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Some New Inequalities for Wavelet Frames on Local Fields

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Abstract. Wavelet frames have gained considerable popularity during the past decade, primarily due to their substantiated applications in diverse and widespread fields of science and engineering. Finding general and verifiable conditions which imply that the wavelet systems are wavelet frames is among the core problems in time-frequency analysis. In this article, we establish some new inequalities for wavelet frames on local fields of positive characteristic by means of the Fourier transform. As an application, an improved version of the Li-Jiang inequality for wavelet frames on local fields is obtained.

Key Words: Frame, inequalities, wavelet frame, local field, Fourier transform.

AMS Subject Classifications: 42C15, 42C40, 43A70, 11S85

1 Introduction

The notion of frame in a general Hilbert space was first introduced by Duffin and Schaeffer [5] to study some deep problems in non-harmonic Fourier series. Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or obtain other desirable features unavailable with orthonormal bases. The idea of Duffin and Schaeffer did not generate much interest outside non-harmonic Fourier series until the seminal work by Daubechies, Grossmann, and Meyer [3]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. After their work, the theory of frames began to be studied widely and deeply. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in

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signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine and so on. An introduction to the frame theory and its applications can be found in [2,4].

The following is the standard definition on frames in a Hilbert space. A sequence $\{f_k: k\in\mathbb{Z}\}$ of elements of a Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist constants $A,B>0$ such that

$$A\|f\|_2^2\le\sum_{k\in\mathbb{Z}}\big|\langle f,f_k\rangle\big|^2\le B\|f\|_2^2 \tag{1.1}$$

holds for every $f\in\mathcal{H}$, and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. A tight frame refers to the case when $A=B$, and a Parseval frame refers to the case when $A=B=1$.

An important example about frame is wavelet frame, which is obtained by translating and dilating a finite number of functions. Wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, wavelet frames become much easier to construct than the orthonormal wavelets. An important problem in practice is therefore to determine conditions on the wavelet function, dilation and translation parameters so that the corresponding wavelet system forms a frame. In her famous book, Daubechies [2] proved the first result on the necessary and sufficient conditions for wavelet frames, and then, Chui and Shi [1] gave an improved result. In recent years, these conditions have been further improved and investigated by many authors [6,9,13,15,16,18].

A field K equipped with a topology is called a local field if both the additive K^+ and multiplicative groups K^* of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. The local field K is a natural model for the structure of wavelet frame systems, as well as a domain upon which one can construct wavelet basis functions. There is a substantial body of work that has been concerned with the construction of wavelets on K, or more generally, on local fields of positive characteristic. For example, Li and Jiang [7] have obtained a necessary condition and a set of sufficient conditions for wavelet frames on local fields of positive characteristic in the frequency domain. The characterizations of tight wavelet frames on local fields were completely established by Shah and Abdullah [11] by virtue of two basic equations in the Fourier domain. These studies were continued by Shah and his colleagues in [8,10,12,14], where they have provided some algorithms for constructing periodic, wave packet frames and semi-orthogonal wavelet frames on local fields of positive characteristic.

In the present paper, we shall present generalized inequalities for wavelet frames on local fields of positive characteristic. In particular, we establish a necessary condition for wavelet frames on local fields of positive characteristic via Fourier transform. The inequalities we proposed are stated in terms of the Fourier transforms of the wavelet system's generating functions, and they improves known ones by Li and Jiang [7].

This paper is organized as follows: In Section 2, we discuss some preliminary facts about local fields of positive characteristic and state the main results. Section 3 gives the proofs of the results.

2 Preliminaries on local fields

Local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in the number theory, but also in the representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics.

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. We use the notation of the book by Taibleson [17]. In the rest of this paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x,y \in K$;

(c) $|x+y| \leq \max\{|x|,|y|\}$ for all $x,y \in K$.

Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [17], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x'$, $x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [17]. Moreover, each $\mathfrak{B}^k=\mathfrak{p}^k\mathfrak{D}=\{x\in K:|x|<q^{-k}\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U}=\{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x\in K$ can be expressed uniquely as $x=\sum_{\ell=k}^{\infty}c_\ell\mathfrak{p}^\ell$ with $c_\ell\in\mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y\in\mathfrak{B}^k$, then $\chi_y(x)=\chi(y,x),x\in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n):n\in\mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [17], the set $\{\chi_{u(n)}:n\in\mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

Definition 2.1. If $f\in L^1(K)$, then the Fourier transform of f is defined by

$$\mathcal{F}\{f(x)\}=\hat{f}(\xi)=\int_K f(x)\overline{\chi_\xi(x)}dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi)=\int_K f(x)\overline{\chi_\xi(x)}dx=\int_K f(x)\chi(-\xi x)dx.$$

The properties of Fourier transform on local field K are much similar to those of on the real line. In fact, the Fourier transform have the following properties:

- The map $f\to\hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty\le\|f\|_1$.
- If $f\in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f\in L^1(K)\cap L^2(K)$, then $\|\hat{f}\|_2=\|f\|_2$.

