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Analysis of Formal and Analytic Solutions for Singularities of the Vector Fractional Differential Equations

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Abstract. In this article, we study on the existence of solution for a singularities of a system of nonlinear fractional differential equations (FDE). We construct a formal power series solution for our considering FDE and prove convergence of formal solutions under conditions. We use the Caputo fractional differential operator and the nonlinearity depends on the fractional derivative of an unknown function.

Key Words: Fractional differential equations, formal power series solution.

AMS Subject Classifications: 26A33, 34B18, 34A08

1 Introduction

Recently, fractional differential equations have been investigated extensively. The motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electrodynamics of complex medium, and so on. For examples and details, see [1–6, 8, 9, 11, 14, 16, 20, 23, 25, 26] and the references therein. Fractional calculus in the complex plane also was done by Osler and et al. [30–33]

The existence of formal and analytic solutions for singularities of ordinary differential equations such as $x\frac{d\vec{y}}{dx}=\vec{f}(x,\vec{y})$ and other statements was discussed in [27–29]. Motivated by the above mentioned work, in this paper we consider a system of singularities nonlinear fractional order differential equation:

$$x^{\alpha}\frac{d^{\alpha}\overrightarrow{y}}{dx^{\alpha}}=\overrightarrow{f}(x,\overrightarrow{y}), \quad \alpha\geq 1. \tag{1.1}$$

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The paper has been organised as follows. In Section 2 we give basic definitions and preliminary. Leibniz rule and chain rule for LFD have been derived in Section 3 and Section 4. Extensions of directional LFDs and local fractional Taylor series to higher orders have been presented in Sections 5 and 6.

2 Preliminaries

In this section, we present some notations, definitions and preliminary that will be useful for our main results. This materials can be found in the literatures [10,17,19,21,22,24].

$$\vec{y}=\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n \end{bmatrix},\quad \vec{f}=\begin{bmatrix} f_1\\ f_2\\ \vdots\\ f_n \end{bmatrix},\quad |\vec{y}|=\max\{|y_1|,|y_2|,\cdots,|y_n|\}.$$

$\vec{f}(x,\vec{y})$ is $\mathbb{C}^n-$valued function in complex variables $(x,\vec{y})\in\mathbb{C}^{n+1}$. We denote by $\mathbb{C}[[x]]$ the set of all formal power series in x with coefficients in $\mathbb{C}$. Also, denote by $\mathbb{C}\{x\}$ the set of all power series in $\mathbb{C}[[x]]$ that have nonzero radii of convergence. Denote by $x^\sigma\mathbb{C}[[x]]$ the set of formal series $x^\sigma f(x)$, where $f(x)\in\mathbb{C}[[x]],\sigma$ is a complex number, and $x^\sigma=\exp(\sigma\ln x)$. Similarly, let $x^\sigma\mathbb{C}\{x\}$ denote the set of convergent series $x^\sigma f(x)$, where $f(x)\in\mathbb{C}\{x\}$.

Definition 2.1. A formal power series

$$\vec{\phi}(x)=\sum_{m=0}^{\infty}x^m\vec{c}_m\in x\mathbb{C}[[x]]^n,\quad \vec{c}_m\in\mathbb{C}^n,$$

is a formal solution of system (1.1) if

$$x^\alpha\frac{d^\alpha\vec{\phi}(x)}{dx^\alpha}=\vec{f}(x,\vec{\phi}(x)).$$

Riemanns modified form of Liouvilles fractional integral operator is a generalization of Cauchys iterated integral formula

$$\int_a^x dx_1\int_a^{x_1}dx_2\cdots\int_a^{x_{n-1}}g(x_n)dx_n=\frac{1}{\Gamma(n)}\int_a^x(x-s)^{n-1}g(s)ds, \tag{2.1}$$

where Γ is Euler's gamma function. Clearly, the right-hand side of Eq. (2.1) is meaningful for any positive real value of n. Hence, it is natural to define the fractional integral as follows:

Definition 2.2. If y be analitic function in $\mathbb{C}$, then the Riemann-Liouville fractional integral is defined by

$$\frac{d^{-\alpha}y(x)}{dx^\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}y(s)ds. \tag{2.2}$$

