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L^∞-Bounds of Solutions for Strongly Nonlinear Elliptic Problems with Two Lower Order Terms

Youssef Akdim[1,*], Mohammed Belayachi[1] and Mostafa El Moumni[2]

[1] *Laboratory LSI, Department of Mathematics, Physics and Informatics, Faculty Polydisciplinary of Taza, University Sidi Mohamed Ben Abdellah, P.O. Box 1223 Taza Gare, Morocco*
[2] *Department of Mathematics, Faculty of Sciences El Jadida, University Chouaib Doukkali, P.O. Box 20, 24000 El jadida, Morocco*

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Abstract. In this work, we will prove the existence of bounded solutions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ for nonlinear elliptic equations $-\operatorname{div}(a(x,u,\nabla u))+g(x,u,\nabla u)+H(x,\nabla u)=f$, where a, g and H are Carathéodory functions which satisfy some conditions, and the right hand side f belongs to $W^{-1,q}(\Omega)$.

Key Words: L^∞-estimate, nonlinear elliptic equations, rearrangement, Sobolev spaces.

AMS Subject Classifications: 35J60, 46E30, 46E35

1 Introduction

Let Ω be a regular bounded domain of $\mathbb{R}^N$, $N>1$ and let us consider the problem:

$$\begin{cases} -\operatorname{div}\big(a(x,u,\nabla u)\big)+g(x,u,\nabla u)+H(x,\nabla u)=f \quad \text{in } \mathcal{D}'(\Omega), \\ u\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \end{cases} \tag{1.1}$$

where $-\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ with $p>1$ and $\frac{1}{p}+\frac{1}{p'}=1$, g is a nonlinearity which satisfies the growth condition and also it satisfies a sign condition (i.e., it is an absorption of a lower order term) and H is a reaction term on which suitable hypothesis are made. Moreover the source term f belongs to $W^{-1,q}(\Omega)$ where $q>\frac{N}{p-1}$ and $q\geq p'$.

When $H\equiv 0$, in [3] the authors were interested by the existence of the $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ solutions of $-(a_{ij}u_{x_j})_{x_i}+a_0u=g(x,u,\nabla u)$ with $|g|\leq C_0+b(|u|)|\nabla u|^2$ where a_{ij} is bounded

*Corresponding author. *Email addresses:* akdimyoussef@yahoo.fr (Y. Akdim), mohammedbelayachi7@gmail.com (M. Belayachi), mostafaelmoumni@gmail.com (M. El Moumni)

measurable, $a_0>0$ and b is a function on $\mathbb{R}^+$, also they were interested by an existence result for $-\triangle_p u+g(x,u,\nabla u)+a_0|u|^{p-1}\text{sign}(u)=f-\text{div}F$ with $|g|\leq C_0+C_1|\nabla u|^p$ where a_0,C_0 and C_1 are strictly positive, f and F are suitably integrable in [5]. For $-(a_{ij}u_{x_j})_{x_i}=g(x,u,\nabla u)-\text{div}f$ with g is satisfying $|g|\leq b+|\nabla u|^p$ for $p=2$, when f and b is suitably integrable, an existence result can be found in [10]. The $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution of $-\triangle_p u=g(x,u,\nabla u)$ where g is satisfying $|g|\leq b+|\nabla u|^p$ and b is a suitably integrable study in [6]. In [11] the existence of the $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ solution of $-\text{div}(|\nabla u|^{p-1}\nabla u)=|\nabla u|^p+g-\text{div}f$ with g and f are suitably integrable. Let us point out that more works in this direction can be found in [4, 18]. Recently in [21] when $H\equiv 0$, the authors have proved the existence of bounded solutions of unilateral problems associated with the Dirichlet problems (1.1) in the setting of Orlicz Sobolev space without any restriction on the N-function of the Orlicz spaces, where the function $g(x,u,\nabla u)$ is not satisfying the sign condition.

In the case H is not necessarily the null function, the existence result for the problem (1.1) where $u\in W_0^{1,p}(\Omega)$ was firstly proved in [8] in the case where the functions g does not depend on the gradient and it was secondly proved in [14] using the rearrangement techniques. The existence result of equations with this type with a measure data have been given in [1] and has also been studied in [20] in the case of unilateral problems with L^1-data.