The Fourier transform of a function $f\in L^2(K)$ is defined by

$$\hat{f}(\xi)=\lim_{k\to\infty}\hat{f}_k(\xi)=\lim_{k\to\infty}\int_{|x|\le q^k}f(x)\overline{\chi_\xi(x)}dx, \tag{2.2}$$

where $f_k=f\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f\in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}(u(n))=\int_{\mathfrak{D}}f(x)\overline{\chi_{u(n)}(x)}dx. \tag{2.3}$$

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have

$$\mathfrak{D}/\mathfrak{B}\cong GF(q),$$

where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1=\zeta_0,\zeta_1,\zeta_2,\cdots,\zeta_{c-1}\}\subset\mathfrak{D}^*$ such that span

$$\{\zeta_j\}_{j=0}^{c-1}\cong GF(q).$$

For $n\in\mathbb{N}_0$ satisfying

$$0\le n<q,\quad n=a_0+a_1p+\cdots+a_{c-1}p^{c-1},\quad 0\le a_k<p,\quad \text{and}\quad k=0,1,\cdots,c-1,$$

we define

$$u(n)=(a_0+a_1\zeta_1+\cdots+a_{c-1}\zeta_{c-1})\mathfrak{p}^{-1}.$$

Also, for $n=b_0+b_1q+b_2q^2+\cdots+b_sq^s$, $n\in\mathbb{N}_0$, $0\le b_k<q$, $k=0,1,2,\cdots,s$, we set

$$u(n)=u(b_0)+u(b_1)\mathfrak{p}^{-1}+\cdots+u(b_s)\mathfrak{p}^{-s}.$$

This defines $u(n)$ for all $n\in\mathbb{N}_0$. In general, it is not true that $u(m+n)=u(m)+u(n)$. But, if $r,k\in\mathbb{N}_0$ and $0\le s<q^k$, then $u(rq^k+s)=u(r)\mathfrak{p}^{-k}+u(s)$. Further, it is also easy to verify that $u(n)=0$ if and only if $n=0$ and $\{u(\ell)+u(k):k\in\mathbb{N}_0\}=\{u(k):k\in\mathbb{N}_0\}$ for a fixed $\ell\in\mathbb{N}_0$. Hereafter we use the notation $\chi_n=\chi_{u(n)}$, $n\ge 0$.

Let the local field K be of characteristic $p>0$ and $\zeta_0,\zeta_1,\zeta_2,\cdots,\zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu\mathfrak{p}^{-j})=\begin{cases}\exp(2\pi i/p), & \mu=0,\quad \text{and}\ \ j=1,\\ 1,\quad \mu=1,\cdots,c-1, & \text{or}\ \ j\neq 1.\end{cases}\tag{2.4}$$

Since $\bigcup_{j\in\mathbb{Z}}\mathfrak{p}^{-j}\mathfrak{D}=K$, we can regard $\mathfrak{p}^{-1}$ as the dilation and since $\{u(n):n\in\mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\Lambda=\{u(n):n\in\mathbb{N}_0\}$ can be treated as the translation set. Note that Λ is a subgroup of K^+ and unlike the standard wavelet theory on the real line, the translation set is not a group.

We also denote the test function space on K by Ω, that is, each function f in Ω is a finite linear combination of functions of the form $\mathbf{\Phi}_k(x-h)$, $h\in K$, $k\in\mathbb{Z}$, where $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. This class of functions can also be described in the following way. A function $g\in\Omega$ if and only if there exist integers k, ℓ such that g is constant on cosets of $\mathfrak{B}^k$ and is supported on $\mathfrak{B}^\ell$. It follows that Ω is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $\mathcal{C}_0(K)$ as well as in $L^p(K)$, $1\le p<\infty$.

For a given $\psi\in L^2(K)$, define the wavelet system

$$\mathcal{F}_\Psi:=\Big\{\psi_{j,k}=:q^{j/2}\psi\big(\mathfrak{p}^j\cdot-u(k)\big),\ j\in\mathbb{Z},\ k\in\mathbb{N}_0\Big\}.\tag{2.5}$$

We call the wavelet system $\mathcal{F}_\Psi$ a wavelet frame for $L^2(K)$, if there exist positive numbers $0<C\le D<\infty$ such that for all $f\in L^2(K)$

$$C\|f\|_2^2\le\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\big|\langle f,\psi_{j,k}\rangle\big|^2\le D\|f\|_2^2.\tag{2.6}$$

Recently, Li and Jiang [7] gave a necessary condition for the wavelet system (2.5) to be a frame in $L^2(K)$. Specifically, they proved the following.

Proposition 2.1. If $\{\psi_{j,k}: j\in\mathbb{Z},\ k\in\mathbb{N}_0\}$ is a wavelet frame for $L^2(K)$ with bounds A and B, then

$$A\le\sum_{j\in\mathbb{Z}}\left|\hat{\psi}(\mathfrak{p}^j\xi)\right|^2\le B,\quad a.e.\ \xi\in K. \tag{2.7}$$

Motivated and inspired by the fundament works in [7,11] on local fields of positive characteristic, in this article, we shall establish generalized inequalities for the wavelet system (2.5) to be a frame in $L^2(K)$. Before stating our result, we introduce some notations.

