Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 2 (2017), pp. 118-133

DOI: 10.4208/ata.2017.v33.n2.3

New Fixed Point Results of Generalized g-Quasi-Contractions in Cone b-Metric Spaces Over Banach Algebras

Shaoyuan Xu[1,*], Suyu Cheng[2], Suzana Aleksić[3] and Yongjie Piao[4]

[1] *School of Mathematics and Statistics, Hanshan Normal University, Chaozhou 521041, China*
[2] *Library, Hanshan Normal University, Chaozhou 521041, China*
[3] *Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia*
[4] *Department of Mathematics, College of Science, Yanbian University, Yanji 1330022, Jilin, China*

Received 1 September 2016; Accepted (in revised version) 13 January 2017

Abstract. In this paper, we introduce the concept of generalized g-quasi-contractions in the setting of cone b-metric spaces over Banach algebras. By omitting the assumption of normality we establish common fixed point theorems for the generalized g-quasi-contractions with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda)\in[0,\frac{1}{s})$ in the setting of cone b-metric spaces over Banach algebras, where the coefficient s satisfies $s\geq 1$. The main results generalize, extend and unify several well-known comparable results in the literature.

Key Words: Cone b-metric spaces over Banach algebras, non-normal cones, c-sequences, generalized g-quasi-contractions, fixed point theorems.

AMS Subject Classifications: 54H25, 47H10

1 Introduction

Huang and Zhang [1] introduced the concept of cone metric space, proved the properties of sequences on cone metric spaces and obtained various fixed point theorems for contractive mappings. The existence of a common fixed point on cone metric space was considered in [2–5]. Also, Ilić and Rakočević [8] introduced quasi-contraction on cone metric space when the underlying cone is normal. Later on, Kadelburg et al. [7] obtained a fixed point result without the normality of the underlying cone, but only in the case of a quasi-contractive constant $\lambda\in[0,1/2)$ (see [7, Theorem 2.2]). However, Gajić and

*Corresponding author. *Email addresses:* `xushaoyuan@126.com` (S. Xu), `chengsuyu1992@126.com` (S. Cheng), `suzanasimic@kg.ac.rs` (S. Aleksić), `sxpyj@ybu.edu.cn` (Y. Piao)

Rakočević [6] proved that result is true for $\lambda\in[0,1)$ on cone metric spaces which answered the open question whether the result is true for $\lambda\in[0,1)$. Recently, Hussain and Shah [13] introduced cone b-metric spaces, as a generalization of b-metric spaces and cone metric spaces, and established some important topological properties in such spaces. Following Hussain and Shah, Huang and Xu [10] obtained some interesting fixed point results for contractive mappings in cone b-metric spaces. Inspired by [6], Shi and Xu [31] presented a similar common fixed point result in the case of the contractive constant $\lambda\in[0,1/s)$ in cone b-metric spaces without the assumption of normality (see [31]. Similar results can be seen in [32].

Let (X,d) be a complete metric space. Recall that a mapping $T:X\to X$ is called a quasi-contraction if, for some $k\in[0,1)$ and for all $x,y\in X$, one has

$$d(Tx,Ty)\le k\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.$$

Ćirić [21] introduced and studied quasi-contractions as one of the most general classes of contractive-type mappings. He proved the well-known theorem that any quasi-contraction T has a unique fixed point. Recently, scholars obtained various similar results on cone metric spaces. See, for instance, [6–8].

Recently, some authors investigated the problem of whether cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings involved. They used to establish the equivalence between some fixed point results in metric and in (topological vector spaces valued) cone metric spaces (see [18,19,26,27,36,37]). Very recently, Liu and Xu [22] introduced the concept of cone metric spaces with Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. Although they proved some fixed point theorems of quasi-contractions, the proof relied strongly on the assumption that the underlying cone is normal. We may state that it is significant to introduce the concept of cone metric spaces with Banach algebras (which we call in this paper cone metric spaces over Banach algebras). This is because there are examples to show that one is unable to conclude that the cone metric space (X,d) with a Banach algebra $\mathcal{A}$ discussed is equivalent to the metric space (X,d^*), where the metric d^* is defined by $d^*=\xi_e\circ d$, here the nonlinear scalarization function $\xi_e:A\to\mathbb{R}$ $(e\in\mathrm{int}P)$ is defined by

$$\xi_e(y)=\inf\{r\in\mathbb{R}:y\in re-P\}.$$

See [18,19,22,26–28] for more details.

In the present paper we introduce the concept of generalized g-quasi-contractions in cone b-metric spaces over Banach algebras and obtain common fixed point theorems for two weakly compatible self-mappings satisfying the g-quasi-contractive condition in the case of the g-quasi-contractive constant vector with $r(\lambda)\in[0,1/s)$ in cone b-metric spaces without the assumption of normality, where the coefficient s satisfies $s\ge 1$. As consequences, our main results not only extend the fixed point theorem of g-quasi-contractions in cone b-metric spaces to the case in cone b-metric spaces over Banach algebras, but also yield new corresponding results concerning the quasi-contractions in cone metric spaces over Banach algebras. So our main results generalize and extend the relevant results in the literature (see, for example, [2–12, 23, 31–35]).

2 Preliminaries

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x,y,z\in\mathcal{A},\alpha\in\mathbb{R}$):

1. $(xy)z=x(yz)$;
2. $x(y+z)=xy+xz$ and $(x+y)z=xz+yz$;
3. $\alpha(xy)=(\alpha x)y=x(\alpha y)$;
4. $\|xy\|\le\|x\|\|y\|$.

