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The Oscillation Inequality of Harmonic Functions on Post Critically Finite Self-Similar Sets

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Abstract. In this paper we establish the oscillation inequality of harmonic functions and Hölder estimate of the functions in the domain of the Laplacian on connected post critically finite (p.c.f.) self-similar sets.

Key Words: p.c.f. Self-similar sets, oscillation inequality, Hölder estimate, harmonic functions.

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1 Introduction

Recently there are considerable interests in studying Hölder estimates of harmonic functions or functions in the domain of Dirichlet forms and the Laplacian on various fractal sets (see [1,2,4–10]). The results above can be used in the upper bound estimates of heat kernel, the transition density estimates, and spaces embedding.

In [6] Strichartz established the Hölder estimates of harmonic functions and the functions in the domain of the Laplacian on a class of Sierpinski gasket type sets with D_3 symmetry. In [9], the authors extended the results for harmonic functions on level n Sierpinski gaskets and n-gaskets.

Let F_i, $i=1,\cdots,N$ be contractive mappings, K is the unique nonempty compact set satisfying

$$K=\bigcup_{i=1}^{N} F_i K,$$

$V_0=\{p_1,\cdots,p_n\}$ is boundary of K with $n\le N$, and f is continuous on K. The initial energy of K can be defined as

$$\varepsilon_0(f,f)=\sum_{1\le i<j\le n}(f(p_i)-f(p_j))^2.$$

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In [4] Kigami showed that there existed the matrices A_j, $j=1,2,\cdots,N$ satisfying $h|_{F_iV_0}=A_jh|_{V_0}$ if h is a harmonic function on K.

For any stochastic matrix $M=(M_{i,j})$, Hajnal [3] defined

$$\lambda(M)=1-\min_{i_1,i_2}\sum_j\min\{M_{i_1,j},M_{i_2,j}\}\in[0,1].$$

Let $A_i=(a^i_{jk})_{n\times n}$, $i=1,2,\cdots,N$ denote the harmonic extension matrices obtained by using Kigami's Theory (see [4] for details). Let $\lambda_i\triangleq\lambda(A_i)$ for $i=1,2,\cdots,N$.

In the present paper we have obtained the main results as follows:

Theorem 1.1. *Let $Osc(f,E)$ denote the difference between the maximum and minimum values of f on a set E. If a continuous function h is harmonic on a p.c.f. self-similar set K, then*

$$Osc(h,F_iK)\le\lambda_i\cdot Osc(h,K)\quad \text{for}\ \ 1\le i\le N. \tag{1.1}$$

Moreover,

$$Osc(h,F_wK)\le\lambda_w\cdot Osc(h,K). \tag{1.2}$$

Where $\lambda_w=\lambda_{w_1}\cdots\lambda_{w_m}$ for the word $w=w_1\cdots w_m$. Furthermore,

$$|h(x)-h(y)|\le 2\lambda_w\cdot\|h\|_\infty,\quad \text{if}\ \ x,y\in F_wK. \tag{1.3}$$

The paper is arranged as follows. In Section 2 we show some basic facts about p.c.f. self-similar sets. In Section 3 we establish the oscillation inequality of harmonic functions and Hölder estimate of the function in the domain of the Laplacian on p.c.f. self-similar sets. In Appendix, we show that if there exists $M_{ij}=1\in M$ for a stochastic matrix $M=(M_{i,j})$, then $\delta(M)=\lambda(M)$.

2 Basic facts about p.c.f. self-similar sets

In this section we summarize some basic facts about p.c.f. self-similar sets from Kigami [4].

Let $S=\{1,2,\cdots,N\}$, and K is the unique nonempty compact set satisfying

$$K=\bigcup_{i=1}^N F_iK.$$

For $m\ge 0$, we define $\Sigma_m=S^m=\{1,2,\cdots,N\}^m$ with $\Sigma_0=\{\emptyset\}$ and call $\emptyset$ the empty word. Also, set

$$\Sigma_*=\bigcup_{m\ge 0}\Sigma_m\quad\text{and}\quad\Sigma_\infty=S^\infty.$$

Denote the length of $w\in\Sigma_*$ by $|w|$ for $w=w_1w_2\cdots w_n$. For $k\le n$, let $w|_k=w_1w_2\cdots w_k$ denote the initial segment of w of length k.

For $w=w_1w_2\cdots w_n\in\Sigma_*$, we let $F_w=F_{w_1}\circ\cdots\circ F_{w_n}$ and $K_w=F_w(K)$.

Let F_i be a continuous injection from K to itself for any $i\in S$. Then $(K,S,\{F_i\}_{i\in S})$ is called a self-similar structure if there exists a continuous surjection $\pi:\Sigma_\infty\to K$, such that

$$F_i\circ\pi=\pi\circ\sigma_i \quad \text{for every } i\in S,$$

where $\sigma_i:\Sigma_\infty\to\Sigma_\infty$ is defined by

$$\sigma_i(w_1w_2w_3\cdots)=iw_1w_2w_3\cdots$$

for each $w_1w_2w_3\cdots\in\Sigma^{\mathbb{N}}$.

