leq d(f_{n-1}(x_{n-2}),x_{n-1}) + d(x_{n-1},y_{n-1}) < \delta + \eta(\varepsilon') < \varepsilon' < r$. Therefore there is $y_{n-2} \in B_{\varepsilon'}(x_{n-2}) \subseteq B_\gamma(x_{n-2})$ such that $f_{n-1}(y_{n-2}) = y_{n-1}$. Hence

$$d(x_{n-2},y_{n-2}) < d(f_{n-1}(x_{n-2}),y_{n-1}) < \delta + \eta(\varepsilon') < \varepsilon' \leq \varepsilon.$$

Repeating this argument, we can find points $y_{n-1}, y_{n-2}, \cdots, y_0$ in $X$ such that for all $0 \leq i \leq n-1$, $f_{i+1}(y_i) = y_{i+1}$ and $d(y_i,x_i) < \varepsilon$. Further more $f_1^n(y_0) = x_n$, hence $f_{1,\infty}$ has $h$-shadowing property. ☐

As a consequence, in the case of a single map we have the following result.

**Proposition 2.3.** Suppose that $f: X \to X$ is a continuous and an open map. If $f$ is weakly expanding small distances then $f$ has $h$-shadowing property.

**Theorem 2.3.** *The following conditions hold:*
*(1) If the sequence $f_{1,\infty}$ is locally expanding, then it has h-shadowing property.*
*(2) If the sequence $f_{1,\infty}$ is uniformly expanding, and for all $i\geq 1$, $f_i$ is surjective, then $f_{1,\infty}$ has h-shadowing property.*

*Proof.* Suppose that $f_{1,\infty}$ is locally expanding, there exist $\lambda>1$ and $\gamma>0$ such that for every $i\geq 1$ and $\varepsilon<\gamma$, we have $B_{\lambda\varepsilon}(f_i(x))\subseteq f_i(B_\varepsilon(x))$. For a fixed $0<\varepsilon<\gamma$, we set $\delta=(\lambda-1)\varepsilon$, therefore for every $x\in X$ and $i\geq 1$

$$B_{\varepsilon+\delta}(f_i(x))\subseteq B_{\varepsilon\lambda}(f_i(x))\subseteq f_i(B_\varepsilon(x)). \tag{2.3}$$

Let $\{x_0,x_1,\cdots,x_m\}\subseteq X$ be a $\delta$-pseudo-orbit for $f_{1,\infty}$. Then $d(f_m(x_{m-1}),x_m)<\delta$ implies $x_m\in B_{\varepsilon+\delta}(f_m(x_{m-1}))$, hence there is a point $y_{m-1}\in B_\varepsilon(x_{m-1})$ such that $f_m(y_{m-1})=x_m$ and so we have:

$$d(f_{m-1}(x_{m-2}),y_{m-1})\leq d(f_{m-1}(x_{m-2}),x_{m-1})+d(x_{m-1},y_{m-1})<\delta+\varepsilon.$$

In other word, $y_{m-1}\in B_{\varepsilon+\delta}(f_{m-1}(x_{m-2}))$ so there exists $y_{m-2}\in B_\varepsilon(x_{m-2})$ such that

$$f_{m-1}(y_{m-2})=y_{m-1}.$$

Repeating this argument, we can find $y_{m-2},y_{m-3},\cdots,y_0$ in $X$ such that for all $0\leq i\leq m-1$,

$$f_{i+1}(y_i)=y_{i+1} \quad \text{and} \quad d(y_i,x_i)<\varepsilon,$$

which proves the $h$-shadowing property of $f_{1,\infty}$.