Definition 2.3. Let $\alpha\in\mathbb{R}$, $n-1<\alpha\le n$, $n\in\mathbb{N}$ and y be analitic function in $\mathbb{C}$, then the Caputo fractional derivative of order α defined by

$$\frac{d^\alpha y(x)}{dx^\alpha}=\frac{d^{-\alpha}}{dx^\alpha}\left(\frac{d^n y(x)}{dx^n}\right). \tag{2.3}$$

The fractional integral of $y(x)=x^\gamma$, $\gamma>-1$ is given as

$$\frac{d^{-\alpha}y(x)}{dx^\alpha}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}x^{\gamma+\alpha}, \tag{2.4}$$

and the fractional derivative of $y(x)=x^\gamma,\gamma>-1$ also is given as

$$\frac{d^\alpha y(x)}{dx^\alpha}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}x^{\gamma-\alpha}. \tag{2.5}$$

Lemma 2.1 (see [14, 21])**.** *For $\alpha>0$, the general solution of the fractional differential equation $\frac{d^\alpha y(x)}{dx^\alpha}=0$ is given by*

$$y(x)=\sum_{i=0}^{r-1}c_i x^i,\quad c_i\in\mathbb{R},\quad i=0,1,2,\cdots,r-1,r=[\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

In view of Lemma 2.1 it follows that

$$\frac{d^{-\alpha}}{dx^\alpha}\left(\frac{d^\alpha y(x)}{dx^\alpha}\right)=y(x)+\sum_{j=0}^{r-1}c_j x^j \quad \text{for some } c_i\in\mathbb{R},\quad i=0,1,\cdots,r-1. \tag{2.6}$$

But in the opposite way we have,

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{d^{-\gamma}y(x)}{dx^\alpha}\right)=\frac{d^{\alpha-\gamma}y(x)}{dx^\alpha}. \tag{2.7}$$

3 Formal solution

In this section, generally speaking, in order to construct a power series solutions

$$\vec{y}(x)=\sum_{m=0}^{\infty}x^m\vec{a}_m,$$

this expression is inserted into Eq. (1.1) to find relationships among the coefficients $\vec{a}_m$, and the coefficients $\vec{a}_m$ are calculated by using these relation. In this stage of the calculation, we do not pay any attention to the convergence of the series.

Theorem 3.1. *Suppose that $A(x)$ is $n\times n$ matrix whose entries are formal power series in x and λ is an eigenvalue of $A(0)$. Assume also that $\lambda+k$ are not eigenvalues of $A(0)$ for all positive integers k. Then, the fractional differential equation*

$$x^{\alpha}\frac{d^{\alpha}\vec{y}}{dx^{\alpha}}=A(x)\vec{y} \tag{3.1}$$

has a nontrivial formal solution

$$\vec{\phi}(x)=x^{\lambda}\vec{f}(x)=x^{\lambda}\sum_{m=0}^{\infty}x^{m}\vec{a}_m\in x^{\lambda}\mathbb{C}[[x]]^{n}. \tag{3.2}$$

Proof. Insert

$$x^{\lambda}\sum_{m=0}^{\infty}x^{m}\vec{a}_m=\sum_{m=0}^{\infty}x^{\lambda+m}\vec{a}_m$$

into Eq. (3.1) and setting

$$A(x)=\sum_{m=0}^{\infty}x^{m}A_m,$$

where $A_m\in\mathcal{M}_n(\mathbb{C})$ and $A_0=A(0)$. Then, using by Eq. (2.5) we obtain

$$x^{\alpha}\sum_{m=0}^{\infty}\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+m-\alpha+1)}\vec{a_m}x^{\lambda+m-\alpha}=\sum_{m=0}^{\infty}x^{\lambda+m}\left[\sum_{h=0}^{m}A_{m-h}\vec{a}_h\right].$$

Therefore, in order to construct a formal solution, the coefficients $\vec{a}_m$ must be determined by the equations

$$\lambda\vec{a}_0=A_0\vec{a}_0\quad\text{and}\quad\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+m-\alpha+1)}\vec{a}_m=A_0\vec{a}_m+\sum_{h=0}^{m-h}A_{m-h}\vec{a}_h,\quad m\geq 1.$$

