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Maximum Modulus of Polynomials

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Abstract. Let

$$P(z)=\sum_{j=0}^{n}a_jz^j$$

be a polynomial of degree n and let $M(P,r)=\max_{|z|=r}|P(z)|$. If $P(z)\neq 0$ in $|z|<1$, then

$$M(P,r)\geq\left(\frac{1+r}{1+\rho}\right)^n M(P,\rho).$$

The result is best possible. In this paper we shall present a refinement of this result and some other related results.

Key Words: Maximum modulus, growth of polynomial, derivative.

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1 Introduction and statement of results

Let

$$P(z)=\sum_{j=0}^{n}a_jz^j$$

be a polynomial of degree n, let

$$M(P,r)=\max_{|z|=r}|P(z)| \quad \text{and} \quad m(P,1)=\min_{|z|=1}|P(z)|,$$

then concerning the size of $M(P,r)$ the following results are well known.

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Theorem 1.1 (Bernstein [3]). *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, then*

$$M(P,R)\leq R^nM(P,1) \quad for \;\; R\geq 1 \tag{1.1}$$

with equality only for $P(z)=\lambda z^n$.

Theorem 1.2 (Zarantauello and Verga [6]). *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, then*

$$M(P,r)\geq r^nM(P,1) \quad for \;\; r\leq 1 \tag{1.2}$$

with equality only for $P(z)=\lambda z^n$.

For polynomials not vanishing in $|z|<1$, Rivilin [5] proved:

Theorem 1.3. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, $P(z)\neq 0$ in $|z|<1$, then*

$$M(P,r)\geq\left(\frac{1+r}{2}\right)^nM(P,1) \quad for \;\; r\leq 1. \tag{1.3}$$

The result is best possible with equality only for the polynomial

$$P(z)=\left(\frac{\lambda+\mu z}{2}\right)^n, \quad |\lambda|=|\mu|.$$

Govil [2] has proved the following generalization of Theorem 1.3.

Theorem 1.4. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zero in $|z|<1$, then for $0\leq r\leq\rho\leq 1$,*

$$M(P,r)\geq\left(\frac{1+r}{1+\rho}\right)^nM(P,\rho). \tag{1.4}$$

The result is best possible and equality holds for the polynomial

$$P(z)=\left(\frac{1+z}{1+\rho}\right)^n.$$

He has shown that the bound can be improved if $P'(0)=0$ and proved:

Theorem 1.5. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, having no zero in $|z|<1$, $P'(0)=0$ then for $0\leq r\leq\rho\leq 1$,*

$$M(P,r)\geq\left(\frac{1+r}{1+\rho}\right)^n\left\{\frac{1}{1-\frac{(1-\rho)(\rho-r)n}{4}\left(\frac{1+r}{1+\rho}\right)^{n-1}}\right\}M(P,\rho). \tag{1.5}$$

In this paper, we shall present the following refinements of Theorems 1.4 and 1.5. Here we prove:

2 Main results

Theorem 2.1. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, having no zero in $|z|<1$, then for $0\le r\le\rho\le 1$,*

$$\max_{|z|=r}|P(z)|\ge\left(\frac{1+r}{1+\rho}\right)^n\max_{|z|=\rho}|P(z)|-\left(\frac{1+r}{1+\rho}\right)^n\min_{|z|=1}|P(z)|. \tag{2.1}$$

The result is best possible and equality holds for

$$P(z)=\Big(\frac{1+z}{1+\rho}\Big)^n.$$

If polynomial $P(z)$ has all its zeros on $|z|=1$, the polynomial $q(z)=z^n\overline{P(\frac{1}{\bar z})}$ also has all its zeros on $|z|=1$. Further if $1\le\rho\le r$ then $\frac{1}{r}\le 1$, $\frac{1}{\rho}\le 1$ and we can apply (2.2) to the polynomial $q(z)$ and obtain

$$M\left(q,\frac{1}{r}\right)\ge\left(\frac{1+\frac{1}{r}}{1+\frac{1}{\rho}}\right)^n M\left(q,\frac{1}{\rho}\right)-\left(\frac{1+\frac{1}{r}}{1+\frac{1}{\rho}}\right)^n m(P,1),$$

which is equivalent to

$$M(P,r)\ge\left(\frac{1+r}{1+\rho}\right)M(P,\rho)-\left(\frac{r+1}{\rho+1}\right)m(P,1),$$

we thus obtain

Corollary 2.1. If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, having all its zeros on $|z|=1$, then for $0\le r\le\rho\le 1$, and $1\le\rho\le r$,

$$M(P,r)\ge\left(\frac{1+r}{1+\rho}\right)^n M(P,\rho)-\left(\frac{r+1}{\rho+1}\right)m(P,1).$$

The result is best possible and equality holds for $P(z)=\big(\frac{1+z}{1+\rho}\big)^n$.