For the proof of (2), it is easy to prove that if for all $i\geq 1$, $f_i$ are surjective maps and $f_{1,\infty}$ is uniformly expanding then $f_{1,\infty}$ is locally expanding, which proves the $h$-shadowing property. □

**Theorem 2.4.** *Suppose there is a continuous map $f$ such that $f_i\to f$ pointwise. If both $f_{1,\infty}$ and $f$ are weakly uniformly expanding, and for all $i\geq 1$, $f_i$ is surjective then $f_{1,\infty}$ has h-shadowing property.*

*Proof.* There is $\gamma>0$ such that

$$d(f_i(x),f_i(y))<\gamma\Rightarrow d(x,y)<d(f_i(x),f_i(y)) \quad \text{for all} \;\; i\geq 1$$

and

$$d(f(x),f(y))<\gamma\Rightarrow d(x,y)<d(f(x),f(y)).$$

For $\varepsilon>0$, let $0<\varepsilon'<\min\{\gamma,\varepsilon\}$, we define:

$$\eta(\varepsilon'):=\sup\{d(x,y):d(f_n(x),f_n(y))<\varepsilon',\, n\geq 1\}.$$

Hence $\eta(\varepsilon')\leq\varepsilon'$. We claim that $\eta(\varepsilon')<\varepsilon'$, indeed if $\eta(\varepsilon')=\varepsilon'$ then there exist sequences $\{d(x_i,y_i)\}_{i=1}^{\infty}$, and $\{k(i)\}_{i=1}^{\infty}\subseteq\mathbb{N}$ such that $d(f_{k(i)}(x_i),f_{k(i)}(y_i))<\varepsilon'$ and

$$\lim_{i\longrightarrow\infty} d(x_i,y_i)=\eta(\varepsilon')=\varepsilon'.$$

Since $X$ is compact there is a subsequence $\{n_i\}_{i=1}^{\infty}\subseteq\mathbb{N}$ such that $x_{n_i}\to x_0$ and $y_{n_i}\to y_0$. If $\{k(i)\}_{i=1}^{\infty}$ is infinite, then

$$\begin{aligned}\varepsilon' =&\eta(\varepsilon)=\lim_{i\to\infty}d(x_{n_i},y_{n_i})=d(x_0,y_0)<d(f(x_0),f(y_0))\\ &=\lim_{i\to\infty}d(f_{k(n_i)}(x_{n_i}),f_{k(n_i)}(y_{n_i}))\leq\varepsilon',\end{aligned}$$

which is impossible. If $\{k(i)\}_{i=1}^{\infty}$ is finite, then there is a subsequence $\{s_i\}_{i=1}^{\infty}\subseteq\{n_i\}_{i=1}^{\infty}$ such that

$$d(f_n(x_0),f_n(y_0))=\lim_{i\to\infty}d(f_{k(s_i)}(x_{s_i}),f_{k(s_i)}(y_{s_i}))\leq\varepsilon'<\gamma\quad\text{for some } n\geq 1,$$

which yields $d(x_0,y_0)<d(f_n(x_0),f_n(y_0))\leq\varepsilon'$. But we have

$$\varepsilon'=\eta(\varepsilon')=\lim_{i\to\infty}d(x_{n_i},y_{n_i})=d(x_0,y_0),$$

which is impossible.

Now let $0<\delta\leq\varepsilon'-\eta(\varepsilon')$ and $\{x_0,x_1,\cdots,x_n\}$ be a $\delta$-pseudo-orbit for $f_{1,\infty}$. Since $f_n$ is surjective, there is $y_{n-1}\in X$ such that $f_n(y_{n-1})=x_n$, therefore we have $d(f_n(x_{n-1}),x_n)<\delta\leq\varepsilon'$, and

$$d(x_{n-1},y_{n-1})\leq\eta(\varepsilon')<\varepsilon'\leq\varepsilon,$$

so it implies

$$d(f_{n-1}(x_{n-2}),y_{n-1})\leq d(f_{n-1}(x_{n-2}),x_{n-1})+d(x_{n-1},y_{n-1})<\delta+\eta(\varepsilon')\leq\varepsilon'.$$