Let σ be the shift map on Σ_∞ defined as $\sigma(w_1w_2w_3\cdots)=w_2w_3\cdots$, and $(K,S,\{F_i\}_{i\in S})$ is a self-similar structure. We define

$$C_K=\bigcup_{i,j\in S,\, i\neq j}K_i\cap K_j,\quad C=\pi^{-1}(C_K),\quad P=\bigcup_{n\geq 1}\sigma^n(C),\quad V_0=\pi(P).$$

C is called the critical set and P is called the post critical set. $(K,S,\{F_i\}_{i\in S})$ is called to be post critically finite (p.c.f.) if and only if the post critical set P is a finite set.

Let p_j are fixed points of the mapping F_j, $j=1,2,\cdots,N$, $V_0=\{p_1,p_2,\cdots,p_n\}$ with $n\leq N$,

$$V_{k+1}=\bigcup_{j\in S}F_jV_k \quad \text{and} \quad V_*=\bigcup_{k\geq 0}V_k.$$

We use a sequence of complete graphs $G_0,G_1,\cdots$, with vertices $V_0\subseteq V_1\subseteq V_2\cdots$ to approximate a p.c.f. fractal K. For $x,y\in V_m$ and $x\neq y$, if there exists a word $w=(w_1,\cdots,w_m)$ with length $|w|=m$ such that $x,y\in F_wV_0$, we call x,y have the edge relation, denoted by $x\sim_m y$.

Let $r=\{r_1,r_2,\cdots,r_N\}$ be the scaling factor on K.

The initial energy is defined by

$$\varepsilon_0(f,f)=\sum_{1\leq i<j\leq n}(f(p_i)-f(p_j))^2. \tag{2.1}$$

On each graph G_m the energy ε_m is defined by

$$\varepsilon_m(f,f)=\sum_{i=1}^{N}\frac{1}{r_i}\varepsilon_{m-1}(f\circ F_i,f\circ F_i) \quad \text{for } m\geq 1, \tag{2.2}$$

here f is a continuous function on K.

Proposition 2.1 (see [4]). (i) For any continuous function f on K, the sequence $\varepsilon_m(f,f)$ is monotone increasing, so

$$\varepsilon(f,f)=\lim_{m\to\infty}\varepsilon_m(f,f) \tag{2.3}$$

is well-defined in $[0,\infty]$, and $\varepsilon(f,f)=0$ if and only if f is constant. The domain of the energy ε consists of continuous functions with finite energy.

(ii) A function h is said to be harmonic on K if it satisfies

$$\varepsilon(h,h)=\min_{u\in C(K)}\{\varepsilon(f,f):f|_{V_0}=h|_{V_0}\}.$$

The space of harmonic functions is n-dimensional with $n=\sharp V_0$ and any harmonic function can be determined uniquely by its boundary values. If h is harmonic on K, and the values of h on $V_0=\{p_1,p_2,\cdots,p_n\}$ are known, then all of the values $h(x)$ for $x\in V_m$ are desired as follow.

$$(h(F_wp_1)\ h(F_wp_2)\cdots h(F_wp_n))^T=A_w(h(p_1)h(p_2)\cdots h(p_n))^T, \tag{2.4}$$

here T means the transition of a matrix, $A_w=A_{w_m}\circ\cdots\circ A_{w_1}$ for a word $w=w_1\cdots w_m$, and A_{w_i} is a stochastic matrix introduced by Kigami [4] for $i=1,2,\cdots,N$.

3 Main results

In this section we will show the oscillation inequality of harmonic functions and Hölder estimate of the functions in the domain of the Laplacian on connected p.c.f. self-similar sets.

For any stochastic matrix $M=(M_{i,j})$, Hajnal [3] defined two parameters λ and δ.

$$\lambda(M):=1-\min_{i_1,i_2}\sum_j\min\{M_{i_1,j},M_{i_2,j}\}\in[0,1],$$

and

$$\delta(M):=\max_j\max_{i_1,i_2}|M_{i_1,j}-M_{i_2,j}|\in[0,1].$$

Generally $\delta(M)\le\lambda(M)$. One can prove that for a 2×2 or 3×3 stochastic matrix M, $\delta(M)=\lambda(M)$. It is known that (see [3])

$$\delta(M_1M_2\cdots M_k)\le\prod_{i=1}^k\lambda(M_i).$$

By Kigami's theory (see [4] for details), we can obtain the harmonic extension matrices $A_i=(a^i_{jk})_{n\times n}$, $i=1,2,\cdots,N$. Let $\delta_i\triangleq\delta(A_i)$, and $\lambda_i\triangleq\lambda(A_i)$, for $i=1,2,\cdots,N$.