An element $u(\alpha)\in K$ is called q-adic number if it has the form $u(\alpha)=\mathfrak{p}^{-j}u(m)$ for some $j\in\mathbb{Z}$ and $m\in\mathbb{N}_0$, where $\alpha=m/q^{-j}$ is called a nonnegative q-adic number. With this concept, we consider the set

$$\Gamma=\left\{\alpha\in K: \text{there exists } (j,m)\in\mathbb{Z}\times\mathbb{N}_0 \text{ such that } \frac{m}{q^{-j}}\right\}, \tag{2.8}$$

and for all $\alpha\in\Gamma$, we define

$$I(\alpha)=\left\{(j,m)\in\mathbb{Z}\times\mathbb{N}_0: \alpha=\frac{m}{q^{-j}}\right\}, \tag{2.9a}$$

$$\Delta_\alpha(\psi,\xi)=\sum_{(j,m)\in I(\alpha)}\hat{\psi}(\mathfrak{p}^j\xi)\overline{\hat{\psi}(\mathfrak{p}^j\xi+u(m))}. \tag{2.9b}$$

It is easy to verify the following facts:

(i) $I(0)=\mathbb{Z}\times\{0\}$;

(ii) For any $\alpha_1,\alpha_2\in\Gamma$, if $u(\alpha_1)\neq u(\alpha_2)$, then $\alpha_1\neq\alpha_2$;

(iii) For any $\alpha_1,\alpha_2\in\Gamma$, if $\alpha_1\neq\alpha_2$, then $I(\alpha_1)\neq I(\alpha_2)$;

(iv) $\mathbb{Z}\times\mathbb{N}_0=\bigcup_{\alpha\in\Gamma}I(\alpha)$, and $\mathbb{Z}\times\mathbb{N}=\bigcup_{\alpha\in\Gamma\setminus\{0\}}I(\alpha)$.

With the notations above, for every $n\in\mathbb{N}_0$ and $\alpha\in\Gamma\setminus\{0\}$, we can define

$$G_{\alpha,n}(\psi,\xi)=\left[\Delta_{(k-\ell)\alpha}\big(\psi,\xi+u(\ell\alpha)\big)\right]_{-n\le k,\ell\le n}. \tag{2.10}$$

We now state a generalized necessary condition for the wavelet system $\{\psi_{j,k}: j\in\mathbb{Z},\ k\in\mathbb{N}_0\}$ to be a frame for $L^2(K)$.

Theorem 2.1. *If $\{\psi_{j,k}: j\in\mathbb{Z},\ k\in\mathbb{N}_0\}$ is a wavelet frame for $L^2(K)$ with bounds A and B, then*

$$A\le G_{\alpha,n}(\psi,\xi)\le B,\quad a.e.\ \forall n\in\mathbb{N},\quad \alpha\in\Gamma\setminus\{0\}, \tag{2.11}$$

where $G_{\alpha,n}(\psi,\xi)$ is given by (2.10).

3 Proof of the main results

In this section, we give the proof of the main result. Since Ω is dense in $L^2(K)$ and closed under the Fourier transform, the set

$$\Omega^0=\Big\{f\in\Omega:\mathrm{supp}\hat{f}\subset K\backslash\{0\}\ \text{ and }\ \|\hat{f}\|_\infty<\infty\Big\}$$

is also dense in $L^2(K)$. Therefore, it is enough to say that the system $\mathcal{F}_\Psi$ given by (2.5) is a wavelet frame for $L^2(K)$ if the inequalities in (2.6) hold for all $f\in\Omega^0$. Moreover, we call a function $\psi\in L^2(K)$ admissible if

$$C_\psi=\int_K\frac{|\hat{\psi}(\xi)|^2}{|\xi|}d\xi<\infty. \tag{3.1}$$

The following lemma is useful in changing the summation order.

Lemma 3.1. *If $f\in\Omega^0$ and $\psi\in L^2(K)$, then*

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\int_K\Big|\hat{f}(\xi)\hat{f}(\xi+\mathfrak{p}^{-j}u(k))\hat{\psi}(\mathfrak{p}^j\xi)\hat{\psi}(\mathfrak{p}^j\xi+u(k))\Big|d\xi<\infty.$$

Proof. For any $f\in\Omega^0$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\int_K\Big|\hat{f}(\xi)\hat{f}(\xi+\mathfrak{p}^{-j}u(k))\hat{\psi}(\mathfrak{p}^j\xi)\hat{\psi}(\mathfrak{p}^j\xi+u(k))\Big|d\xi\\
=&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}q^j\int_K\Big|\hat{f}(\mathfrak{p}^{-j}\xi)\hat{f}(\mathfrak{p}^{-j}+\mathfrak{p}^{-j}u(k))\hat{\psi}(\xi)\hat{\psi}(\xi+u(k))\Big|d\xi\\
\le&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}q^j\left\{\int_K\Big|\hat{f}(\mathfrak{p}^{-j})\hat{f}\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big||\hat{\psi}(\xi)|^2d\xi\right\}^{1/2}\\
&\qquad\times\left\{\int_K\Big|\hat{f}(\mathfrak{p}^{-j})\hat{f}\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big||\hat{\psi}(\xi+u(k))|^2d\xi\right\}^{1/2}\\
\le&\sum_{j\in\mathbb{Z}}q^j\left\{\sum_{k\in\mathbb{N}_0}\int_K\Big|\hat{f}(\mathfrak{p}^{-j})\hat{f}\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big||\hat{\psi}(\xi)|^2d\xi\right\}^{1/2}\\
&\qquad\times\left\{\sum_{k\in\mathbb{N}_0}\int_K\Big|\hat{f}(\mathfrak{p}^{-j})\hat{f}\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big||\hat{\psi}(\xi+u(k))|^2d\xi\right\}^{1/2}\\
=&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}q^j\int_K\Big|\hat{f}(\mathfrak{p}^{-j}\xi)\hat{f}\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big||\hat{\psi}(\xi)|^2d\xi\\
=&\int_K\sum_{j\in\mathbb{Z}}q^j\Big|\hat{f}(\mathfrak{p}^{-j}\xi)\Big|\sum_{k\in\mathbb{N}_0}\Big|\hat{f}\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big||\hat{\psi}(\xi)|^2d\xi.
\end{aligned}$$