Throughout this paper, we shall assume that a Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $ex=xe=x$ for all $x\in\mathcal{A}$. An element $x\in\mathcal{A}$ is said to be invertible if there is an inverse element $y\in\mathcal{A}$ such that $xy=yx=e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [20].

The following proposition is well known (see [20]).

Proposition 2.1. Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x\in\mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x)=\lim_{n\to\infty}\|x^n\|^{\frac{1}{n}}=\inf_{n\ge1}\|x^n\|^{\frac{1}{n}}<1,$$

then $e-x$ is invertible. Actually,

$$(e-x)^{-1}=\sum_{i=0}^{\infty}x^i.$$

Now let us recall the concepts of cone and semi-order for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone if

1. P is non-empty closed and $\{\theta,e\}\subset P$;
2. $\alpha P+\beta P\subset P$ for all non-negative real numbers α,β;
3. $P^2=PP\subset P$;
4. $P\cap(-P)=\{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P\subset\mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x\preceq y$ if and only if $y-x\in P$. $x\prec y$ will stand for $x\preceq y$ and $x\neq y$, while $x\ll y$ will stand for $y-x\in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

The cone P is called normal if there is a number $M>0$ such that for all $x,y\in\mathcal{A}$,

$$\theta\preceq x\preceq y\Rightarrow \|x\|\leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

In the following we always assume that P is a cone in Banach algebra $\mathcal{A}$ with $\mathrm{int}P\neq\emptyset$ and $\preceq$ is the partial ordering with respect to P.

Definition 2.1 (see [1,22,23])**.** Let X be a non-empty set. Suppose the mapping $d:X\times X\to\mathcal{A}$ satisfies

1. $0\preceq d(x,y)$ for all $x,y\in X$ and $d(x,y)=0$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$ for all $x,y\in X$;
3. $d(x,y)\preceq d(x,z)+d(z,x)$ for all $x,y,z\in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 2.2 (see [1,22,23])**.** Let (X,d) be a cone metric space over a Banach algebra $\mathcal{A}$, $x\in X$ and $\{x_n\}$ a sequence in X. Then

1. $\{x_n\}$ converges to x whenever for each $c\in\mathcal{A}$ with $\theta\ll c$ there is a natural number N such that $d(x_n,x)\ll c$ for all $n\geq N$. We denote this by $\lim_{n\to\infty}x_n=x$ or $x_n\to x$.
2. $\{x_n\}$ is a Cauchy sequence whenever for each $c\in\mathcal{A}$ with $\theta\ll c$ there is a natural number N such that $d(x_n,x_m)\ll c$ for all $n,m\geq N$.
3. (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Now, we shall appeal to the following lemmas in the sequel.

Lemma 2.1 (see [28])**.** *If E is a real Banach space with a cone P and if $a\preceq\lambda a$ with $a\in P$ and $0\leq\lambda<1$, then $a=\theta$.*

Lemma 2.2 (see [24])**.** *If E is a real Banach space with a solid cone P and if $\theta<u\ll c$ for each $\theta\ll c$, then $u=\theta$.*

Lemma 2.3 (see [24])**.** *If E is a real Banach space with a solid cone P and if $\|x_n\|\to 0(n\to\infty)$, then for any $\theta\ll\epsilon$, there exists $N\in\mathbb{N}$ such that for any $n>N$, we have $x_n\ll\epsilon$.*

Finally, let us recall the concept of quasi-contraction defining on the cone metric s-paces over Banach algebras, which is introduced in [23].

Definition 2.3 (see [23])**.** Let (X,d) be a cone metric space over a Banach algebra $\mathcal{A}$. A mapping $T:X\to X$ is called a quasi-contraction if for some $k\in P$ with $r(k)<1$ and for all $x,y\in X$, one has

$$d(Tx,Ty)\preceq ku,$$

where

$$u\in\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}.$$

Remark 2.1. If $r(k)<1$, then $\|k^m\|\to 0(m\to\infty)$.

Definition 2.4 (see [13])**.** Let X be a nonempty set and $s\geq 1$ a given real number. A mapping $d:X\times X\to\mathcal{A}$ is said to be a cone b-metric if and only if for all $x,y,z\in X$ the following conditions are satisfied:

(i) $\theta\prec d(x,y)$ with $x\neq y$ and $d(x,y)=\theta$ if and only if $x=y$;

(ii) $d(x,y)=d(y,x)$;

(iii) $d(x,y)\preceq s[d(x,z)+d(z,y)]$.

The pair (X,d) is called a cone b-metric space over a Banach algebra $\mathcal{A}$.

Example 2.1. Denote by L_p $(0<p<1)$ the set of all real measurable functions $x(t)$ $(t\in[0,1])$ such that $\int_0^1|x(t)|^p dt<\infty$. Let $X=L_p$, $E=\mathbb{R}^2$, $P=\{(x,y)\in E|x,y\geq 0\}\subset\mathbb{R}^2$ and $d:X\times X\to E$ such that

$$d(x,y)=\left(\alpha\left\{\int_0^1|x(t)-y(t)|^p dt\right\}^{\frac{1}{p}},\beta\left\{\int_0^1|x(t)-y(t)|^p dt\right\}^{\frac{1}{p}}\right),$$

where $\alpha,\beta\geq 0$ are constants. Then (X,d) is a cone b-metric space over a Banach algebra with the coefficient $s=2^{\frac{1}{p}-1}$ (see the subsequent Example 2.2 for details).