The scope of the present work is to obtain the uniform $L^\infty-$estimates for the solutions of strongly nonlinear elliptic equations (1.1), we based on rearrangement properties [13]. This method has been successfully applied to nonlinear elliptic problems with p-growth in the gradient by Ferone et al. [11]. Such an estimate allows us to prove the existence of a solution of (1.1) see [14]. The smallness conditions on the measure of Ω and some norm of b_1, b, and f are essential in the $L^\infty-$estimates.

Let us briefly summarize the contents of this article. Section 2, contains some preliminary results concerned with the rearrangement propriety. In Section 3, we give the assumption on the data and we show the existence of our result (Theorem 3.1).

2 Preliminary results

We recall here some standard notations and properties which will be used through the paper. Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, and let $\omega:\Omega\to\mathbb{R}$ be a measurable function. If one denotes by $|E|$ the lebesgue measure of a set E, one can define the distribution function $\mu_\omega(t)$ of ω as:

$$\mu_\omega(t)=|\{x\in\Omega:\omega>t\}|,\quad t\geq 0.$$

The decreasing rearrangement ω^* of ω is defined as the generalized inverse function of μ_ω:

$$\omega^*(s)=\inf\{t\geq 0:\mu_\omega(t)\leq s\},\quad s\in[0,|\Omega|].$$

We recall that ω and ω^* are equimeasurable, i.e.,

$$\mu_\omega(t)=\mu_{\omega^*}(t), \quad t\in\mathbb{R}^+.$$

This implies that for any Borel function ψ it holds that

$$\int_\Omega \psi(\omega(x))dx=\int_0^{|\Omega|}\psi(\omega^*(s))ds,$$

and, in particular,

$$||\omega^*||_{L^p(0,|\Omega|)}=||\omega||_{L^p(\Omega)}, \quad 1\le p<\infty. \tag{2.1}$$

The theory of rearrangements is well known and exhaustive treatments of it can be found for example in [9, 13, 15]. Now we recall two notions which allow us to define a "generalized" concept of rearrangement of a function f with respect to a given function ω.

Definition 2.1 (see [2]). Let $f\in L^1(\Omega)$ and $\omega\in L^1(\Omega)$. We will say that a function $\overline{f}_\omega\in L^1(0,|\Omega|)$ is a pseudo-rearrangement of f with respect to ω if there exists a family $\{D(s)\}_{s\in(0,|\Omega|)}$ of subsets of Ω satisfying the properties:

(i) $|D(s)|=s$,

(ii) $s_1<s_2\Rightarrow D(s_1)\subset D(s_2)$,

(iii) $D(s)=\{x\in\Omega:\omega(x)>t\}$ if $s=\mu_\omega(t)$,

such that

$$\overline{f}_\omega(s)=\frac{d}{ds}\int_{D(s)}f(x) \quad \text{in } \mathcal{D}'(\Omega).$$

Definition 2.2 (see [16]). Let $f\in L^1(\Omega)$ and $\omega\in L^1(\Omega)$. The following limit exists:

$$\lim_{\lambda\searrow 0}\frac{(\omega+\lambda f)^*-\omega^*}{\lambda}=f_\omega^*,$$

where the convergence is in $L^p(\Omega)$-weak, if $f\in L^p(\Omega), 1\le p<\infty$, and in $L^\infty(\Omega)-\text{weak}^*$, if $f\in L^\infty(\Omega)$. The function f_ω^* is called the relative rearrangement of f with respect to ω. Moreover, one has

$$f_\omega^*(s)=\frac{dG}{ds} \quad \text{in } \mathcal{D}'(\Omega),$$

where

$$G(s)=\int_{\{\omega>\omega^*(s)\}}f(x)dx+\int_0^{s-|\{\omega>\omega^*(s)\}|}(f|_{\{\omega=\omega^*(s)\}})(\sigma)d\sigma.$$

The two notions are equivalent in some precise sense (see [9]). For this reason we will denote $\overline{f}_\omega$ and f^*_ω by F_ω. We only recall a few results which hold for both the pseudo- and the relative rearrangements.