Since $f\in\Omega^0$, hence the number of elements of the set

$$\#\Big\{k\in\mathbb{N}_0 : \hat f\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\neq 0\Big\}\le(2q^{-j}+1),\quad \forall\xi\in K.$$

Therefore, we have

$$\sum_{k\in\mathbb{N}_0}\Big|\hat f\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big|\le(2q^{-j}+1)\left\|\hat f\right\|_\infty,\quad \forall\xi\in K.$$

Consequently, we have

$$\begin{aligned}&\int_K\sum_{j\in\mathbb{Z}}q^j\left|\hat f(\mathfrak{p}^{-j}\xi)\right|\sum_{k\in\mathbb{N}_0}\Big|\hat f\Big(\mathfrak{p}^{-j}(\xi+u(k))\Big)\Big|\,|\hat\psi(\xi)|^2d\xi\\ \le&\left\|\hat f\right\|_\infty\int_K\sum_{j\in\mathbb{Z}}q^j\left|\hat f(\mathfrak{p}^{-j}\xi)\right|(2q^{-j}+1)\,|\hat\psi(\xi)|^2d\xi\\ =&\left\|\hat f\right\|_\infty\int_K\sum_{j\in\mathbb{Z}}\left|\hat f(\mathfrak{p}^{-j}\xi)\right|\,|\hat\psi(\xi)|^2d\xi+\left\|\hat f\right\|_\infty\int_K\sum_{j\in\mathbb{Z}}\left|\hat f(\mathfrak{p}^{-j}\xi)\right|q^j|\xi|\frac{|\hat\psi(\xi)|^2}{|\xi|}d\xi.\end{aligned}\tag{3.2}$$

Using the fact that $f\in\Omega^0$ and for all $\xi\in K$, we have

$$\#\Big\{j\in\mathbb{Z} : \hat f(\mathfrak{p}^{-j}\xi)\neq 0\Big\}\le\left(\frac{2}{\ln q}+1\right)\|\hat f\|_\infty.$$

Hence

$$\sum_{j\in\mathbb{Z}}\left|\hat f(\mathfrak{p}^{-j}\xi)\right|\le\left(\frac{2}{\ln q}+1\right)\|\hat f\|_\infty\quad\text{and}\quad\sum_{j\in\mathbb{Z}}\left|\hat f(\mathfrak{p}^{-j}\xi)\right|q^j|\xi|\le q^j\left(\frac{2}{\ln q}+1\right)\|\hat f\|_\infty.\tag{3.3}$$

Combining the estimates (3.1)-(3.3), we get the desired result. □

Proof of Theorem 2.1. Implementing Proposition 2.1 for some fixed $\alpha\in\Gamma\setminus\{0\}$, it is obvious that both the series

$$\sum_{j\in\mathbb{Z}}\left|\hat\psi(\mathfrak{p}^j\xi)\right|^2\quad\text{and}\quad\sum_{(j,s)\in I(\alpha)}\hat\psi(\mathfrak{p}^j\xi)\,\overline{\hat\psi\Big(\mathfrak{p}^j(\xi+u(\alpha))\Big)}$$

are locally integrable on K. Assume E_j is the set of regular points of $\left|\hat\psi(\mathfrak{p}^j(\xi+u(\ell\alpha)))\right|^2$, which means that for each $x\in E_j$, we have

$$q^n\int_{\xi-x\in\mathfrak{B}^n}\Big|\hat\psi\Big(\mathfrak{p}^j(\xi+u(\ell\alpha))\Big)\Big|^2d\xi\to\Big|\hat\psi\Big(\mathfrak{p}^j(\xi+u(\ell\alpha))\Big)\Big|^2\quad\text{as } n\to\infty.$$

Then, $K\setminus\bigcup_{j\in\mathbb{Z}}E_j^c$ is a zero-measured set [14, Theorem 1.14]. Choose any ξ_0 in E_j such that $\xi_0+u(\ell\alpha)\neq 0$, $|\ell|\le n$. For sufficiently small ε, $0<\varepsilon<|\alpha|$ and $\beta\in K$, we denote

$H_\varepsilon(\beta)=\big[\xi_0+\beta-\varepsilon/2,\xi_0+\beta+\varepsilon/2\big]$ a ball of radius $\varepsilon>0$ with centre $\xi_0+\beta$. For this choice of ball, we define f_ε by

$$\hat{f}_\epsilon(\xi)=\frac{1}{\sqrt{\varepsilon}}\sum_{|\ell|\le n} d_\ell \mathbf{\Phi}_{H_\varepsilon(\ell\alpha)}(\xi), \tag{3.4}$$

where

$$d=\{d_\ell\}_{|\ell|\le n}\in\mathbb{C}^{2n+1}.$$

Therefore, we obtain

$$\Big\|\hat{f}_\epsilon\Big\|_2=\|f_\epsilon\|_2=1.$$

Let

$$I_1=\sum_{j\le -M}\sum_{k\in\mathbb{N}_0}\big|\langle f_\varepsilon,\psi_{j,k}\rangle\big|^2 \quad\text{and}\quad I_2=\sum_{j>-M}\sum_{k\in\mathbb{N}_0}\big|\langle f_\varepsilon,\psi_{j,k}\rangle\big|^2,$$