Hence, $\vec{a}_0$ must be eigenvector of A_0 associated with the eigenvalue λ, whereas

$$\vec{a}_m=\left(\frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+m-\alpha+1)}I_n-A_0\right)^{-1}\left[\sum_{h=0}^{m-h}A_{m-h}\vec{a}_h\right]\quad\text{for }\ m\geq 1.$$

Thus, we complete the proof. ☐

Example 3.1. Consider a system of nonlinear fractional differential equation:

$$x^{\frac{3}{2}}\frac{d^{\frac{3}{2}}\vec{y}}{dx^{\frac{3}{2}}}=\vec{f}(x,\vec{y}),\quad \vec{f}=(f_1,f_2)^{t},\quad \vec{y}=(y_1,y_2)^{t}, \tag{3.3}$$

where $f_1(x,\vec{y}) = xy_2 - 2y_1\exp(x)$ and $f_2(x,\vec{y}) = (1+x^2)y_1 + 2y_2\cosh(x)$. In other words, (3.1) equivalent to

$$\begin{cases} x^{\frac{3}{2}}\left(\dfrac{d^{\frac{3}{2}}y_1}{dx^{\frac{3}{2}}}\right) = xy_2 - 2y_1\exp(x), \\ x^{\frac{3}{2}}\left(\dfrac{d^{\frac{3}{2}}y_2}{dx^{\frac{3}{2}}}\right) = (1+x^2)y_1 + 2y_2\cosh(x). \end{cases} \tag{3.4}$$

In this example we have

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} xy_2 - 2y_1\exp(x) \\ (1+x^2)y_1 + 2y_2\cosh(x) \end{bmatrix},$$

or

$$\vec{f} = A(x)\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \text{where } A(x) = \begin{bmatrix} -2\exp(x) & x \\ 1+x^2 & 2\cosh(x) \end{bmatrix}.$$

Then, entries of matrix $A(x)$ haveing formal power series in the form

$$A(x) = \begin{bmatrix} -2\displaystyle\sum_{m=0}^{\infty}\frac{x^m}{m!} & x \\ 1+x^2 & 2\displaystyle\sum_{m=0}^{\infty}\frac{x^{2m}}{(2m)!} \end{bmatrix}.$$

Hence,

$$A(0) = \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \lambda = \pm 2.$$

If $\lambda = 2$ and $\vec{y}(x) = x^2\sum_{m=0}^{\infty} x^m\vec{a}_m$ is a formal solution system (3.2), then $\vec{a}_0 = (0,1)$. Note that $\vec{a}_0 = (0,1)$ is an eigenvector of $A(0)$ associated with the eigenvalue $\lambda = 2$ and the eigenvector of $A(0)$ associated with the eigenvalue $\lambda = 2$ is not unique. Therefore

$$\vec{a}_m = \left(\frac{\Gamma(3+m)}{\Gamma(\frac{3}{2}+m)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix}\right)^{-1}\left[\sum_{h=0}^{m-h} A_{m-h}\vec{a}_h\right] \quad \text{for } m \geq 1.$$

Note that we can not select $\lambda = -2$, as $-2+4 = 2$ is an eigenvalue of $A(0)$ and hence $\vec{a}_m$ are not exist for some $m \geq 1$.

Remark 3.1. For an eigenvalue λ_0 of $A(0)$, let h be the maximum integer such that $\lambda_0 + h$ is also an eigenvalue of $A(0)$. Then, Theorem 3.1 applies to $\lambda = \lambda_0 + h$.

4 Convergence of formal solution

In this section, we prove convergence of formal solutions of a system of fractional differential equations (1.1). To achieve our main goal, we need some preparations.

Lemma 4.1. *Suppose that the entries of the* $\mathbb{C}^n-$*valued function* $\vec{f}$ *are convergent power series in* $(x,\vec{y})$ *with coefficients in* $\mathbb{C}$. *If matrices* $\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}I_n - A(0)$ *be invertible for positive integers* m *sufficiently large, where* $A(0)=\frac{\partial \vec{f}}{\partial \vec{y}}(0,0)$. *Then the formal solution*

$$\vec{\phi}(x)=\sum_{m=1}^{\infty} x^m \vec{a}_m \in x\mathbb{C}[[x]]^n, \tag{4.1}$$

of system (1.1) *is unique.*

Proof. Since formal power series (4.1) satisfy (1.1), i.e.,

$$x^{\alpha}\frac{d^{\alpha}\vec{\phi}(x)}{dx^{\alpha}}=\vec{f}(x,\vec{\phi}(x)). \tag{4.2}$$