If f and ω are non-negative and $\omega \in W^{1,1}_0(\Omega)$ it is possible to prove the following properties:

$$-\frac{d}{dt}\int_{\{\omega>t\}} f(x)dx = F_\omega(\mu_\omega(t))(-\mu'_\omega(t)) \quad \text{for a.e. } t>0, \tag{2.2a}$$

$$||F_\omega||_{L^p(0,|\Omega|)} \le ||f||_{L^p(\Omega)}, \quad 1\le p<\infty. \tag{2.2b}$$

The proofs of (2.2a) and (2.2b) can be found in [2] (for pseudo-rearrangements) and in [17, 19] (for relative rearrangements). We finally recall the following chain of inequalities which holds for any non-negative $\omega \in W^{1,p}_0(\Omega)$:

$$\begin{aligned} NC_N^{1/N}\mu_\omega(t)^{1-1/N} &\le -\frac{d}{dt}\int_{\{\omega>t\}} |\nabla\omega|dx \\ &\le (-\mu'_\omega(t))^{1/p'}\left(-\frac{d}{dt}\int_{\{\omega>t\}}|\nabla\omega|^p dx\right)^{1/p}, \end{aligned} \tag{2.3}$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$. It is a consequence of the Fleming-Rishel formula [12], the isoperimetric inequality [7] and the Hölder's inequality.

3 Main results

Let us now give the precise hypotheses on the problem (1.1), we assume that the following assumptions: Ω is a bounded open set of $\mathbb{R}^N$ ($N>1$), $1<p<+\infty$, let $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ be a Carathéodory function, such that

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)>0 \text{ for all } (\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N, \text{ with } \xi\neq\eta, \tag{3.1a}$$

$$a(x,s,\xi)\xi \ge \alpha|\xi|^p, \tag{3.1b}$$

where α is a strictly positive constant

$$|a(x,s,\xi)| \le \beta(d(x)+|s|^{p-1}+|\xi|^{p-1}), \tag{3.2}$$

for a.e. $x\in\Omega$, all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, a positive function $d(x)\in L^{p'}(\Omega)$, $1<p\le N$, and $\beta>0$.

Furthermore, let $g(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $H(x,\xi):\Omega\times\mathbb{R}^N\to\mathbb{R}$ are two Carathéodory functions which satisfy, for almost every $x\in\Omega$ and for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$, the following conditions:

$$g(x,s,\xi)s\ge 0, \tag{3.3a}$$

$$|g(x,s,\xi)| \le b_1(x)+b_2(x)|\xi|^p, \tag{3.3b}$$

where $b_1(x)\in L^m(\Omega)$, $m>\frac{N}{p}$, and $b_1(x)\geq 0$ a.e. $|b_2(x)|\leq\lambda$ a.e. in Ω, where λ is a strictly positive constant.

$$|H(x,\xi)|\leq b(x)|\xi|^{p-1}, \tag{3.4}$$

where $b(x)$ is positive and belongs to $L^r(\Omega)$ with $r>N$.

Finally, for the right hand side, we assume that

$$f\in W^{-1,q}(\Omega),\quad q\geq p'\quad\text{and}\quad q>\frac{N}{p-1}. \tag{3.5}$$

We recall that if $f\in W^{-1,p'}(\Omega)$ there exist $f_0,f_1,\cdots,f_N\in L^{p'}(\Omega)$ such that :

$$\langle f,v\rangle=\int_\Omega f_0 v+\sum_i\int_\Omega f_i\frac{\partial v}{\partial x_i},\quad \forall v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega),$$

since Ω is bounded, we can assume that $f_0=0$ and we set $f=(f_1,f_2,\cdots,f_N)$, $|f|=(\sum f_i^2)^{\frac{1}{2}}$.

Now, we give the following results which will be used in our main result.

Lemma 3.1. *When f satisfies* (3.5) *and when assumptions* (3.1a)-(3.4) *are satisfied, there exists a weak solution u of* (1.1) *in the following sense:*

$$\int_\Omega a(x,u,\nabla u)\nabla v dx+\int_\Omega\Big(g(x,u,\nabla u)+H(x,\nabla u)\Big)v dx=\langle f,v\rangle,$$

for all $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

The proof of this Lemma see [14].

Remark 3.1. The duality product between f and v make sense since $q\geq p'$ so $W^{-1,q}(\Omega)\subset W^{-1,p'}(\Omega)$. Note also that (3.1a), (3.3a) are not required to obtain our results. However, these hypotheses are needed for the existence result (see [14]).