Example 2.2. Let $X=\mathbb{R}$, $E=C^1_{\mathbb{R}}[0,1]$ and $P=\{f\in E: f\geq 0\}$. Define $d:X\times X\to E$ by $d(x,y)=|x-y|^{1.5}\varphi(t)$ where $\varphi:[0,1]\to\mathbb{R}$ is a function such that $\varphi(t)=e^t$. It is easy to see that (X,d) is a cone b-metric space over a Banach algebra with the coefficient $s=2^{0.5}$, but it is not a cone metric space.

Definition 2.5 (see [13])**.** Let (X,d) be a cone b-metric space over a Banach algebra $\mathcal{A}$, $x\in X$ and $\{x_n\}$ be a sequence in X. We say

(i) $\{x_n\}$ converges to x whenever for every $c\in E$ with $\theta\ll c$ there is a natural number N such that $d(x_n,x)\ll c$ for all $n\geq N$. We denote this by $\lim\limits_{n\to\infty}x_n=x$ or $x_n\to x$ $(n\to\infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c\in E$ with $\theta\ll c$ there is a natural number N such that $d(x_n,x_m)\ll c$ for all $n,m\geq N$.

(iii) (X,d) is a complete cone b-metric space over a Banach algebra $\mathcal{A}$ if every Cauchy sequence is convergent.

Lemma 2.4 (see [13]). *Let $\preceq$ be the partial ordering with respect to P, where P is the given cone P of the Banach algebra $\mathcal{A}$. The following properties are often used while dealing with cone b-metric spaces where the underlying cone is not necessarily normal:*

(1) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(2) If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.

(3) If $a \preceq b + c$ for each $c \in intP$, then $a \preceq b$.

(4) If $c \in intP$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n > n_0$.

(5) Let (X,d) be a cone b-metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X. If $d(x_n,x) \preceq b_n$ and $b_n \to \theta$, then $x_n \to x$.

Lemma 2.5 (see [13]). *The limit of a convergent sequence in cone b-metric space is unique.*

Definition 2.6 (see [2]). The mappings $f,g: X \to X$ are called weakly compatible, if for every $x \in X$ holds $fgx = gfx$ whenever $fx = gx$.

Definition 2.7 (see [3]). Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 2.6 (see [3]). *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 2.8 (see [12]). Let (X,d) be a cone metric space. A mapping $f: X \to X$ such that, for some constant $\lambda \in [0,1)$ and for every $x,y \in X$, there exists an element

$$u \in C(g;x,y) = \{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\}$$

for which $d(fx,fy) \preceq \lambda u$ is called a g-quasi-contraction, where $g: X \to X$, $f(X) \subset g(X)$.

Definition 2.9 (see [31]). Let (X,d) be a cone b-metric space with the coefficient $s \geq 1$. A mapping $f: X \to X$ is called a g-quasi-contraction where $g: X \to X$, $f(X) \subset g(X)$, if for some real number λ with $\lambda \in [0,1/s)$ and for all $x,y \in X$, one has

$$d(fx,fy) \preceq \lambda u,$$

where

$$u \in C(g;x,y) = \{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\}.$$

Definition 2.10. Let (X,d) be a cone b-metric space with the coefficient $s \geq 1$ over a Banach algebra $\mathcal{A}$. A mapping $f: X \to X$ is called a generalized g-quasi-contraction where $g: X \to X$, $f(X) \subset g(X)$, if for some $\lambda \in P$ with $r(\lambda) \in [0,1/s)$ and for all $x,y \in X$, one has

$$d(fx,fy) \preceq \lambda u,$$

where

$$u \in C(g;x,y) = \{d(gx,gy), d(gx,fx), d(gy,fy), d(gx,fy), d(gy,fx)\}. \tag{2.1}$$

Definition 2.11 (see [29, 30]). Let P be a solid cone in a Banach space $\mathcal{A}$. A sequence $\{u_n\}\subset P$ is a c-sequence if for each $c\gg\theta$ there exists $n_0\in\mathbb{N}$ such that $u_n\ll c$ for $n\geq n_0$.

It is easy to show the following propositions:

Proposition 2.2 (see [29]). Let P be a solid cone in a Banach space $\mathcal{A}$ and let $\{u_n\}$ and $\{v_n\}$ be sequences in P. If $\{u_n\}$ and $\{v_n\}$ are c-sequences and $\alpha,\beta>0$, then $\{\alpha u_n+\beta v_n\}$ is a c-sequence.

In addition to Proposition 2.2 above, the following propositions are crucial to the proof of our main result.

Proposition 2.3 (see [29]). Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Then the following conditions are equivalent:

(1) $\{u_n\}$ is a c-sequence.

(2) For each $c\gg\theta$ there exists $n_0\in\mathbb{N}$ such that $u_n\prec c$ for $n\geq n_0$.

(3) For each $c\gg\theta$ there exists $n_1\in\mathbb{N}$ such that $u_n\preceq c$ for $n\geq n_1$.

Proposition 2.4 (see [38]). Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Suppose that $k\in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Proposition 2.5 (see [38]). Let $\mathcal{A}$ be a Banach algebra with a unit e, P be a cone in $\mathcal{A}$ and $\preceq$ be the semi-order be yielded by the cone P. Let $\lambda\in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold true.

(i) If $\lambda\succeq\theta$, then we have $(e-\lambda)^{-1}\succeq\theta$. In addition, we have $\theta\preceq(e-\lambda)^{-1}\lambda^n\preceq(e-\lambda)^{-1}\lambda$ for any integer $n\geq 1$.