It is necessary that $\vec{f}(0,0)=\vec{0}$. Therefore, write $\vec{f}$ in the form

$$\vec{f}(x,\vec{y})=\vec{f}_0(x)+A(x)\vec{y}+\sum_{|P|\geq 2}\vec{f}_P(x), \tag{4.3}$$

where

(1) $P=(p_1,\cdots,p_2)$ and p_j are non-negative integers,

(2) $|P|=p_1+\cdots+p_n$ and $\vec{y}^P=y_1^{p_1}\cdots y_n^{p_n}$,

(3) $\vec{f}_0\in x\mathbb{C}\{x\}^n$ and $\vec{f}_P\in\mathbb{C}\{x\}^n$,

(4) $A(x)$ is $n\times n$ matrix with the entries in $\mathbb{C}\{x\}$.

Note that $\vec{f}_0(x)=\vec{f}(x,0), A(x)=\frac{\partial \vec{f}}{\partial \vec{y}}(x,0)$ and $A(x)=\sum_0^{\infty}x^m A_m$, where A_m are in $\mathcal{M}_n(\mathbb{C})$, write Eq. (4.2) in the form

$$x^{\alpha}\frac{d^{\alpha}\vec{y}}{dx^{\alpha}}=A_0\vec{\phi}+\vec{f}(x,\vec{\phi})-A_0\vec{\phi}.$$

Then,

$$\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}\vec{c}_m=A_0\vec{c}_m+\vec{\gamma}_m \quad \text{for} \ \ m=1,2,\cdots, \tag{4.4}$$

where

$$\begin{aligned}\vec{f}(x,\vec{\phi})-A_0\vec{\phi}=&\vec{f}_0(x)+[A(x)-A_0]\vec{\phi}(x)+\sum_{|P|\geq 2}\left(\vec{\phi}(x)\right)^P\vec{f}_P(x)\\ =&\sum_{m=1}^{\infty}x^m\vec{\gamma}_m\end{aligned}$$

and $\vec{\gamma}_m\in\mathbb{C}^n$. Note that $\vec{\gamma}_m$ is determined when $\vec{c}_1,\cdots,\vec{c}_{m-1}$ are determined and that the matrices $\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}I_n-A_0$ are invertible if positive integers m are sufficiently large. This implies that there exists a positive integers m_0 such that if $\vec{c}_1,\cdots,\vec{c}_{m_0}$ are determined, then $\vec{c}_m$ is uniquely determined for all integers m greater than m_0. Therefor, the system of a finite number of equations

$$\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}\vec{c}_m=A_0\vec{c}_m+\vec{\gamma}_m,\quad m=1,2,\cdots,m_0, \tag{4.5}$$

decides whether a formal solution $\vec{\phi}(x)$ exists. If system (4.5) has a solution $\{\vec{c}_1,\cdots,\vec{c}_{m_0}\}$, those m_0 constants vectors determine a formal solution $\vec{\phi}(x)$ uniquely. Thus, the proof is completed. □

Remark 4.1. Supposing that formal power series (4.1) is a formal solution of (1.1), set

$$\vec{\phi}_N(x)=\sum_{m=1}^{N}x^m\vec{c}_m. \tag{4.6}$$

Since

$$\vec{f}(x,\vec{\phi}(x))-\vec{f}(x,\vec{\phi}_N(x))=A(x)(\vec{\phi}(x)-\vec{\phi}_N(x))+\sum_{|P|\geq 2}\left[\vec{\phi}(x)^P-\vec{\phi}_N(x)^P\right]\vec{f}_P(x),$$

it follows that $\vec{f}(x,\vec{\phi}(x))-\vec{f}(x,\vec{\phi}_N(x))\in x^{N+1}\mathbb{C}[[x]]^n$. Also,

$$\vec{f}(x,\vec{\phi}(x))-x^\alpha\frac{d^\alpha\vec{\phi}_N(x)}{dx^\alpha}=x^\alpha\frac{d^\alpha\vec{\phi}(x)}{dx^\alpha}-x^\alpha\frac{d^\alpha\vec{\phi}_N(x)}{dx^\alpha}\in x^{N+1}\mathbb{C}[[x]]^n.$$