where M is positive integer to be determined later. Then we have

$$\begin{aligned}
I_1&=\sum_{j\le -M}q^{-j}\sum_{k\in\mathbb{N}_0}\left|\int_K \hat{f}_\varepsilon(\xi)\overline{\hat{\psi}(\mathfrak{p}^j\xi)}\chi_k(\mathfrak{p}^j\xi)\,d\xi\right|^2\\
&=\sum_{j\le -M}q^{j}\sum_{k\in\mathbb{N}_0}\left|\int_K \hat{f}_\varepsilon(\mathfrak{p}^{-j}\xi)\overline{\hat{\psi}(\xi)}\chi_k(\xi)\,d\xi\right|^2\\
&=\sum_{j\le -M}q^{j}\sum_{k\in\mathbb{N}_0}\left|\int_{\mathfrak{D}}\sum_{s\in\mathbb{N}_0} \hat{f}_\varepsilon\Big(\mathfrak{p}^{-j}\big(\xi+u(s)\big)\Big)\overline{\hat{\psi}(\xi+u(s))}\chi_k(\xi)\,d\xi\right|^2\\
&=\sum_{j\le -M}q^{j}\int_{\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0} \hat{f}_\varepsilon\Big(\mathfrak{p}^{-j}\big(\xi+u(s)\big)\Big)\overline{\hat{\psi}(\xi+u(s))}\chi_k(\xi)\,d\xi\right|^2\\
&=\sum_{j\le -M}q^{j}\int_{(\mathfrak{p}^j{}_\circ)\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0} \hat{f}_\varepsilon\Big(\mathfrak{p}^{-j}\big(\xi+u(s)\big)\Big)\overline{\hat{\psi}(\xi+u(s))}\chi_k(\xi)\,d\xi\right|^2.
\end{aligned}$$

By the Cauchy inequality, we have

$$\begin{aligned}
I_1&\le\sum_{j\le -M}q^{j}\int_{(\mathfrak{p}^j{}_\circ)\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0}\hat{\psi}(\xi+u(s))\right|^2\cdot\Big|\hat{f}_\varepsilon\Big(\mathfrak{p}^{-j}\big(\xi+u(s)\big)\Big)\Big|\sum_{m\in\mathbb{N}_0}\Big|\hat{f}_\varepsilon\Big(\mathfrak{p}^{-j}\big(\xi+u(m)\big)\Big)\Big|\,d\xi\\
&\le\sum_{j\le -M}\frac{q^j}{\varepsilon}\int_{(\mathfrak{p}^j{}_\circ)\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0}\hat{\psi}(\xi+u(s))\right|^2\sum_{|\ell|\le n}|d_\ell|\mathbf{\Phi}_{H_\varepsilon(\ell\alpha)}\Big(\mathfrak{p}^{-j}\big(\xi+u(s)\big)\Big)\\
&\qquad\times\sum_{m\in\mathbb{N}_0}\sum_{|\ell'|\le n}|d'_\ell|\mathbf{\Phi}_{H_\varepsilon(\ell'\alpha)}\Big(\mathfrak{p}^{-j}\big(\xi+u(m)\big)\Big)\,d\xi\\
&\le\|d\|_\infty^2\sum_{j\le -M}\frac{q^j}{\varepsilon}\int_{(\mathfrak{p}^j{}_\circ)\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0}\hat{\psi}(\xi+u(s))\right|^2\sum_{|\ell|\le n}\mathbf{\Phi}_{H_\varepsilon(\ell\alpha)}\Big(\mathfrak{p}^{-j}\big(\xi+u(s)\big)\Big)
\end{aligned}$$

$$\times\sum_{m\in\mathbb{N}_0}\sum_{|\ell'|\le n}\mathbf{\Phi}_{H_\varepsilon(\ell'\alpha)}\Big(\mathfrak{p}^{-j}\big(\xi+u(m)\big)\Big)d\xi.$$

It is easy to verify that whenever

$$\xi\in\big[\mathfrak{p}^j\xi_0-1/2,\mathfrak{p}^j\xi_0-1/2\big]\quad\text{ and }\quad\big|u(m)-\mathfrak{p}^j u(\ell'\alpha)\big|>\big(\varepsilon/2q^j+1/2\big),$$

we have

$$\mathbf{\Phi}_{H_\varepsilon(\ell'\alpha)}\Big(\mathfrak{p}^{-j}\big(\xi+u(m)\big)\Big)=0.$$

Hence,

$$\begin{aligned}
I_1\le&(2n+1)\|d\|_\infty^2\sum_{j\le -M}\frac{q^j}{\varepsilon}\int_{(\mathfrak{p}^j\xi_0)\mathfrak{D}}\Big|\sum_{s\in\mathbb{N}_0}\hat\psi\big(\xi+u(s)\big)\Big|^2\\
&\times\sum_{|\ell|\le n}\mathbf{\Phi}_{H_\varepsilon(\ell\alpha)}\Big(\mathfrak{p}^{-j}\big(\xi+u(s)\big)\Big)\big(\varepsilon\mathfrak{p}^j+2\big)d\xi\\
=&C\sum_{j\le -M}\frac{q^j}{\varepsilon}\int_K\big|\hat\psi(\xi)\big|^2\sum_{|\ell|\le n}\mathbf{\Phi}_{H_\varepsilon(\ell\alpha)}\big(\mathfrak{p}^{-j}\xi\big)\big(\varepsilon\mathfrak{p}^j+2\big)d\xi\\
\le&C_1\sum_{j\le -M}\sum_{|\ell|\le n}\int_{\mathfrak{p}^jH_\varepsilon(\ell\alpha)}\left(\big|\hat\psi(\xi)\big|^2+\frac{q^j}{\varepsilon}\big|\hat\psi(\xi)\big|^2\right)d\xi.
\end{aligned}\tag{3.5}$$