Theorem 2.2. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n, having no zero in $|z|<1$, then for $0\le r\le\rho\le 1$,*

$$\begin{aligned}M(P,r)\ge&\left(\frac{1+r}{1+\rho}\right)^n\left[1-\frac{(1-\rho)(\rho-r)n}{4}\left(\frac{1+r}{1+\rho}\right)^{n-1}\right]^{-1}M(P,\rho)\\&+\frac{n}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right)m(P,1).\end{aligned} \tag{2.2}$$

The result is best possible and equality holds for $P(z)=\big(\frac{1+z}{1+\rho}\big)^n$.

For the proofs of these theorems we need the following lemmas.

3 Lemmas

The first Lemma is due to Aziz and Mohammad [1].

Lemma 3.1. *If $P(z)$ is a polynomial of degree n with $\max_{|z|=1}|P(z)|=1$, then*

$$|P(Re^{i\theta})|+|Q(Re^{i\theta})|\le R^n+1, \quad 0\le\theta\le 2\pi,$$

where

$$Q(z)=z^n\overline{P\Big(\frac{1}{\overline{z}}\Big)}, \quad R\ge 1.$$

The following Lemma is of independent interest.

Lemma 3.2. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, having no zero on $|z|\le k$, $k\ge 1$, $P'(0)=0$, then*

$$\max_{|z|=1}|P'(z)|\le \frac{n}{1+k^2}\max_{|z|=1}|P(z)|-\frac{n}{1+k^2}\min_{|z|=1}|P(z)|. \tag{3.1}$$

The result is best possible and equality holds for

$$P(z)=\Big(\frac{z^2+k^2}{1+k^2}\Big)^{\frac{n}{2}},$$

n being even.

Proof. If $m=\min_{|z|=k}|P(z)|$, then $m\le|P(z)|$ for $|z|=k$, $k\ge 1$. Since all the zeros of $P(z)$ lie in $|z|>k$, therefore for every complex number α such that $|\alpha|<1$. It follows by Rouche's Theorem for $m\ge 0$,that the polynomial $F(z)=P(z)-\alpha m$ does not vanish in $|z|<k$.

$$F'(z)=P'(z), \quad \text{which gives} \quad F'(0)=P'(0).$$

If we define $Q(z)=z^n\overline{P(\frac{1}{\overline{z}})}$ and $G(z)=z^n\overline{F(\frac{1}{\overline{z}})}$, then we have $G(z)=Q(z)-\overline{\alpha}mz^n$.

Since

$$k^2|F'(z)|\le|G'(z)|,$$

which gives,

$$k^2|P'(z)|\le|Q'(z)-n\overline{\alpha}z^{n-1}m| \quad \text{for} \quad |z|=1.$$

Choosing an argument of α such that

$$|Q'(z)-n\overline{\alpha}z^{n-1}m|=|Q'(z)|-n|\alpha|m \quad \text{for} \quad |z|=1.$$

It follows that

$$k^2|P'(z)| \le |Q'(z)| - n|\alpha|m.$$

Letting $|\alpha| \to 1$, we obtain

$$k^2|P'(z)| \le |Q'(z)| - nm. \tag{3.2}$$

If $P(z)$ is a polynomial of degree n, then it follows by Lemma 3.1,

$$|P'(z)| + |nP(z) - zP'(z)| \le n \max_{|z|=1} |P(z)|. \tag{3.3}$$

Since

$$|Q'(z)| = |nP(z) - zP'(z)| \quad \text{for} \quad |z| = 1.$$

It follows from (3.3) that

$$|P'(z)| + |Q'(z)| \le n \max_{|z|=1} |P(z)|. \tag{3.4}$$

Now inequality (3.2) gives with the help of (3.4) that

$$\begin{aligned}(1+k^2)|P'(z)| &\le |P'(z)| + Q'(z)| - nm \\ &\le n\Big\{\max_{|z|=1}|P(z)| - \min_{|z|=k}|P(z)|\Big\}.\end{aligned}$$

Hence

$$|P'(z)| \le \frac{n}{(1+k^2)} \max_{|z|=1}|P(z)| - \frac{n}{(1+k^2)} \min_{|z|=k}|P(z)|.$$