Hence,

$$\vec{f}(x,\vec{\phi}_N(x))-x^\alpha\frac{d^\alpha\vec{\phi}_N(x)}{dx^\alpha}\in x^{N+1}\mathbb{C}[[x]]^n.$$

Set

$$\vec{g}_{N,0}(x)=\vec{f}(x,\vec{\phi}_N(x))-x^\alpha\frac{d^\alpha\vec{\phi}_N(x)}{dx^\alpha}. \tag{4.7}$$

Now, by means of the transformation $\vec{y}=\vec{z}+\vec{\phi}_N(x)$, change system (1.1) to the system

$$x^\alpha \frac{d^\alpha \vec{z}}{dx^\alpha}=\vec{g}_N(x,\vec{z}), \quad \vec{z}\in\mathbb{C}, \tag{4.8}$$

where

$$\begin{aligned}\vec{g}_N(x,\vec{z})=&\vec{f}(x,\vec{z}+\vec{\phi}_N(x))-x^\alpha\frac{d^\alpha\vec{\phi}_N(x)}{dx^\alpha}\\ =&\vec{g}_{N,0}(x)+\vec{f}(x,\vec{z}+\vec{\phi}_N(x))-\vec{f}(x,\vec{\phi}_N(x))\\ &+\vec{g}_{N,0}(x)+A(x)\vec{z}+\sum_{|P|\geq 2}\Big[\big(\vec{z}+\vec{\phi}_N(x)\big)^P-\vec{\phi}_N(x)^P\Big]\vec{f}_P(x).\end{aligned}$$

As in Lemma 4.1, write $\vec{g}_N(x)$ in the form

$$\vec{g}_N(x,\vec{z})=\vec{g}_{N,0}(x)+B_N(x)\vec{z}+\sum_{|P|\geq 2}(\vec{z})^P\vec{g}_{N,P}(x),$$

where

(1) $\vec{g}_{N,0}(x)\in x^{N+1}\mathbb{C}\{x\}^n$ and $\vec{g}_{N,P}\in\mathbb{C}\{x\}^n$,

(2) $B_N(x)$ is an $n\times n$ matrix with the entries in $\mathbb{C}\{x\}$,

(3) the entries of the matrix $B_N(x)-A_0$ are contained in $x\mathbb{C}\{x\}$.

Remark 4.2. Supposing that system (4.8) has a formal solution, then using (4.6) we obtain

$$\vec{\psi}_N(x):=\vec{\phi}(x)-\vec{\phi}_N(x)=\sum_{m=N+1}^{\infty}x^m\vec{c}_m\in x^{N+1}\mathbb{C}[[x]]^n.$$

If we substituting $\vec{\psi}_N(x)$ into system (4.8). Then the coefficients $\vec{c}_m$ determined recursively by

$$\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}\vec{c}_m=A_0\vec{c}_m+\vec{\gamma}_m, \quad m=N+1,N+2,\cdots,$$

where

$$\begin{aligned}&\vec{g}_N(x,\vec{\psi}(x))-A_0\vec{\psi}_N(x)\\ =&\vec{f}(x,\phi(x))-A_0\vec{\psi}_N(x)-x^\alpha\frac{d^\alpha\vec{\phi}_N(x)}{dx^\alpha}\\ =&\vec{f}(x,\phi(x))-A_0\vec{\phi}(x)+A_0\vec{\phi}_N(x)-x^\alpha\frac{d^\alpha\vec{\phi}_N(x)}{dx^\alpha}\\ =&\vec{g}_{N,0}(x)+[B_N(x)-A_0]\vec{\psi}_N(x)+\sum_{|P|\geq 2}\big(\vec{\psi}_N(x)\big)^P\vec{g}_{N,P}(x)\\ =&\sum_{m=N+1}^{\infty}x^m\vec{\gamma}_m.\end{aligned}$$

Note that $\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}I_n-A_0$ are invertible for $m=N+1,N+2,\cdots$ if N is sufficiently large.