(ii) For any $u\succ\theta$, we have $u\npreceq\lambda u$. Moreover, we have $u\npreceq\lambda^n u$ for any integer $n\geq 1$.

Proposition 2.6 (see [38]). Let (X,d) be a complete cone metric space with a Banach algebra $\mathcal{A}$ and let P be the underlying solid cone in Banach algebra $\mathcal{A}$. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x\in X$, then we have

(i) $\{d(x_n,x)\}$ is a c-sequence;

(ii) for any $p\in\mathbb{N}$, $\{d(x_n,x_{n+p})\}$ is a c-sequence.

3 Main results

In this section, we give some common fixed point results for generalized g-quasi-contractions with the quasi-contractive constant vector satisfying $r(\lambda)\in[0,1/s)$ in the setting of cone b-metric spaces over Banach algebras without the assumption of normality.

Theorem 3.1. *Let (X,d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s\geq 1$ and the underlying solid cone P. Let the mapping $f:X\to X$ be the g-quasi-contraction with the g-quasi-contractive constant vector satisfying $r(\lambda)\in[0,1/s)$. If the range of g contains the range of f and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

We begin the proof of Theorem 3.1 with a useful lemma. For each $x_0\in X$, set $gx_1=fx_0$ and $gx_{n+1}=fx_n$. We will prove that $\{gx_n\}$ is a Cauchy sequence. First, we shall show the following lemmas. Note that for these lemmas, we suppose that all the conditions in Theorem 3.1 are satisfied.

Lemma 3.1. *For any $N\geq 2$ and $1\leq m\leq N-1$, one has that*

$$d(gx_N,gx_m)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0). \tag{3.1}$$

Proof. We now prove Lemma 3.1 by induction. When $N=2$, $m=1$, since $f:X\to X$ is a g-quasi-contraction, there exists

$$u_1\in C(g;x_1,x_0)=\{d(gx_1,gx_0),d(gx_1,gx_2),d(gx_0,gx_1),d(gx_1,gx_1),d(gx_0,gx_2)\}$$

such that

$$d(gx_2,gx_1)\preceq\lambda u_1.$$

Hence, $u_1=d(gx_1,gx_0)$ or $u_1=d(gx_0,gx_2)$ (note that it is obvious that $u_1\neq d(gx_1,gx_2)$ since $d(gx_2,gx_1)\npreceq\lambda d(gx_1,gx_2)$ and $u_1\neq d(gx_1,gx_1)$ since $d(gx_1,gx_2)\neq\theta$).

When $u_1=d(gx_1,gx_0)$, then we have

$$\begin{aligned}d(gx_2,gx_1)&\preceq\lambda d(gx_0,gx_1)\\&\preceq s\lambda d(gx_0,gx_1)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0).\end{aligned}$$

When $u_1=d(gx_2,gx_0)$, then we have

$$d(gx_2,gx_1)\preceq\lambda d(gx_2,gx_0)\preceq s\lambda[d(gx_2,gx_1)+d(gx_1,gx_0)].$$

So we get

$$(e-s\lambda)d(gx_2,gx_1)\preceq s\lambda d(gx_1,gx_0),$$

which implies that

$$d(gx_2,gx_1)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0).$$

Hence, (3.1) holds for $N=2$ and $m=1$.

Suppose that for some $N\geq 2$ and for any $2\leq p\leq N$ and $1\leq n\leq p$, one has

$$d(gx_p,gx_n)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0). \tag{3.2}$$

That is,

$$d(gx_p,gx_1)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0), \tag{3.3a}$$

$$d(gx_p,gx_2)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0),\cdots, \tag{3.3b}$$

$$d(gx_p,gx_{p-1})\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0). \tag{3.3c}$$

Then, we need to prove that for $N+1\geq 2$ and any $1\leq n<N+1$, one has

$$d(gx_{N+1},gx_n)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0). \tag{3.4}$$

That is,

$$d(gx_{N+1},gx_1)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0), \tag{3.5a}$$

$$d(gx_{N+1},gx_2)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0),\cdots, \tag{3.5b}$$

$$d(gx_{N+1},gx_{N-1})\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0), \tag{3.5c}$$

$$d(gx_{N+1},gx_N)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0). \tag{3.5d}$$

In fact, since $f:X\to X$ is a g-quasi-contraction, there exists

$$u_1\in C(g;x_N,x_0)=\{d(gx_N,gx_0),d(gx_N,gx_{N+1}),d(gx_0,gx_1),d(gx_N,gx_1),d(gx_0,gx_{N+1})\},$$

such that

$$d(gx_{N+1},gx_1)\preceq\lambda u_1.$$

If $u_1=d(gx_N,gx_1)$, then by (3.3a) we have

$$\begin{aligned}d(gx_{N+1},gx_1)&\preceq s\lambda^2(e-s\lambda)^{-1}d(gx_1,gx_0)\preceq(s\lambda)^2(e-s\lambda)^{-1}d(gx_1,gx_0)\\&\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0).\end{aligned}$$

If $u_1=d(gx_0,gx_1)$, then we have

$$d(gx_{N+1},gx_1)\preceq\lambda d(gx_1,gx_0)\preceq s\lambda d(gx_1,gx_0)\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0).$$

If $u_1=d(gx_N,gx_0)$, then by (3.3a) we have

$$\begin{aligned}d(gx_{N+1},gx_1)&\preceq\lambda d(gx_N,gx_0)\preceq s\lambda(d(gx_N,gx_1)+d(gx_1,gx_0))\\&\preceq s\lambda(s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0)+d(gx_1,gx_0))\\&=s\lambda((s\lambda)^{-1}+e)d(gx_1,gx_0)\\&=s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0).\end{aligned}$$