Our main results are collected in the following theorem:

Theorem 3.1. *Let u be a solution of* (1.1) *under the assumptions* (3.1b)-(3.2) *and* (3.3b)-(3.5). *If b_1, b and f satisfy the inequality*

$$\begin{aligned}&\frac{1}{(NC_N^{1/N})^{p'}}\frac{N\sigma(p-1)}{p\sigma-N}\Big(\frac{p'}{\alpha}|\Omega|^{\frac{p}{N}-\frac{1}{m}}||b_1||_m+\frac{p'}{p}|\Omega|^{\frac{p}{N}-\frac{p}{r}}||b||_r^p+\Big(\frac{\lambda p'}{\alpha^{p'+1}}\\&+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{\frac{p}{N}-\frac{p'}{q}}||f||_q^q\Big)^{p'/p}+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\left(\frac{N(q(p-1)-1)}{q(p-1)+N}\right)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||f||_q^{p'/p}\\&\leq\frac{\alpha^{p'}(p-1)}{\lambda p'\alpha^{p'-1}+1},\end{aligned} \tag{3.6}$$

where $\sigma=\min(m,\frac{r}{p},\frac{q}{p'})$, then there exists a constant $M>0$, which depends only on N, p, p', q, r, $|\Omega|$, $||f||_{L^q(\Omega)}$, $||b||_{L^r(\Omega)}$ and $||b_1||_{L^m(\Omega)}$, such that

$$||u||_{L^\infty(\Omega)}\leq M. \tag{3.7}$$

Proof. The proof of Theorem 3.1 is done in two steps. In this step, we prove the decreasing rearrangement of $\omega=\frac{e^{k|u|}-1}{k}$ satisfies the following differential inequality:

$$(-\omega^*(s))' \leq \frac{[(-\omega^*(s))']^{1/p}}{NC_N^{1/N}s^{1-1/N}}\Big(\int_0^s \psi^*(\tau)(k\omega^*(\tau)+1)^{p-1}d\tau\Big)^{1/p} + \frac{k\omega^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\big(F_\omega(s)\big)^{1/p}, \tag{3.8}$$

where ψ^* is the decreasing rearrangement of

$$\psi=\frac{p'}{\alpha}b_1(x)+\frac{p'}{p}|b|^p+\left(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\right)|f|^{p'}$$

and F_ω is a pseudo-rearrangement (or the relative rearrangement) of $|f|^{p'}$ with respect to ω.

Let us define two real functions $\phi_1(z)$, $\phi_2(z)$, $z\in\mathbb{R}$, as follows:

$$\begin{cases} \phi_1(z)=e^{k(p-1)|z|}\ \text{sign}(z), \\ \phi_2(z)=\dfrac{(e^{kz}-1)}{k}, \end{cases} \tag{3.9}$$

where

$$k=\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)},$$

we observe that $\phi_2(0)=0$ and for $z\neq 0$, $\phi_1'(z)>0$, $\phi_2'(z)>0$,

$$\phi_1(z)\phi_2'(|z|)\ \text{sign}(z)=|\phi_2'(|z|)|^p, \tag{3.10a}$$

$$\phi_1'(z)-\left(\frac{\lambda p'}{\alpha}+\frac{1}{\alpha^{p'}}\right)|\phi_1(z)|=0. \tag{3.10b}$$

Furthermore, for $t>0$, $h>0$, let us put

$$S_{t,h}(z)=\begin{cases} \text{sign}(z), & \text{if } |z|>t+h, \\ ((|z|-t)/h)\ \text{sign}(z), & \text{if } t<|z|\leq t+h, \\ 0, & \text{if } |z|\leq t. \end{cases} \tag{3.11}$$

We use in (1.1) the test function $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ defined by

$$v=\phi_1(u)S_{t,h}(\omega)=\phi_1(u)S_{t,h}(\phi_2(|u|)), \tag{3.12}$$

where $\omega=\frac{e^{k|u|}-1}{k}$. Using (3.11) we have

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<\omega\leq t+h\}} a(x,u,\nabla u)\nabla u\phi_1(u)\phi_2'(|u|)\text{sign }(u)dx\\
&+\int_{\{\omega>t\}}\Big((g(x,u,\nabla u)+H(x,\nabla u))\phi_1(u)+a(x,u,\nabla u)\nabla u\phi_1'(u)\Big)S_{t,h}(\omega)dx\\
&=\int_{\{\omega>t\}}\sum_{i=1}^{N}f_i\frac{\partial u}{\partial x_i}\phi_1'(u)S_{t,h}(\omega)dx+\frac{1}{h}\int_{\{t<\omega\leq t+h\}}\sum_{i=1}^{N}f_i\frac{\partial u}{\partial x_i}\phi_1(u)\phi_2'(|u|)\text{sign }(u)dx. \quad (3.13)
\end{aligned}$$