Remark 4.3. Suppose that system (1.1) has an actual solution $\vec{\eta}(x)$ such that the entries of $\vec{\eta}(x)$ are analytic at $x=0$ and that $\vec{\eta}(0)=0$. Then, the Taylor expansion

$$\vec{\phi}(x)=\sum_{m=1}^{\infty}\frac{d^m\vec{\eta}(0)}{dx^m}\frac{x^m}{m!}$$

of $\vec{\eta}(x)$ at $x=0$ is a formal solution of system (1.1). Furthermore, $\vec{\phi}$ is convergent and $\vec{\phi}\in x\mathbb{C}\{x\}^n$.

Kipping these Lemma and Remarks from above, let us prove the following main theorem.

Theorem 4.1. *Suppose that* $\vec{f}_0(x)=\vec{f}(x,\vec{0})\in x^{N+1}\mathbb{C}\{x\}^n$ *and that the matrices* $\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}I_n-A_0,m=N+1,N+2,\cdots$ *are invertible, where* $A_0=\frac{\partial\vec{f}}{\partial\vec{y}}(0,\vec{0})$*. Then, system* (1.1) *has a unique formal solution*

$$\vec{\psi}=\sum_{m=N+1}^{\infty}x^m\vec{c_m}\in x^{N+1}\mathbb{C}[[x]]^n. \tag{4.9}$$

Furthermore, $\vec{\phi}(x)\in x^{N+1}\mathbb{C}\{x\}^n$*.*

Proof. We prove this theorem in six steps.

Step 1. Using the argument of Lemma 4.1, we can prove the existence and uniqueness of formal solution (4.9). In fact,

$$\begin{aligned}\vec{f}(x,\vec{\phi})-A_0\vec{\phi}&=\vec{f}_0(x)+[A(x)-A_0]\vec{\phi}(x)+\sum_{|P|\ge 2}\left(\vec{\phi}(x)\right)^P\vec{f}_P(x)\\&=\sum_{m=1}^{\infty}x^m\vec{\gamma}_m.\end{aligned}$$

This implies that $\vec{\gamma}=0$ for $m=1,2,\cdots,N$. Hence, $\vec{c}_m,m=N+1,N+2,\cdots$ are uniquely determined by

$$\frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}\vec{c}_m=A_0\vec{c}_m+\vec{\gamma}_m\quad\text{for } m=N+1,N+2,\cdots.$$

Step 2. Suppose that system (1.1) has an actual solution $\vec{\eta}(x)$ satisfying the following conditions:

(i) The entries of $\vec{\eta}(x)$ are analytic at $x=0$,

(ii) There exist two positive numbers K and δ such that

$$|\vec{\eta}(x)|\le K|x|^{N+1}\quad\text{for } |x|\le\delta.$$

Then, the Taylor expansion

$$\sum_{m=N+1}^{\infty} \frac{d^m \vec{\eta}(0)}{dx^m} \frac{x^m}{m!}$$

of $\vec{\eta}(x)$ at $x=0$ is a formal solution of Eq. (1.1). Since, such a formal solution is unique, it follows that

$$\vec{\phi}(x) = \sum_{m=N+1}^{\infty} \frac{d^m \vec{\eta}(0)}{dx^m} \frac{x^m}{m!}.$$

Because the Taylor expansion of $\vec{\eta}(x)$ at $x=0$ is convergent, the formal solution $\vec{\phi}$ convergent and $\vec{\phi} \in x^{N+1}\mathbb{C}\{x\}^n$.

Step 3. Hereafter, we shall construct an actual solution $\vec{\eta}(x)$ of (1.1) that satisfies conditions (i) and (ii) of Step 2. To do this, first notice that there exist three positive numbers H, δ and ρ such that

$$|\vec{f}(x,\vec{0})| \leq H|x|^{N+1}, \quad |x| \leq \delta, \tag{4.10}$$

and

$$|\vec{f}(x,\vec{y}_1) - \vec{f}(x,\vec{y}_2)| \leq (|A_0|+1)|\vec{y}_1 - \vec{y}_2|, \quad |x| \leq \delta, \quad |\vec{y}_j| \leq \rho, \quad j=1,2. \tag{4.11}$$

Hence,

$$|\vec{f}(x,\vec{y}) \leq H|x|^{N+1} + (|A_0|+1)|\vec{y} \quad \text{for } |x| \leq \delta \quad \text{and} \quad |\vec{y}| \leq \rho. \tag{4.12}$$