If $u_1=d(gx_0,gx_{N+1})$, then we have

$$d(gx_{N+1},gx_1)\preceq\lambda d(gx_0,gx_{N+1})\preceq s\lambda(d(gx_0,gx_1)+d(gx_1,gx_{N+1})).$$

Hence, we see

$$(e-s\lambda)d(gx_{N+1},gx_1)\preceq s\lambda d(gx_0,gx_1),$$

which implies that

$$d(gx_{N+1},gx_1)\preceq (e-s\lambda)^{-1}s\lambda d(gx_0,gx_1).$$

Without loss of generality, suppose that $u_1=d(gx_N,gx_{N+1})$. Since $f:X\to X$ is a g-quasi-contraction, there exists $u_2\in C(g;x_{N-1},x_N)$ such that

$$u_1=d(gx_N,gx_{N+1})\preceq\lambda u_2,$$

where

$$\begin{aligned}&C(g;x_{N-1},x_N)\\=&\{d(gx_{N-1},gx_N),d(gx_{N-1},gx_N),d(gx_N,gx_{N+1}),d(gx_{N-1},gx_{N+1}),d(gx_N,gx_N)\}.\end{aligned}$$

So, we get

$$d(gx_{N+1},gx_1)\preceq\lambda u_1\preceq\lambda^2u_2.$$

Similarly, it is easy to see that $u_2\neq d(gx_N,gx_N)$ since $u_2\neq\theta$ and $u_2\neq d(gx_N,gx_{N+1})$, since $d(gx_N,gx_{N+1})\npreceq\lambda^2d(gx_N,gx_{N+1})$.

If $u_2=d(gx_{N-1},gx_N)$, then by the induction assumption (3.2), we have

$$\begin{aligned}d(gx_{N+1},gx_1)\preceq&\lambda^2u_2\preceq s\lambda^3(e-s\lambda)^{-1}d(gx_1,gx_0)\\\preceq&(s\lambda)^3(e-s\lambda)^{-1}d(gx_1,gx_0)\\\preceq&s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0).\end{aligned}$$

Without loss of generality, suppose that $u_2=d(gx_{N-1},gx_{N+1})$. There exists $u_3\in C(g;x_{N-2},x_N)$, such that

$$u_2=d(gx_{N-1},gx_{N+1})\preceq\lambda u_3,$$

where

$$\begin{aligned}&C(g;x_{N-2},x_N)\\=&\{d(gx_{N-2},gx_N),d(gx_{N-2},gx_{N-1}),d(gx_N,gx_{N+1}),d(gx_{N-2},gx_{N+1}),d(gx_N,gx_{N-1})\}.\end{aligned}$$

In general, suppose that $u_{i-1}=d(gx_{N-i+2},gx_{N+1})$. Since $f:X\to X$ is a g-quasi-contraction, by the similar arguments above, there exists $u_i\in C(g;x_{N-i+1},x_N)$ such that

$$u_{i-1}=d(gx_{N-i+2},gx_{N+1})\preceq\lambda u_i,$$

for which we obtain

$$d(gx_{N+1},gx_1)\preceq\lambda u_1\preceq\lambda^2u_2\preceq\cdots\preceq\lambda^iu_i,$$

where

$$C(g;x_{N-i+1},x_N)=\{d(gx_{N-i+1},gx_N),d(gx_{N-i+1},gx_{N-i+2}),d(gx_N,gx_{N+1}),\\ d(gx_{N-i+1},gx_{N+1}),d(gx_N,gx_{N-i+2})\}.$$

Similarly, it is easy to see that $u_i\neq d(gx_N,gx_{N+1})$. This is because by Proposition 2.5(iii) we have

$$u_1=d(gx_N,gx_{N+1})\npreceq\lambda^{i-1}d(gx_N,gx_{N+1}).$$

So we know that if $u_i=d(gx_{N-i+1},gx_N)$ or $u_i=d(gx_{N-i+1},gx_{N-i+2})$ or $u_i=d(gx_N,gx_{N-i+2})$ then by the induction assumption (3.2), we have $u_i\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0)$. Hence,

$$\begin{aligned} d(gx_{N+1},gx_1)&\preceq\lambda^i u_i\preceq s\lambda^{i+1}(e-s\lambda)^{-1}d(gx_1,gx_0)\\ &\preceq(s\lambda)^{i+1}(e-s\lambda)^{-1}d(gx_1,gx_0)\\ &\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0), \end{aligned}$$

which means that (3.5a) holds true. Without loss of generality, suppose that $u_i=d(gx_{N-i+1},gx_{N+1})$. Then by the similar arguments as above we have $u_i\preceq\lambda u_{i+1}$, where $u_{i+1}\in C(g;x_{N-i},x_N)$. Hence, there is a sequence $\{u_n\}$ such that

$$d(gx_{N+1},gx_1)\preceq\lambda u_1\preceq\lambda^2u_2\preceq\cdots\preceq\lambda^{N-1}u_{N-1}\preceq\lambda^N u_N,$$

where

$$u_{N-1}=d(gx_2,gx_{N+1})\preceq\lambda u_N$$

and

$$\begin{aligned} &u_N\in C(g;x_1,x_N)\\ &=\{d(gx_1,gx_N),d(gx_1,gx_2),d(gx_N,gx_{N+1}),d(gx_N,gx_2),d(gx_1,gx_{N+1})\}. \end{aligned}$$