Which proves Lemma 3.2. □

4 Proofs of theorems

Proof of Theorem 2.1. By hypothesis polynomial $P(z)$ has all its zeros in $|z| \ge 1$ and $m = \min_{|z|=1}|P(z)|$, therefore $m \le |P(z)|$ for $|z| \le 1$. we show for any given complex number α with $|\alpha| \le 1$, the polynomial $F(z) = P(z) + \alpha m$ has all its zeros in $|z| \ge 1$. This is obvious if $m = 0$ that is if $P(z)$ has a zero on $|z| = 1$, we now suppose that all the zeros of $P(z)$lie in $|z| > 1$, so that $m = \min_{|z|=1}|P(z)| > 0$. Hence $\frac{m}{P(z)}$ is analytic for $|z| \le 1$ and $|\frac{m}{P(z)}| \le 1$ for $|z| = 1$. Since $\frac{m}{P(z)}$ is not a constant, it follows by Maximum Modulus Principle that

$$m < |P(z)| \quad \text{for} \quad |z| < 1. \tag{4.1}$$

Now assume that $F(z)=P(z)+\alpha m$ has a zero in $|z|<1$ say at $z=z_0$ with $|z_0|<1$, then

$$P(z_0)+\alpha m=F(z_0)=0.$$

This implies,

$$|P(z_0)|=|\alpha m|\leq m.$$

Which is a contradiction to (4.1). Hence we conclude that in any case $F(z)=P(z)+\alpha m$ has all its zeros in $|z|\geq 1$.

Let $R_1e^{i\theta_1},R_2e^{i\theta_2},\cdots,R_ne^{i\theta_n}$ be the zeros of $F(z)$, then $R_j\geq 1$, $j=1,2,\cdots,n$ and we have

$$F(z)=c\prod_{j=1}^{n}(z-R_je^{i\theta_j}),$$

where $R_j\geq 1$, $j=1,2,\cdots,n$, therefore for $0\leq\theta<2\pi$ and $0\leq r\leq\rho\leq 1$, we have clearly

$$\begin{aligned}\left|\frac{F(re^{i\theta})}{F(\rho e^{i\theta})}\right|&=\prod_{j=1}^{n}\left|\frac{re^{i\theta}-R_je^{i\theta_j}}{\rho e^{i\theta}-R_je^{i\theta_j}}\right|=\prod_{j=1}^{n}\left|\frac{re^{i(\theta-\theta_j)}-R_j}{\rho e^{i(\theta-\theta_j)}-R_j}\right|\\&=\prod_{j=1}^{n}\left\{\frac{(r^2+R_j^2-2rR_jcos(\theta-\theta_j))}{(\rho^2+R_j^2-2\rho R_jcos(\theta-\theta_j))}\right\}^{\frac{1}{2}}\\&\geq\prod_{j=1}^{n}\left(\frac{r+R}{\rho+R}\right)=\left(\frac{r+R}{\rho+R}\right)^n.\end{aligned}$$

This implies

$$|F(re^{i\theta})|\geq\left(\frac{r+R}{\rho+R}\right)^n|F(\rho e^{i\theta})|\quad\text{for }\ 0\leq\theta<2\pi\quad\text{and}\quad 0\leq r\leq\rho\leq 1.$$

Replacing $F(z)$ by $P(z)-\alpha m$, we get

$$|P(rz)-m\alpha|\geq\left(\frac{r+1}{\rho+1}\right)^n|P(\rho z)-m\alpha|. \tag{4.2}$$

Now for every θ with $0\leq\theta<2\pi$ and for every α with $|\alpha|<1$, we choose the argument of α such that

$$|P(re^{i\theta})-m\alpha|=|P(re^{i\theta})|-m|\alpha|, \tag{4.3}$$

which is possible because $P(z)\neq 0$ in $|z|<1$, and hence

$$m\leq|P(re^{i\theta})|\quad\text{for every }\ \theta,\quad 0\leq\theta<2\pi,\quad 0\leq r\leq 1.$$

Using (4.3) in (4.2), we get

$$|P(rz)|-m|\alpha|\geq\left(\frac{r+1}{\rho+1}\right)^n\Big\{|P(\rho z)|-m|\alpha|\Big\}$$

or

$$\big|P(rz)\big|\geq\left(\frac{r+1}{\rho+1}\right)^n\big|P(\rho z)\big|-m|\alpha|\left[1-\left(\frac{r+1}{\rho+1}\right)^n\right].$$

Letting $|\alpha|\to 1$, we get

$$\big|P(rz)\big|\geq\left(\frac{r+1}{\rho+1}\right)^n\big|P(\rho z)\big|-\left[1-\left(\frac{r+1}{\rho+1}\right)^n\right]m.$$