Using the transformation of Remark 4.2, N can be made as large as we want without changing the matrix A_0. Hence, assume without loss of any generality that

$$\frac{|A_0|+1}{(N-\alpha+2)\Gamma(\alpha)} < \frac{1}{2}. \tag{4.13}$$

Also, fix two positive numbers K and δ so that

$$K > \frac{H + (|A_0|+1)K}{(N+2-\alpha)\Gamma(\alpha)} \quad \text{and} \quad K\delta^{N+1} \leq \rho. \tag{4.14}$$

Step 4. Using Theorem 4.3 in [14] and $\vec{\eta}(0)=0$, the system (1.1) is equivalent to integral equation

$$\vec{\eta}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\vec{f}(s,\vec{\eta}(s))}{s^\alpha (x-s)^{1-\alpha}} ds. \tag{4.15}$$

Define successive approximations

$$\vec{\eta}_0(x)=0 \quad \text{and} \quad \vec{\eta}_{k+1}(x)=\frac{1}{\Gamma(\alpha)}\int_0^x \frac{\vec{f}(s,\vec{\eta}_k(s))}{s^\alpha(x-s)^{1-\alpha}}ds, \quad k=0,1,\cdots.$$

Now, we shall show that

$$|\vec{\eta}_k(x)\le K|x|^{N+1} \quad \text{for} \ |x|\le\delta \quad \text{and} \quad k=0,1,\cdots.$$

Since this is true for $k=0$, we show this recursively with respect to k as follows.

First if this is true for k, then

$$|\vec{\eta}_k(x)\le K|x|^{N+1}\le K\delta^{N+1}\le\rho, \quad |x|\le\delta.$$

Hence, (4.12) with assuming $\alpha\ge 1$ yields

$$\begin{aligned}\left|\frac{1}{s^\alpha(x-s)^{1-\alpha}}\vec{f}(s,\vec{\eta}_k(s))\right| &\le\frac{1}{|s|^\alpha|x-s|^{1-\alpha}}\{H|s|^{N+1}+(|A_0|+1)|\vec{\eta}_k|\}\\ &\le\frac{|x-s|^{\alpha-1}}{|s|^\alpha}\{H|s|^{N+1}+(|A_0|+1)K|s|^{N+1}\}\\ &\le|x|^{\alpha-1}\{H+(|A_0|+1)K\}|s|^{N+1-\alpha}, \quad |s|\le\delta.\end{aligned}$$

Therefore, using (4.14) we have

$$|\vec{\eta}_{k+1}|\le|x|^{\alpha-1}\frac{(H+(|A_0|+1)K}{(N+2-\alpha)\Gamma(\alpha)}|x|^{N+2-\alpha}\le K|x|^{N+1}, \quad |x|\le\delta.$$

Step 5. Set

$$\|\vec{\eta}_{k+1}-\vec{\eta}_k\|=\max\left\{\frac{|\vec{\eta}_{k+1}-\vec{\eta}_k|}{|x|^{N+1}}:|x|\le\delta\right\}.$$

Thus, using (4.11) since

$$\begin{aligned}|\vec{\eta}_{k+1}-\vec{\eta}_k| &=\left|\int_0^x\frac{(x-s)^{\alpha-1}}{s^\alpha\Gamma(\alpha)}\Big(\vec{f}(s,\vec{\eta}_k(s))-\vec{f}(s,\vec{\eta}_{k-1}(s))\Big)ds\right|\\ &\le\int_0^x\frac{(x-s)^{\alpha-1}}{s^\alpha\Gamma(\alpha)}\Big|\vec{f}(s,\vec{\eta}_k(s))-\vec{f}(s,\vec{\eta}_{k-1}(s))\Big|ds\\ &\le|A_0+1|\int_0^x\frac{|x-s|^{\alpha-1}}{|s|^\alpha}|\vec{\eta}_k(s)-\vec{\eta}_{k-1}(s)|ds\\ &\le|A_0+1||x|^{\alpha-1}\int_0^x\frac{1}{|s|^\alpha\Gamma(\alpha)}|\vec{\eta}_k(s))-\vec{\eta}_{k-1}(s))|ds\\ &\le\frac{|A_0+1|}{\Gamma(\alpha)}|x|^{\alpha-1}\int_0^x|s|^{N-\alpha+1}\max\left\{\frac{|\vec{\eta}_k(s))-\vec{\eta}_{k-1}(s))|}{|s|^{N+1}}\right\}ds\\ &\le\frac{|A_0+1|}{(N-\alpha+2)\Gamma(\alpha)}\|\vec{\eta}_k-\vec{\eta}_{k-1}\||x|^{N+1},\end{aligned}$$