Recall that we choose the regular point ξ_0 in E_j such that $\xi_0+u(\ell\alpha)\neq 0$, $|\ell|\le n$. Therefore, without loss of generality, we assume that

$$0<\varepsilon<\frac{2(q-1)}{(q+1)}\min\Big\{\big|\xi_0+u(\ell\alpha)\big|:|\ell|\le n\Big\}.$$

Then, we have

$$\sum_{j\le -M}\sum_{|\ell|\le n}\int_{\mathfrak{p}^jH_\varepsilon(\ell\alpha)}\big|\hat\psi(\xi)\big|^2d\xi\le\sum_{|\ell|\le n}\int_{|\xi|>2\mathfrak{p}^{-M}|\xi_0+u(\ell\alpha)|/(q+1)}\big|\hat\psi(\xi)\big|^2d\xi.\tag{3.6}$$

Using (3.6) in (3.5), we obtain

$$I_1\le C_1\sum_{|\ell|\le n}\left\{\int_{|\xi|>2\mathfrak{p}^{-M}|\xi_0+u(\ell\alpha)|/(q+1)}\big|\hat\psi(\xi)\big|^2d\xi+\frac{1}{\varepsilon}\int_{H_\varepsilon(\ell\alpha)}\sum_{j\le -M}\Big|\hat\psi\big(\mathfrak{p}^j\xi\big)\Big|^2d\xi\right\}.$$

Since ξ_0 is a regular point of

$$\sum_{j\le -M}\Big|\hat\psi\Big(\mathfrak{p}^{-j}\big(\xi+u(\ell\alpha)\big)\Big)\Big|^2,$$

so we have

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\left|I_1\right|\\ \le\,& C_1\sum_{|\ell|\le n}\int_{|\xi|>2\mathfrak{p}^{-M}|\xi_0+u(\ell\alpha)|/(q+1)}\left|\hat{\psi}(\xi)\right|^2 d\xi\\ &+C_1\sum_{|\ell|\le n}\sum_{j\le -M}\left|\hat{\psi}\Big(\mathfrak{p}^j\big(\xi_0+u(\ell\alpha)\big)\Big)\right|^2. \end{aligned}\tag{3.7}$$

Now we estimate I_2. Since $f\in S_0$, by Lemma 3.1, we have

$$\begin{aligned}I_2=&\sum_{j>-M}q^{-j}\sum_{k\in\mathbb{N}_0}\left|\int_K\hat{f}_\varepsilon(\xi)\overline{\hat{\psi}(\mathfrak{p}^j)}\chi_k(\mathfrak{p}^j\xi)d\xi\right|^2\\ =&\sum_{j>-M}q^{-j}\sum_{k\in\mathbb{N}_0}\left|\int_{\mathfrak{p}^{-j}\mathfrak{D}}\sum_{s\in\mathbb{N}_0}\hat{f}_\epsilon\big(\xi-\mathfrak{p}^{-j}u(s)\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi-u(s)\big)}\chi_k\big(\mathfrak{p}^j\xi\big)d\xi\right|^2\\ =&\sum_{j>-M}\int_{\mathfrak{p}^{-j}\mathfrak{D}}\left|\sum_{s\in\mathbb{N}_0}\hat{f}_\varepsilon\big(\xi-\mathfrak{p}^{-j}u(s)\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi-u(s)\big)}\right|^2 d\xi\\ =&\sum_{j>-M}\int_{\mathfrak{p}^{-j}\mathfrak{D}}\sum_{\ell\in\mathbb{N}_0}\overline{\hat{f}_\varepsilon\big(\xi-\mathfrak{p}^{-j}u(\ell)\big)}\hat{\psi}\big(\mathfrak{p}^j\xi-u(\ell)\big)\sum_{s\in\mathbb{N}_0}\hat{f}_\varepsilon\big(\xi-\mathfrak{p}^{-j}u(s)\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi-u(s)\big)}\\ =&\sum_{j>-M}\sum_{s\in\mathbb{N}_0}\int_K\overline{\hat{f}_\varepsilon(\xi)}\hat{\psi}\big(\mathfrak{p}^j\xi\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(s)\big)}\hat{f}_\varepsilon\big(\xi+\mathfrak{p}^{-j}u(s)\big)\\ =&\sum_{\alpha'\in\Gamma}\sum_{\substack{(-j,s)\in I(\alpha')\\ j>-M}}\int_K\overline{\hat{f}_\varepsilon(\xi)}\hat{f}_\varepsilon\big(\xi+u(\alpha')\big)\hat{\psi}\big(\mathfrak{p}^j\xi\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(s)\big)}d\xi\\ =&\sum_{\substack{(-j,s)\in I(0)\\ j>-M}}\int_K\left|\hat{f}_\varepsilon(\xi)\right|^2\left|\hat{\psi}\big(\mathfrak{p}^j\xi\big)\right|^2 d\xi\\ &+\sum_{1\le\ell\le 2n}\sum_{\substack{(-j,s)\in I(\ell\alpha)\\ j>-M}}\int_K\overline{\hat{f}_\varepsilon(\xi)}\hat{f}_\varepsilon\big(\xi+u(\ell\alpha)\big)\hat{\psi}\big(\mathfrak{p}^j\xi\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(s)\big)}d\xi\\ &+\sum_{1\le\ell\le 2n}\sum_{\substack{(-j,s)\in I(-\ell\alpha)\\ j>-M}}\int_K\overline{\hat{f}_\varepsilon(\xi)}\hat{f}_\varepsilon\big(\xi-u(\ell\alpha)\big)\hat{\psi}\big(\mathfrak{p}^j\xi\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(s)\big)}d\xi\\ &+\sum_{\substack{(-j,s)\in I(\alpha')\\ j>-M}}\sum_{\alpha'\in\Gamma\backslash\{\pm\ell\alpha,0\le\ell\le 2n\}}\int_K\overline{\hat{f}_\varepsilon(\xi)}\hat{f}_\varepsilon\big(\xi+u(\alpha')\big)\hat{\psi}\big(\mathfrak{p}^j\xi\big)\overline{\hat{\psi}\big(\mathfrak{p}^j\xi+u(s)\big)}d\xi\\ =&I_{21}+I_{22}+I_{23}+I_{24}.\end{aligned}$$