So, we complete the proof. □

Proof of Theorem 2.2. If $P(z)$ has no zeros in $|z|<1$ and $0\leq t<1$, then $P(z)=p(tz)$, has no zeros in $|z|<\frac{1}{t}$ where $t\geq 1$. Also $P'(0)=tp'(0)=0$. Hence by the above Lemma

$$\max_{|z|=1}\big|tp'(tz)\big|\leq\frac{n}{1+\frac{1}{t^2}}\max_{|z|=1}\big|p(tz)\big|-\frac{n}{1+\frac{1}{t^2}}\min_{|z|=\frac{1}{t}}\big|p(tz)\big|.$$

For $0\leq r\leq\rho\leq 1$, we have

$$\begin{aligned}\big|p(\rho e^{i\theta})-p(re^{i\theta})\big|&=\left|\int_r^\rho p'(te^{i\theta})e^{i\theta}dt\right|\leq\int_r^\rho\big|p'(te^{i\theta})\big|dt\\&\leq\int_r^\rho\frac{nt}{1+t^2}\left\{\max_{|z|=t}\big|P(z)\big|-\min_{|z|=1}\big|P(z)\big|\right\}\\&\leq\int_r^\rho\frac{nt}{1+t^2}\left\{\left(\frac{1+t}{1+r}\right)^nM(P,r)-m(P,1)\right\}\\&\leq\int_r^\rho\frac{t(1+t)^n}{1+t^2}dt\left\{\frac{n}{(1+r)^n}M(P,r)\right\}-m(P,1)\int_r^\rho\frac{nt}{1+t^2}dt\\&\leq\frac{nM(P,r)\rho(1+\rho)}{(1+r)^n(1+\rho^2)}\int_r^\rho(1+t)^{n-1}dt-n\frac{m(P,1)}{2}\int_r^\rho\frac{2t}{1+t^2}dt\\&=\frac{nM(P,r)\rho(1+\rho)}{(1+r)^n(1+\rho^2)}\left\{\frac{(1+\rho)^n-(1+r)^n}{n}\right\}-n\frac{m(P,1)}{2}\Big|\log(1+t^2)\Big|_r^\rho\\&=\frac{nM(P,r)\rho(1+\rho)}{(1+r)^n(1+\rho^2)}\left\{\frac{(1+\rho)^n-(1+r)^n}{n}\right\}-n\frac{m(P,1)}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right).\end{aligned}$$

Which gives for $0\leq r\leq\rho\leq 1$,

$$M(P,\rho)\leq\left\{1+\frac{\rho(1+\rho)^{n+1}}{(1+r)^n(1+\rho^2)}\left[1-\left(\frac{1+r}{1+\rho}\right)^n\right]\right\}M(P,r)-\frac{n}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right)m(P,1)$$

$$
\begin{aligned}
&=\left[\frac{1-\rho}{1+\rho^2}+\frac{(\rho+\rho^2)}{1+\rho^2}\left(\frac{1+\rho}{1+r}\right)^n\right]M(P,r)-\frac{n}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right)m(P,1)\\
&=\left(\frac{1+\rho}{1+r}\right)^n\left[1-\left(\frac{1-\rho}{1+\rho^2}\right)\left\{1-\left(\frac{1+r}{1+\rho}\right)^n\right\}\right]M(P,r)-\frac{n}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right)m(P,1)\\
&=\left(\frac{1+\rho}{1+r}\right)^n\left[1-\frac{(1-\rho)(\rho-r)}{(1+\rho^2)(1+\rho)\big(1-(\frac{1+r}{1+\rho})\big)}\left\{1-\left(\frac{1+r}{1+\rho}\right)^n\right\}\right]M(P,r)\\
&\qquad-\frac{n}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right)m(P,1)\\
&\leq\left(\frac{1+\rho}{1+r}\right)^n\left[1-\frac{(1-\rho)(\rho-r)n}{(1+\rho^2)(1+\rho)}\left(\frac{1+r}{1+\rho}\right)^{n-1}\right]M(P,r)-\frac{n}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right)m(P,1)\\
&\leq\left(\frac{1+\rho}{1+r}\right)^n\left[1-\frac{(1-\rho)(\rho-r)n}{4}\left(\frac{1+r}{1+\rho}\right)^{n-1}\right]M(P,r)-\frac{n}{2}\log\left(\frac{1+\rho^2}{1+r^2}\right)m(P,1).
\end{aligned}
$$

From which the result follows. □

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