Now since $f_{n-1}$ is surjective, there is $y_{n-2}\in X$ such that $f_{n-1}(y_{n-2})=y_{n-1}$ and

$$d(x_{n-2},y_{n-2})\leq\eta(\varepsilon')<\varepsilon'\leq\varepsilon.$$

Repeating this argument, we can find $y_{n-1},y_{n-2},\cdots,y_0$ in X such that for $0\leq i\leq n-1$, we have $f_{i+1}(y_i)=y_{i+1}$ and $d(y_i,x_i)<\varepsilon'<\varepsilon$. □

**Example 2.1.** Consider the finite set $A$ of symbols and define $X=A^{\mathbb{N}}$, the set of all infinite sequences $(a_1,a_2,\cdots)$ with $a_i\in A$. We consider metric $d$ on $X$ as follows, for $x=(x_1,x_2,\cdots),y=(y_1,y_2,\cdots)\in X$, let $d(x,y)=\frac{1}{2^k}$ where $k$ is the smallest positive integer for which $x_k\neq y_k$. The metric space $(X,d)$ is compact. We consider the sequence of shift maps defined on $X$ as follows:

$$\sigma_i((x_1,x_2,\cdots))=(x_{i+1},x_{i+2},\cdots)$$

this mappings are continuous and

$$d(x,y) = \frac{1}{2^i} d(\sigma_i(x),\sigma_i(y)),$$

therefore the sequence $\{\sigma_i\}$ is uniformly expanding. We prove directly that the sequence $\{\sigma_i\}$ has shadowing property. For $\varepsilon > 0$, let $\delta \le \epsilon$ and $\frac{1}{2^n} < \delta \le \frac{1}{2^{n-1}}$. Suppose that $\{x_i\}_{i=0}^{\infty}$ is a $\delta$-pseudo orbit. If $x_0 = (a_1,a_2,\cdots)$, the relation $d(\sigma_1(x_0),x_1) < \delta$ implies that there exists $n_1 \ge n+1$ such that $b_{n_1} \ne a_{n_1}$ and $x_1 = (a_2,\cdots,a_{n_1-1},b_{n_1},a^1_{n_1+1},a^1_{n_1+2},\cdots)$. Since $d(\sigma_2(x_1),x_2) < \delta$ there exists $n_2 \ge n_1+1$ such that, $b_{n_2} \ne a_{n_1}$ and

$$x_2 = (a_4,\cdots,a_{n_1-1},b_{n_1},a^1_{n_1+1},\cdots,a^1_{n_2-1},b_{n_2},a^2_{n_2+1},\cdots).$$

By continuing this procedure, we obtain $n_i \ge n_{i-1}+i$ and $b_{n_i} \ne a^{i-1}_{n_i}$ and an appropriate representation for $x_i$ as above. Now we consider

$$Z = (a_1,\cdots,a_{n_1-1},b_{n_1},a^1_{n_1+1},\cdots,a^1_{n_2-1},b_{n_2},a^2_{n_2+1},\cdots,a^{i-1}_{n_i-1},b_{n_i},a^i_{n_i+1},\cdots).$$

We have $d(\sigma_1(Z),x_1) < \epsilon$, $d(\sigma_3(Z),x_2) < \epsilon$, $\cdots$, $d(\sigma_{\frac{i(i+1)}{2}}(Z),x_i) < \epsilon$, $\cdots$, which is the same as $d(\sigma_1^k(Z),x_k) < \epsilon$, $k = 1,2,\cdots,i,\cdots$, therefore the sequence $\{\sigma_i\}$ has shadowing property.

**Example 2.2.** Consider the sequence

$$f_n : S^1 \to S^1, \quad f_n(e^{i\theta}) = e^{i\frac{2n+1}{n}\theta}.$$

For $\lambda = 2$ and for every $n$, $B_{\lambda\epsilon}(f_n(e^{i\theta})) \subseteq f_n(B_\epsilon(e^{i\theta}))$, and hence $\{f_n\}$ has locally expanding property. Therefore $\{f_n\}$ has $h$-shadowing property.

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