Taking into account (3.10a) and Young's inequality, it follows that

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<\omega\leq t+h\}} a(x,u,\nabla u)\nabla u\phi_1(u)\phi_2'(|u|)\text{sign}(u)dx\\
&+\int_{\{\omega>t\}}\big((g(x,u,\nabla u)+H(x,\nabla u))\phi_1(u)+a(x,u,\nabla u)\nabla u\phi_1'(u)\big)S_{t,h}(\omega)dx\\
&\leq\frac{\alpha^{-p'/p}}{p'}\int_{\{\omega>t\}}|f|^{p'}\phi_1'(u)S_{t,h}(\omega)dx+\frac{\alpha}{p}\int_{\{\omega>t\}}|\nabla u|^p\phi_1'(u)S_{t,h}(\omega)dx\\
&+\frac{\alpha^{-p'/p}}{p'h}\int_{\{t<\omega\leq t+h\}}|f|^{p'}|\phi_2'(|u|)|^pdx+\frac{\alpha}{ph}\int_{\{t<\omega\leq t+h\}}|\nabla u|^p|\phi_2'(|u|)|^pdx,
\end{aligned}$$

using (3.3b), (3.4) and the ellipticity condition (3.1b), we obtain

$$\begin{aligned}
&\frac{\alpha}{h}\int_{\{t<\omega\leq t+h\}}|\nabla u|^p|\phi_2'(|u|)|^pdx\\
&\leq\int_{\{\omega>t\}}\Big(\Big(b_1(x)+b_2(x)|\nabla u|^p+b(x)|\nabla u|^{p-1}\Big)\phi_1(u)-\alpha|\nabla u|^p\phi_1'(u)\Big)S_{t,h}(\omega)dx\\
&+\frac{\alpha^{-p'/p}}{p'}\int_{\{\omega>t\}}|f|^{p'}\phi_1'(u)S_{t,h}(\omega)dx+\frac{\alpha}{p}\int_{\{\omega>t\}}|\nabla u|^p\phi_1'(u)S_{t,h}(\omega)dx\\
&+\frac{\alpha^{-p'/p}}{p'h}\int_{\{t<\omega\leq t+h\}}|f|^{p'}|\phi_2'(|u|)|^pdx+\frac{\alpha}{ph}\int_{\{t<\omega\leq t+h\}}|\nabla u|^p|\phi_2'(|u|)|^pdx.
\end{aligned}$$

By (3.10b) and Young's inequality, it follows that

$$\begin{aligned}
&\frac{1}{h}\int_{\{t<\omega\leq t+h\}}|\nabla u|^p|\phi_2'(|u|)|^pdx\\
&\leq\int_{\{\omega>t\}}\left(\frac{\lambda p'}{\alpha}|\phi_1(u)|-\phi_1'(u)+\frac{1}{\alpha^{p'}}|\phi_1(u)|\right)|\nabla u|^pS_{t,h}(\omega)dx\\
&+\int_{\{\omega>t\}}\left(\frac{p'}{\alpha}b_1(x)+\frac{p'}{p}|b|^p+\left(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\right)|f|^{p'}\right)|\phi_1(u)|S_{t,h}(\omega)dx\\
&+\frac{1}{\alpha^{p'}h}\int_{\{t<\omega\leq t+h\}}|f|^{p'}|\phi_2'(|u|)|^pdx.
\end{aligned}$$

Using (3.10b) and the definition of ϕ_1, ϕ_2 in (3.9), the above inequality gives:

$$\begin{aligned}&\frac{1}{h}\int_{\{t<\omega\leq t+h\}}|\nabla\omega|^p dx\\ &\leq\int_{\{\omega>t\}}\psi(k\omega+1)^{p-1}S_{t,h}(\omega)dx+\frac{1}{\alpha^{p'}h}\int_{\{t<\omega\leq t+h\}}|f|^{p'}(k\omega+1)^p dx,\end{aligned}\tag{3.14}$$

where

$$\psi=\frac{p'}{\alpha}b_1(x)+\frac{p'}{p}|b|^p+\left(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\right)|f|^{p'}.$$