Obviously, $u_N\neq d(gx_1,gx_{N+1})$ and $u_N\neq d(gx_N,gx_{N+1})$. On the contrary, if $u_N=d(gx_1,gx_{N+1})$, then $u_N\preceq\lambda^N u_N$, a contradiction. If $u_N=d(gx_N,gx_{N+1})=u_1$, then we have

$$u_1=d(gx_N,gx_{N+1})\preceq\lambda^2u_2\preceq\cdots\preceq\lambda^{N-1}u_{N-1}\preceq\lambda^{N-1}u_1,$$

a contradiction. Hence, it follows that $u_N=d(gx_1,gx_N)$, $u_N=d(gx_1,gx_2)$ or $u_N=d(gx_N,gx_2)$. By the induction assumption (3.2), in any case, we have

$$u_N\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0). \tag{3.6}$$

Therefore, we get

$$\begin{aligned} d(gx_{N+1},gx_1)&\preceq\lambda u_1\preceq\lambda^2u_2\preceq\cdots\preceq\lambda^N u_N\\ &\preceq\lambda^N(e-s\lambda)^{-1}s\lambda d(gx_1,gx_0)\\ &\preceq(s\lambda)^{N+1}(e-s\lambda)^{-1}d(gx_1,gx_0)\\ &\preceq s\lambda(e-s\lambda)^{-1}d(gx_1,gx_0). \end{aligned} \tag{3.7}$$

That is to say, (3.5a) is true. By (3.7), we have

$$u_1 \preceq \lambda^{N-1} s\lambda (e-s\lambda)^{-1} d(gx_1, gx_0).$$

Thus,

$$\begin{aligned} d(gx_N, gx_{N+1}) = u_1 &\preceq \lambda^{N-1} s\lambda (e-s\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq (s\lambda)^N (e-s\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq s\lambda (e-s\lambda)^{-1} d(gx_1, gx_0), \end{aligned}$$

which implies that (3.5d) is true. Similarly, since

$$u_2 = d(gx_{N-1}, gx_{N+1}), \cdots, u_i = d(gx_{N-i+1}, gx_{N+1}), \cdots,$$

by (3.6) and (3.7), we get

$$u_i \preceq \lambda^{N-i} u_N \preceq s\lambda^{n-i+1} (e-s\lambda)^{-1} d(gx_1, gx_0). \tag{3.8}$$

Hence, it follows from (3.8) that (3.5b)-(3.5c) are all true. That is, (3.4) is true. Therefore, we conclude that Lemma 3.1 holds true. □

By Lemma 3.1, we immediately obtain the following result.

Lemma 3.2. *We have that for all $i, j \in \mathbb{N}_+$*

$$d(gx_i, gx_j) \preceq s\lambda (e-s\lambda)^{-1} d(gx_0, gx_1). \tag{3.9}$$

Now, we begin to prove Theorem 3.1. First, we need to show that $\{gx_n\}$ is a Cauchy sequence. For all $n > m$, there exists

$$\begin{aligned} v_1 \in C(g; x_{n-1}, x_{m-1}) = \{ & d(gx_{n-1}, gx_{m-1}), d(gx_{n-1}, gx_n), \\ & d(gx_{m-1}, gx_m), d(gx_{n-1}, gx_m), d(gx_{m-1}, gx_n)\}, \end{aligned}$$

such that

$$d(fx_{n-1}, fx_{m-1}) \preceq \lambda v_1.$$

Using the g-quasi-contractive condition repeatedly, we easily show by induction that there must exist

$$v_k \in \{d(gx_i, gx_j) : 0 \le i < j \le n\}, \quad (k = 2, 3, \cdots, m),$$

such that

$$v_k \preceq \lambda v_{k+1}, \quad (k = 1, 2, \cdots, m-1). \tag{3.10}$$

For convenience, we write $v_m = d(gx_i, gx_j)$, where $0 \le i < j \le n$.

Using the triangular inequality, we have

$$d(gx_i,gx_j)\preceq sd(gx_i,gx_0)+sd(gx_0,gx_j),\quad (0\le i,j\le n),$$

and by Lemma 3.2, we obtain

$$\begin{aligned} d(gx_n,gx_m)&=d(fx_{n-1},fx_{m-1})\preceq\lambda v_1\preceq\lambda^2 v_2\preceq\cdots\preceq\lambda^m v_m\\ &\preceq\lambda^m d(gx_i,gx_j)\\ &=s\lambda^{m+1}(e-s\lambda)^{-1}d(gx_1,gx_0). \end{aligned}$$

Since $r(\lambda)<1/s\le 1$, by Remark 2.1, we have that $s\lambda^{m+1}(e-s\lambda)^{-1}d(gx_1,gx_0)\to\theta$ as $m\to\infty$, so by Proposition 2.4, it is easy to see that for any $c\in intP$, there exists $n_0\in\mathbb{N}$ such that for all $n>m>n_0$,

$$d(gx_n,gx_m)\preceq s\lambda^{m+1}(e-s\lambda)^{-1}d(gx_1,gx_0)\ll c.$$

So $\{gx_n\}$ is a Cauchy sequence in $g(X)$. If $g(X)\subset X$ is complete, there exist $q\in g(X)$ and $p\in X$ such that $gx_n\to q$ as $n\to\infty$ and $gp=q$.

Now, from (2.1), we get

$$d(fx_n,fp)\preceq\lambda v,$$

where

$$v\in C(g;x_n,p)=\{d(gx_n,gp),d(gx_n,fx_n),d(gp,fp),d(gx_n,fp),d(fx_n,gp)\}.$$

Clearly at least one of the following five cases holds for infinitely many n.