First, we estimate I_{21}. It is easy to verify that

$$\begin{aligned}\lim_{\varepsilon\to 0} I_{21} &= \lim_{\varepsilon\to 0} \sum_{\substack{(-j,s)\in I(0)\\ j>-M}} \frac{1}{\varepsilon}\sum_{|k|\le n} |d_k|^2 \int_{H_\varepsilon(k\alpha)} \left|\hat{\psi}(\mathfrak{p}^{\mathrm{j}}\xi)\right|^2 d\xi \\ &= \sum_{|k|\le n} |d_k|^2 \left(\sum_{j>-M}\left|\hat{\psi}\Big(\mathfrak{p}^{\mathrm{j}}\big(\xi_0+u(k\alpha)\big)\Big)\right|^2\right). \end{aligned} \tag{3.8}$$

Next we proceed to estimate I_{22}. We have

$$\begin{aligned} I_{22} &= \sum_{1\le\ell\le 2n}\sum_{\substack{(-j,s)\in I(\ell\alpha)\\ j>-M}} \frac{1}{\sqrt{\varepsilon}}\sum_{|k|\le n}\overline{d_k}\int_{H_\varepsilon(k\alpha)} \hat{f}\big(\xi+u(\ell\alpha)\big)\hat{\psi}(\mathfrak{p}^{\mathrm{j}}\xi)\overline{\hat{\psi}\big(\mathfrak{p}^{\mathrm{j}}\xi+u(s)\big)}d\xi \\ &= \sum_{1\le\ell\le 2n}\sum_{\substack{(-j,s)\in I(\ell\alpha)\\ j>-M}} \frac{1}{\varepsilon}\sum_{|k|\le n}\overline{d_k}\int_{H_\varepsilon(k\alpha)}\sum_{|m|\le n} d_m\mathbf{\Phi}_{H_\varepsilon(m\alpha)}\big(\xi+u(\ell\alpha)\big)\hat{\psi}(\mathfrak{p}^{\mathrm{j}}\xi)\overline{\hat{\psi}\big(\mathfrak{p}^{\mathrm{j}}\xi+u(s)\big)}d\xi \\ &= \sum_{1\le\ell\le 2n}\sum_{\substack{(-j,s)\in I(\ell\alpha)\\ j>-M}} \frac{1}{\varepsilon}\sum_{|k|\le n}\overline{d_k}d_{k+\ell}\int_{H_\varepsilon(k\alpha)}\hat{\psi}(\mathfrak{p}^{\mathrm{j}}\xi)\overline{\hat{\psi}\big(\mathfrak{p}^{\mathrm{j}}\xi+u(s)\big)}d\xi. \end{aligned}$$

Thus, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0} I_{22} \\ &= \sum_{1\le\ell\le 2n}\sum_{-n\le k\le n-\ell}\overline{d_k}d_{k+\ell}\sum_{\substack{(-j,s)\in I(\ell\alpha)\\ j>-M}}\hat{\psi}\Big(\mathfrak{p}^{\mathrm{j}}\big(\xi_0+u(k\alpha)\big)\Big)\overline{\hat{\psi}\Big(\mathfrak{p}^{\mathrm{j}}\big(\xi_0+u(k\alpha)\big)+u(s)\Big)}. \end{aligned}\tag{3.9}$$

Similarly, we can prove that

$$\begin{aligned}&\lim_{\varepsilon\to 0} I_{23} \\ &= \sum_{1\le\ell\le 2n}\sum_{\ell-n\le k\le n}\overline{d_k}d_{k+\ell}\sum_{\substack{(-j,s)\in I(-\ell\alpha)\\ j>-M}}\hat{\psi}\Big(\mathfrak{p}^{\mathrm{j}}\big(\xi_0+u(k\alpha)\big)\Big)\overline{\hat{\psi}\Big(\mathfrak{p}^{\mathrm{j}}\big(\xi_0+u(k\alpha)\big)+u(s)\Big)}. \end{aligned}\tag{3.10}$$

Finally, we estimate I_{24}. It is evident that

$$\overline{\hat{f}_\varepsilon(\xi)}\hat{f}_\varepsilon\big(\xi+u(\alpha')\big)\neq 0$$

only if at least one of

$$u(\alpha')\pm u(\ell\alpha)\in G_\varepsilon(\alpha)=\{\alpha:|\alpha|<\varepsilon\},\quad 0\le\ell\le n,$$

hold, where $\alpha'=q^js$. For $j>0$, we consider

$$\mathcal{S}_j=\Big\{s\in\mathbb{N}_0:\text{there is some } |\ell|\le n \text{ such that } \Big|q^ju(s)\pm u(\ell\alpha)\Big|\le\varepsilon\Big\}.$$