Letting h go to 0 in a standard way we get:

$$-\frac{d}{dt}\int_{\{\omega>t\}}|\nabla\omega|^p dx\leq\int_{\{\omega>t\}}\psi(k\omega+1)^{p-1}dx+\frac{(kt+1)^p}{\alpha^{p'}}\left(-\frac{d}{dt}\int_{\{\omega>t\}}|f|^{p'}dx\right).$$

Using Hardy-Littlewood's inequality and the inequality (2.2a). It follows that

$$\begin{aligned}&-\frac{d}{dt}\int_{\{\omega>t\}}|\nabla\omega|^p dx\\ &\leq\int_0^{\mu_\omega(t)}\psi^*(s)(k\omega^*(s)+1)^{p-1}ds+\frac{(kt+1)^p}{\alpha^{p'}}(-\mu'_\omega(t))F_\omega(\mu_\omega(t)),\end{aligned}\tag{3.15}$$

where F_ω is a pseudo-rearrangement (or the relative rearrangement) of $|f|^{p'}$ with respect to ω. Combining (2.3) and (3.15), we obtain

$$\begin{aligned}&NC_N^{1/N}\mu_\omega(t)^{1-1/N}\\ &\leq(-\mu'_\omega(t))^{1/p'}\Big(\int_0^{\mu_\omega(t)}\psi^*(s)(k\omega^*(s)+1)^{p-1}ds\Big)^{1/p}+\frac{kt+1}{\alpha^{p'/p}}(-\mu'_\omega(t))\big(F_\omega(\mu_\omega(t))\big)^{1/p}\end{aligned}$$

and then, using the definition of $\omega^*(s)$, we have:

$$\begin{aligned}&(-\omega^*(s))'\\ &\leq\frac{[(-\omega^*(s))']^{1/p}}{NC_N^{1/N}s^{1-1/N}}\Big(\int_0^s\psi^*(\tau)(k\omega^*(\tau)+1)^{p-1}d\tau\Big)^{1/p}+\frac{k\omega^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\big(F_\omega(s)\big)^{1/p}\end{aligned}$$

that is (3.8).

We prove the following inequality (3.7) under assumption (3.6). By Young's inequality and (3.8) implies:

$$\begin{aligned}&(-\omega^*(s))'\\ &\leq\frac{1}{p}(-\omega^*(s))'+\frac{1}{p'(NC_N^{1/N}s^{1-1/N})^{p'}}\\ &\quad\times\left(\int_0^s\psi^*(\tau)\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(\tau)+1\right)^{p-1}d\tau\right)^{p'/p}\\ &\quad+\frac{1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(s)+1\right)(F_\omega(s))^{1/p},\end{aligned}$$

we deduce that,

$$\begin{aligned}
&(-\omega^*(s))'\\
\leq&\frac{1}{(NC_N^{1/N}s^{1-1/N})^{p'}}\left(\int_0^s\psi^*(\tau)\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(\tau)+1\right)^{p-1}d\tau\right)^{p'/p}\\
&+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(s)+1\right)(F_\omega(s))^{1/p}.
\end{aligned}$$

Integrating between 0 and $|\Omega|$ we have

$$\begin{aligned}
&\int_0^{|\Omega|}(-\omega^*(s))'ds\\
\leq&\int_0^{|\Omega|}\frac{1}{(NC_N^{1/N}s^{1-1/N})^{p'}}\\
&\times\left(\int_0^s\psi^*(\tau)\left(\big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\big)\omega^*(\tau)+1\right)^{p-1}d\tau\right)^{p'/p}ds\\
&+\int_0^{|\Omega|}\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(s)+1\right)(F_\omega(s))^{1/p}ds.
\end{aligned}$$

Since $\omega^*(|\Omega|)=0$, we obtain

$$\begin{aligned}
\omega^*(0)\leq&\int_0^{|\Omega|}\left(\frac{1}{(NC_N^{1/N}s^{1-1/N})^p}\int_0^s\psi^*(\tau)d\tau\right)^{p'/p}ds\times\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(0)+1\right)^{(p-1)p'/p}\\
&+\int_0^{|\Omega|}\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}(F_\omega(s))^{1/p}ds\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(0)+1\right)\\
\leq&\left[\int_0^{|\Omega|}\left(\frac{1}{(NC_N^{1/N}s^{1-1/N})^p}\int_0^s\psi^*(\tau)d\tau\right)^{p'/p}ds+\int_0^{|\Omega|}\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}(F_\omega(s))^{1/p}ds\right]\\
&\times\left(\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)\omega^*(0)+1\right),
\end{aligned}$$

since ω^* attains its maximum at 0, we can write

$$||\omega||_{L^\infty(\Omega)}\leq\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)A||\omega||_{L^\infty(\Omega)}+A, \tag{3.16}$$

where

$$A=\int_0^{|\Omega|}\left[\left(\frac{1}{(NC_N^{1/N}s^{1-1/N})^p}\int_0^s\psi^*(\tau)d\tau\right)^{p'/p}+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}(F_\omega(s))^{1/p}\right]ds.$$