(1) $d(fx_n,fp)\preceq\lambda d(gx_n,gp)\preceq s\lambda d(gx_{n+1},gp)+s\lambda d(gx_{n+1},gx_n)$;

(2) $d(fx_n,fp)\preceq\lambda d(gx_n,fx_n)=\lambda d(gx_n,gx_{n+1})$;

(3) $d(fx_n,fp)\preceq\lambda d(gp,fp)\preceq s\lambda d(gx_{n+1},gp)+s\lambda d(gx_{n+1},fp)$, that is, $d(fx_n,fp)\preceq s\lambda(e-s\lambda)^{-1}d(gx_{n+1},gp)$;

(4) $d(fx_n,fp)\preceq\lambda d(gx_n,fp)\preceq s\lambda d(gx_{n+1},fp)+s\lambda d(gx_{n+1},gx_n)$, that is, $d(fx_n,fp)\preceq s\lambda(e-s\lambda)^{-1}d(gx_{n+1},gx_n)$;

(5) $d(fx_n,fp)\preceq\lambda d(fx_n,gp)=\lambda d(gx_{n+1},gp)$.

As $s\lambda\preceq s\lambda(e-s\lambda)^{-1}$ (since $\theta\preceq s\lambda$ and $r\ (s\lambda)<1$), we obtain that

$$d(gx_{n+1},fp)\preceq s\lambda(e-s\lambda)^{-1}[d(gx_{n+1},gx_n)+d(gx_{n+1},q)].$$

Since $gx_n\to q$ as $n\to\infty$, we get that for any $c\in intP$, there exists $n_1\in\mathbb{N}$ such that for all $n>n_1$, one has

$$d(gx_{n+1},fp)\ll c.$$

By Lemmas 2.4 and 2.5, we have $gx_n\to fp$ as $n\to\infty$ and $q=fp$.

Now if w is another point such that $gu=fu=w$, hence

$$d(w,q)=d(fu,fp)\preceq\lambda v,$$

where $r(\lambda)\in[0,1/s)$ and

$$v\in C(g;u,p)=\{d(gu,gp),d(gu,fu),d(gp,fp),d(gu,fp),d(fu,gp)\}.$$

It is obvious that $d(w,q)=\theta$, i.e., $w=q$. Therefore, q is the unique point of coincidence of f and g in X. Moreover, the mappings f and g are weakly compatible, by Lemma 2.6, we know that q is the unique common fixed point of f and g.

Similarly, if $f(X)$ is complete, the above conclusion is also established. □

Corollary 3.1. Let (X,d) be a complete cone b-metric space with a Banach algebra $\mathcal{A}$ and let P be the underlying cone with $k\in P$. If the mapping $T:X\to X$ is a quasi-contraction, then T has a unique fixed point in X. And for any $x\in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Set $g=I_X$, the identity mapping from X to X. It is obvious to see that Theorem 3.1 yields Corollary 3.1. □

Remark 3.1. Theorem 3.1 generalizes [31, Theorem 2.6] since the latter is a special case when we take $s=1$ in the cone b-metric space X with a Banach algebra $\mathcal{A}$ discussed in Theorem 3.1.

Remark 3.2. Corollary 3.1 does not need to require the assumption of normality of the cone P. So Corollary 3.1 improves and generalizes Theorem 9 in [23].

Remark 3.3. From the proof of Lemma 3.1, we note that the technique of induction appearing in Theorem 3.1 is somewhat different from that in [23, Theorem 9], and also different from that in [31, Theorem 2.6], which is more interesting and easily to understood.

Corollary 3.2. Taking $E=\mathbb{R}$, $P=[0,+\infty)$, $\lambda\in[0,1/s)$ in Theorem 3.1, we get Das-Naik's result from (a), that is, if $g=I_X$ we get Ćirić's result from [21], both in the setting of b-metric spaces.

The following corollary is the Jungck's result in the setting of cone b-metric spaces.

Corollary 3.3. Let (X,d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s\geq 1$ and the underlying solid cone P. Let the mappings f, $g:X\to X$ satisfy the condition that for $\lambda\in P$ with $r(\lambda)\in[0,1/s)$ and for every $x,y\in X$ holds $d(fx,fy)\preceq\lambda d(gx,gy)$. If $g(X)\subset f(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

The next result is the Banach contraction principle in the setting of cone b-metric spaces.

Corollary 3.4 (see [38])**.** Let (X,d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s\geq 1$ and the underlying solid cone P. Let the mapping $f:X\to X$ satisfy the condition that for $\lambda\in P$ with $r(\lambda)\in[0,1/s)$ and for every $x,y\in X$ holds $d(fx,fy)\preceq\lambda d(x,y)$ (namely, f is a generalized quasi-contraction). If $f(X)$ is a complete subspace of X, then f has a unique point in X.

Acknowledgments

The research is partially is partially supported by the National Natural Science Foundation of China (No. 11361064) and the project No. 174024 of the Ministry of Education, Science and Technological Department of the Republic of Serbia.

References

[1] L.-G. Huang and X. Zhang, Cone metric space and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2007), 1468–1476.

[2] G.-X. Song et al., New common fixed point theorems for maps on cone metric spaces, Appl. Math. Lett., 32 (2010), 1033–1037.

[3] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341 (2008), 416–420.

[4] S. Radenović, V. Rakočević and Sh. Rezapour, Common fixed points for (g,f) type maps in cone metric spaces, Appl. Math. Comput., 218 (2011), 480–491.