Then for $s\in\mathcal{S}_j$, we have

$$|s|\le q^{-j}\big(u(\ell\alpha)+\varepsilon\big)\le q^{-j}\big(u(\ell\alpha)|+1\big),\quad \text{for } \varepsilon<1. \tag{3.11}$$

Since $s\neq 0$, we have $|s|\ge 1$ and $\mathcal{S}_j=\emptyset$ for sufficiently large j. Therefore, for some positive integer M, we have

$$\begin{aligned}&I_{24}\\ =&\sum_{\substack{(-j,s)\in I(\alpha')\\ |j|\le M}}\sum_{\alpha'\in\Gamma\setminus\{\pm\ell\alpha,0\le\ell\le 2n\}}\int_K\overline{\hat f_\varepsilon(\xi)}\hat f_\varepsilon\big(\xi+u(\alpha')\big)\hat\psi(\mathfrak{p}^j\xi)\overline{\hat\psi\big(\mathfrak{p}^j\xi+u(s)\big)}d\xi.\end{aligned}$$

By implementing (3.11) in the above series, we observe that it contains only finitely many terms which are not equal to zero, where as for every $\alpha'\in\Gamma\setminus\{\pm\ell\alpha,\ 0\le\ell\le 2n\}$, the integration in I_{24} tends to zero as $\varepsilon\to 0$. Thus, we conclude that

$$\lim_{\varepsilon\to 0}I_{24}=0. \tag{3.12}$$

Combining Eqs. (3.7)-(3.10) and (3.12), we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\Bigg|I_1+I_2-\sum_{|k|\le n}\big|d_k\big|^2\Bigg(\sum_{j>-M}\Big|\hat\psi\Big(\mathfrak{p}^j\big(\xi_0+u(k\alpha)\big)\Big)\Big|^2\Bigg)\\
&\quad-\sum_{1\le\ell\le 2n}\sum_{-n\le k\le n-\ell}\overline{d_k}d_{k+\ell}\sum_{\substack{(-j,s)\in I(\ell\alpha)\\ j>-M}}\hat\psi\Big(\mathfrak{p}^j\big(\xi_0+u(k\alpha)\big)\Big)\overline{\hat\psi\Big(\mathfrak{p}^j\big(\xi_0+u(k\alpha)\big)+u(s)\Big)}\\
&\quad-\sum_{1\le\ell\le 2n}\sum_{\ell-n\le k\le n}\overline{d_k}d_{k+\ell}\sum_{\substack{(-j,s)\in I(-\ell\alpha)\\ j>-M}}\hat\psi\Big(\mathfrak{p}^j\big(\xi_0+u(k\alpha)\big)\Big)\overline{\hat\psi\Big(\mathfrak{p}^j\big(\xi_0+u(k\alpha)\big)+u(s)\Big)}\Bigg|\\
\le& C_1\sum_{|\ell|\le n}\int_{|\xi|>2\mathfrak{p}^{-M}|\xi_0+u(\ell\alpha)|/(q+1)}\big|\hat\psi(\xi)\big|^2d\xi+C_1\sum_{|\ell|\le n}\sum_{j\le -M}\Big|\hat\psi\Big(\mathfrak{p}^j\big(\xi_0+u(\ell\alpha)\big)\Big)\Big|^2.\end{aligned}$$

By letting $M\to\infty$, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\big(I_1+I_2\big)\\
=&\sum_{|k|\le n}\big|d_k\big|^2\Delta_0\big(\psi,\xi_0+u(k\alpha)\big)\\
&+\sum_{1\le\ell\le 2n}\Big[\sum_{-n\le k\le n-\ell}\overline{d_k}d_{k+\ell}\Delta_{\ell\alpha}\big(\psi,\xi_0+u(k\alpha)\big)\\
&+\sum_{\ell-n\le k\le n}\overline{d_k}d_{k-\ell}\Delta_{-\ell\alpha}\big(\psi,\xi_0+u(k\alpha)\big)\Big].\end{aligned}$$

Since $\|\hat{f}_\varepsilon\| = \|d\|^2$, it follows that

$$\begin{aligned} A\|d\|^2 \le & \sum_{|k|\le n} |d_k|^2 \Delta_0(\psi,\xi_0+u(k\alpha)) \\ & + \sum_{1\le \ell\le 2n} \Big[\sum_{-n\le k\le n-\ell} \overline{d_k} d_{k+\ell} \Delta_{\ell\alpha}(\psi,\xi_0+u(k\alpha)) \\ & + \sum_{-n+\ell\le k\le n} \overline{d_k} d_{k-\ell} \Delta_{-\ell\alpha}(\psi,\xi_0+u(k\alpha)) \Big] \\ \le & B\|d\|^2. \end{aligned}$$

Since d is arbitrary, the above inequalities are equivalent to

$$A \le G_{\alpha,n}(\psi,\xi) \le B.$$

This completes the proof of Theorem 2.1. □

Remark 3.1. Since

$$\Delta_0(\psi,\xi) = \sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathfrak{p}^j\xi)|^2,$$

therefore Eq. (2.11) is a generalization of Eq. (2.7).

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