Lemma 3.2. *Under assumption* (3.6)*, we have*

$$\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)A<1,$$

where A is bounded, more precisely

$$\begin{aligned}A\leq&\frac{1}{(NC_N^{1/N})^{p'}}\frac{N\sigma(p-1)}{p\sigma-N}\\&\times\left(\frac{p'}{\alpha}|\Omega|^{\frac{p}{N}-\frac{1}{m}}||b_1||_{L^m(\Omega)}+\frac{p'}{p}|\Omega|^{\frac{p}{N}-\frac{p}{r}}||b||^p_{L^r(\Omega)}+\Big(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{\frac{p}{N}-\frac{p'}{q}}||f||^q_{L^q(\Omega)}\right)^{p'/p}\\&+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\left(\frac{N(q(p-1)-1)}{q(p-1)-N}\right)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||f||^{\frac{p'}{p}}_{L^q(\Omega)}.\end{aligned}$$

This lemma will be proved below.

In view of Lemma 3.2 and (3.16), we have (3.7). This complete the proof of Theorem 3.1. □

Proof of Lemma 3.2. By using the Hölder's inequality and (2.1), we obtain

$$\begin{aligned}&\int_0^s\psi^*(\tau)d\tau\leq||\psi||_{L^\sigma(\Omega)}s^{1-1/\sigma}\\\leq&\Big(\frac{p'}{\alpha}|\Omega|^{\frac{1}{\sigma}-\frac{1}{m}}||b_1||_{L^m(\Omega)}+\frac{p'}{p}|\Omega|^{\frac{1}{\sigma}-\frac{p}{r}}||b||^p_{L^r(\Omega)}\\&+\Big(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{1/\sigma-\frac{p'}{q}}||f||^{p'}_{L^{p'}(\Omega)}\Big)s^{1-1/\sigma}\\\leq&\Big(\frac{p'}{\alpha}|\Omega|^{\frac{1}{\sigma}-\frac{1}{m}}||b_1||_{L^m(\Omega)}+\frac{p'}{p}|\Omega|^{\frac{1}{\sigma}-\frac{p}{r}}||b||^p_{L^r(\Omega)}\\&+\Big(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{1/\sigma-\frac{p'}{q}}||f||^q_{L^q(\Omega)}\Big)s^{1-1/\sigma},\end{aligned}\tag{3.17}$$

where $\sigma=\min(m,\frac{r}{p},\frac{q}{p'})$. Furthermore taking into account the fact that $q>p'$ property (2.2b) gives

$$\begin{aligned}&\int_0^{|\Omega|}\frac{1}{s^{1-1/N}}(F_\omega(s))^{1/p}ds\\\leq&\Big(\frac{N(q(p-1)-1)}{q(p-1)-N}\Big)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||F_\omega||^{\frac{1}{p}}_{L^{q/p'}(0,|\Omega|)}\\\leq&\Big(\frac{N(q(p-1)-1)}{q(p-1)-N}\Big)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||f||^{\frac{p'}{p}}_{L^q(\Omega)}.\end{aligned}\tag{3.18}$$