[5] S. H. Cho and J. S. Bae, Common fixed points theorems for mappings satisfying property (E,A) on cone metric spaces, Math. Comput. Model., 53 (2011), 945–951.

[6] Lj. Gajić and V. Rakočević, Quasi-contractions on a nonnormal cone metric space, Funct. Anal. Appl., 46(1) (2012), 75–79.

[7] Z. Kadelburg, S. Radenović and V. Rakočević, Remarks on "Quasi-contraction on a cone metric space", Appl. Math. Lett., 22(1) (2009), 1674–1679.

[8] D. Ilić and V. Rakočević, Quasi-contraction on a cone metric space, Appl. Math. Lett., 22(5) (2009), 728–731.

[9] K. M. Das and K. Naik, Common fixed point theorems for commuting maps on a metric space, Proc. Amer. Math. Soc., 77 (1979), 369–373.

[10] H. Huang and S. Xu, Fixed point theorems of contractive mappings in cone b-metric spaces and applications, Fixed Point Theory Appl., 2013 (2013), 112.

[11] I. Marian and A. Branga, Common fixed point results in b-cone metric spaces over topological vector spaces, General Math., 20(1) (2012).

[12] D. llić and V. Rakočević, Common fixed points for maps on cone metric space, J. Math. Anal. Appl., 341 (2008), 876–882.

[13] N. Hussian and M. H. Shah, KKM mappings in cone b-metric spaces, Comput. Math. Appl., 62 (2011), 1677–1684.

[14] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in b-metric spaces, Cen. Euro. J. Math., 8(2) (2010), 367–377.

[15] Sh. Rezapour and R. Hamlbarani, Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 345 (2008), 719–724.

[16] S. Janković, Z. Golubović and S. Radenović, Compatible and weakly compatible mappings in cone metric spaces, Math. Comput. Model., 52 (2010), 1728–1738.

[17] M. Abbas, V. Ć. Rajić, T. Nazir and S. Radenović, Common fixed point of mappings satisfying rational inequalities in ordered complex valued generalized metric spaces, Afr. Mat., DOI 10.1007/s13370-013-0185-z.

[18] H. Çakallı, A. Sönmez and Ç. Genç, On an equivalence of topological vector space valued cone metric spaces and metric spaces, Appl. Math. Lett., 25 (2012), 429–433.

[19] W. S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal., 72(5) (2010), 2259–2261.

[20] Walter Rudin, Functional Analysis, 2nd ed., McGraw-Hill, 1991.

[21] Lj. B. Ćirić, A generalization of Banach's contraction priciple, Proc. Amer. Math. Soc., 45 (1974), 267–273.

[22] H. Liu and S. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory Appl., 2013 (2013), 320.

[23] H. Liu and S. Xu, Fixed point theorem of quasi-contractions on cone metric spaces with Banach algebras, Abstract Appl. Anal., Volume 2013, Article ID 187348, 5 pages.

[24] S. Radenović and B. E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, Comput. Math. Appl., 57 (2009), 1701–1707.

[25] S. Janković, Z. Kadelburg and S. Radenović, On the cone metric space: A survey, Nonlinear Anal., 74 (2011), 2591–2601.

[26] Z. Kadelburg, S. Radenović and V. Rakočević, A note on the equivalence of some metric and cone metric fixed point results, Appl. Math. Lett., 24 (2011), 370–374.

[27] Y. Feng and W. Mao, The equivalence of cone metric spaces and metric spaces, Fixed Point Theory, 11(2) (2010), 259–264.

[28] Z. Kadelburg, M. Pavlović and S. Radenović, Common Fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces, Comput. Math. Appl., 59 (2010), 3148–3159.

[29] Z. Kadelburg, and S. Radenović, A note on various types of cones and fixed point results in cone metric spaces, Asian J. Math. Appl., Volume 2013, Article ID ama0104, 7 pages.

[30] M. Dordević, D. Dorić, Z. Kadelburg, S. Radenović and D. Spasić, Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory Appl., 2011 (2011), 29.

[31] L. Shi and S. Xu, Common fixed point theorems for two weakly compatible self-mappings in cone b-metric spaces, Fixed Point Theory Appl., 2013 (2013), 120.

[32] M.H. Shah, S. Simić, N. Hussain, A. Sretenović and S. Radenović, Common fixed points theorems for occasionally weakly compatible pairs on cone metric type spaces, J. Comput. Anal. Appl., 14(2) (2012), 290–297.

[33] M. Jovanović, Z. Kadelburg and S. Radenović, Common fixed point results in metric-type spaces, Fixed Point Theory Appl., Volume 2010, Article ID 978121.

[34] A. S. Cvetković, M. P. Stanić, S. Dimitrijević and S. Simić, Common fixed point theorems for four mappings on cone metric type spaces, Fixed Point Theory Appl., Volume 2011, Article ID 589725.

[35] M. P. Stanić, A. S. Cvetković, S. Simić and S. Dimitrijević, Common fixed point under contractive condition of Ćirić type on cone metric type spaces, Fixed Point Theory Appl., 2011 (2012), 35, Article ID 589725.

[36] P. Kumam, N. V. Dung and V. T. L. Hang, Some equivalence between cone b-metric spaces and b-metric spaces, Abstract Appl. Anal., Volume 2013, Article ID 573740.

[37] E. Karapinar and W. S. Du, A note on b-cone metric and its related results: Generalizations or equivalence?, Fixed Point Theory Appl., 2013 (2013), 210.

[38] S. Xu and S. Radenović, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, Fixed Point Theory Appl., 2014 (2014), 102.