using (4.13) we obtain

$$\|\vec{\eta}_{k+1}-\vec{\eta}_k\| \leq \frac{|A_0+1|}{(N-\alpha+2)\Gamma(\alpha)}\|\vec{\eta}_k-\vec{\eta}_{k-1}\| \leq \frac{1}{2}\|\vec{\eta}_k-\vec{\eta}_{k-1}\|.$$

This implies that

$$\lim_{k\to+\infty}\frac{\vec{\eta}_k(x)}{x^{N+1}} = \sum_{t=0}^{\infty}\frac{\vec{\eta}_{l+1}(x)-\vec{\eta}_l(x)}{x^{N+1}}$$

exists uniformly for $|x|\leq\delta$.

Step 6. Setting

$$\vec{\eta}(x) = x^{N+1}\left\{\lim_{k\to+\infty}\frac{\vec{\eta}_k(x)}{x^{N+1}}\right\} = \lim_{k\to+\infty}\vec{\eta}_k(x),$$

it is easy to show that $\vec{\eta}(x)$ satisfies integral equation (4.15). It is easy evident that $\vec{\eta}(x)$ is analytic for $|x|<\delta$. Thus, the proof is completed. □

Now, finally, by using the argument given in Remark 4.2 and Remark 4.3 we obtain the following theorem.

Theorem 4.2. *Every formal solution* $\vec{\phi}\in x\mathbb{C}[[x]]^n$ *of system* (1.1) *is convergent, i.e.,* $\vec{\phi}\in x\mathbb{C}\{x\}^n$.

Remark 4.4. In general, system (1.1) may not have any formal solution. However, Theorem (4.2) states that if system (1.1) has formal solution, then every formal solution is convergent.

Example 4.1. The following system of fractional differential equation

$$\begin{cases} x^{\frac{3}{2}}\left(\dfrac{d^{\frac{3}{2}}y_1}{dx^{\frac{3}{2}}}\right) = x^{15}\cos(x)+\frac{\Gamma(15)}{\Gamma(\frac{31}{2})}y_1+x\sin(y_2), \\ x^{\frac{3}{2}}\left(\dfrac{d^{\frac{3}{2}}y_2}{dx^{\frac{3}{2}}}\right) = y_1+10\sinh(y_2)+x^{20}\sinh(x), \end{cases} \tag{4.16}$$

satisfy in Theorem 4.1.

In this example we have

$$\vec{f}(x\,\vec{y}) = \left[\begin{array}{c} f_1 \\ f_2 \end{array}\right] = \left[\begin{array}{c} x^{15}\cos(x)+\dfrac{\Gamma(14)}{\Gamma(\frac{31}{2})}y_1+x\sin(y_2) \\ y_1+10\sinh(y_2)+x^{20}\sinh(x) \end{array}\right].$$

Hence,

$$\vec{f}_0(x) = \vec{f}(x,\vec{0}) = \left[\begin{array}{c} x^{15}\cos(x) \\ x^{20}\sinh(x) \end{array}\right] \in x^{15}\mathbb{C}\{x\}^2,$$

and

$$A_0=\frac{\partial \vec{f}}{\partial \vec{y}}(0,\vec{0})=\begin{bmatrix} \frac{\Gamma(14)}{\Gamma\left(\frac{31}{2}\right)} & 0 \\ 1 & 10 \end{bmatrix},$$

such that

$$\frac{\Gamma(m+1)}{\Gamma(m-\frac{3}{2}+1)}I_2-A_0=\frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})}I_2-A_0$$

is invertible for all $m\geq 15$.

5 Conclusions

The existence of solution for a singularities of a system of fractional differential equations (FDE) comprising of standard Caputo derivatives have been discussed. A formal power series solution for our considering FDE and its convergence under conditions have been worked out.

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