Or we have

$$\begin{aligned}
A&=\int_0^{|\Omega|}\Big[\Big(\frac{1}{(NC_N^{1/N}s^{1-1/N})^p}\int_0^s\psi^*(\tau)d\tau\Big)^{p'/p}+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}(F_\omega(s))^{1/p}\Big]ds\\
&=\int_0^{|\Omega|}\Big(\frac{1}{NC_N^{1/N}}\Big)^{p'}\frac{1}{s^{(1-1/N)p'}}\Big(\int_0^s\psi^*(\tau)d\tau\Big)^{p'/p}ds\\
&\quad+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\int_0^{|\Omega|}\frac{1}{s^{1-1/N}}(F_\omega(s))^{1/p}ds\\
&\le\frac{1}{(NC_N^{1/N})^{p'}}\int_0^{|\Omega|}\Big[\frac{1}{s^{(1-1/N)p'}}\times\Big(\Big(\frac{p'}{\alpha}|\Omega|^{\frac{1}{\sigma}-\frac{1}{m}}||b_1||_{L^m(\Omega)}+\frac{p'}{p}|\Omega|^{\frac{1}{\sigma}-\frac{p}{r}}||b||^p_{L^r(\Omega)}\\
&\quad+\Big(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{1/\sigma-\frac{p'}{q}}||f||^q_{L^q(\Omega)}\Big)s^{1-1/\sigma}\Big)^{p'/p}\Big]ds\\
&\quad+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\Big(\frac{N(q(p-1)-1)}{q(p-1)-N}\Big)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||f||^{\frac{p'}{p}}_{L^q(\Omega)}.
\end{aligned}$$

Using (3.17) and (3.18), we can estimate the quantity A in (3.16), obtaining that under assumption (3.6) the following inequality holds:

$$\Big(\frac{\lambda p'}{\alpha(p-1)}+\frac{1}{\alpha^{p'}(p-1)}\Big)A<1.$$

Then, we have

$$\begin{aligned}
A&\le\frac{1}{(NC_N^{1/N})^{p'}}\frac{N\sigma(p-1)}{p\sigma-N}|\Omega|^{\frac{p\sigma-N}{N\sigma(p-1)}}\\
&\quad\times\left(\frac{p'}{\alpha}|\Omega|^{\frac{1}{\sigma}-\frac{1}{m}}||b_1||_{L^m(\Omega)}+\frac{p'}{p}|\Omega|^{\frac{1}{\sigma}-\frac{p}{r}}||b||^p_{L^r(\Omega)}\right.\\
&\quad\left.+\Big(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{1/\sigma-\frac{p'}{q}}||f||^q_{L^q(\Omega)}\right)^{p'/p}\\
&\quad+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\left(\frac{N(q(p-1)-1)}{q(p-1)-N}\right)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||f||^{\frac{p'}{p}}_{L^q(\Omega)}\\
&\le\frac{1}{(NC_N^{1/N})^{p'}}\frac{N\sigma(p-1)}{p\sigma-N}\left(\frac{p'}{\alpha}|\Omega|^{(\frac{p\sigma-N}{N\sigma(p-1)}\frac{p}{p'})+(\frac{1}{\sigma}-\frac{1}{m})}||b_1||_{L^m(\Omega)}\right.\\
&\quad+\frac{p'}{p}|\Omega|^{(\frac{p\sigma-N}{N\sigma(p-1)}\frac{p}{p'})+(\frac{1}{\sigma}-\frac{p}{r})}||b||^p_{L^r(\Omega)}\\
&\quad\left.+\Big(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{(\frac{p\sigma-N}{N\sigma(p-1)}\frac{p}{p'})+(1/\sigma-\frac{p'}{q})}||f||^q_{L^q(\Omega)}\right)^{p'/p}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\left(\frac{N(q(p-1)-1)}{q(p-1)-N}\right)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||f||^{\frac{p'}{p}}_{L^q(\Omega)}\\
\leq&\frac{1}{(NC_N^{1/N})^{p'}}\frac{N\sigma(p-1)}{p\sigma-N}\\
&\times\left(\frac{p'}{\alpha}|\Omega|^{\frac{p}{N}-\frac{1}{m}}||b_1||_{L^m(\Omega)}+\frac{p'}{p}|\Omega|^{\frac{p}{N}-\frac{p}{r}}||b||^p_{L^r(\Omega)}\right.\\
&\left.+\Big(\frac{\lambda p'}{\alpha^{p'+1}}+\frac{1}{\alpha^{2p'}}\Big)|\Omega|^{\frac{p}{N}-\frac{p'}{q}}||f||^q_{L^q(\Omega)}\right)^{p'/p}\\
&+\frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\left(\frac{N(q(p-1)-1)}{q(p-1)-N}\right)^{1-\frac{p'}{qp}}|\Omega|^{\frac{1}{N}-\frac{p'}{qp}}||f||^{\frac{p'}{p}}_{L^q(\Omega)}.
\end{aligned}
$$

Thus, we complete the proof. □

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