Theorem 3.1. *Let $Osc(f,E)$ denote the difference between the maximum and minimum values of f on a set E. If a continuous function h is harmonic on a p.c.f. self-similar set K, then*

$$Osc(h,F_iK)\le\lambda_i\cdot Osc(h,K)\quad for\quad 1\le i\le N. \tag{3.1}$$

Moreover,

$$Osc(h,F_wK)\le\lambda_w\cdot Osc(h,K). \tag{3.2}$$

Where $\lambda_w=\lambda_{w_1}\cdots\lambda_{w_m}$ for the word $w=w_1\cdots w_m$. Furthermore,

$$|h(x)-h(y)|\le2\lambda_w\cdot\|h\|_\infty,\quad if\ \ x,y\in F_wK. \tag{3.3}$$

Proof. For a p.c.f. self-similar set K, suppose that h is a harmonic function on K with boundary values $h(p_1),h(p_2),\cdots,h(p_n)$. Without loss of generality, we assume that $h(p_1)\leq h(p_2)\leq\cdots\leq h(p_n)$. Then $h\circ F_i$, $i=1,2,\cdots,N$ has boundary values

$$h(F_ip_1)=\sum_{k=1}^{n}a_{1k}^{i}h(p_k),\quad h(F_ip_2)=\sum_{k=1}^{n}a_{2k}^{i}h(p_k),\quad \cdots,\quad h(F_ip_n)=\sum_{k=1}^{n}a_{nk}^{i}h(p_k).$$

By the Maximum principle (see [4, Theorem 3.2.5 and Theorem 3.2.15]), we have

$$\begin{aligned}Osc(h,K)&=\max_{1\leq k<j\leq n}|h(p_k)-h(p_j)|=|h(p_n)-h(p_1)|,\\ Osc(h,F_iK)&=\max_{1\leq l<m\leq n}|h(F_ip_l)-h(F_ip_m)|.\end{aligned}$$

Note that the harmonic extension matrices $A_i=(a_{jk}^i)_{n\times n}$, $i=1,2,\cdots,N$ are stochastic matrices, we have

$$\sum_{k=1}^{n}a_{jk}^{i}=1,\quad a_{jk}^{i}\geq 0,$$

for $j=1,2,\cdots,n$.

Without loss of generality, we only need to prove for $i=1$. Note that A_1 is of the form as follows.

$$A_1=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}.$$

Then, $h(F_1p_l)=a_{l1}h(p_1)+a_{l2}h(p_2)+\cdots+a_{ln}h(p_n)$, $h(F_1p_m)=a_{m1}h(p_1)+a_{m2}h(p_2)+\cdots+a_{mn}h(p_n)$, so,

$$\begin{aligned}&h(F_1p_l)-h(F_1p_m)\\ =&[a_{l1}h(p_1)+a_{l2}h(p_2)+\cdots+a_{ln}h(p_n)]-[a_{m1}h(p_1)+a_{m2}h(p_2)+\cdots\\ &\quad+a_{mn}h(p_n)]\\ =&[a_{l1}h(p_1)+a_{l2}h(p_2)+\cdots+(1-a_{l1}-a_{l2}-a_{l(n-1)})h(p_n)]-\\ &\quad-[a_{m1}h(p_1)+a_{m2}h(p_2)+\cdots+(1-a_{m1}-a_{m2}-a_{m(n-1)})h(p_n)]\\ =&(a_{m1}-a_{l1})[h(p_n)-h(p_1)]+(a_{m2}-a_{l2})[h(p_n)-h(p_2)]+\cdots\\ &\quad+(a_{m(n-1)}-a_{l(n-1)})[h(p_n)-h(p_{n-1})].\end{aligned}$$

Let $\{i|a_{mi}\geq a_{li}\}=\{i_1,i_2,\cdots,i_k\}$, $\{i|a_{mi}<a_{li}\}=\{i_{k+1},i_{k+2},\cdots,i_n\}$, then

$$\begin{aligned}&h(F_1p_l)-h(F_1p_m)\\ \leq&(a_{mi_1}-a_{li_1})[h(p_n)-h(p_{i_1})]+(a_{mi_2}-a_{li_2})[h(p_n)-h(p_{i_2})]+\cdots\\ &+(a_{mi_k}-a_{li_k})[h(p_n)-h(p_{i_k})]\\ \leq&(a_{mi_1}-a_{li_1})[h(p_n)-h(p_1)]+(a_{mi_2}-a_{li_2})[h(p_n)-h(p_1)]+\cdots\\ &+(a_{mi_k}-a_{li_k})[h(p_n)-h(p_1)]\\ =&[(a_{mi_1}+a_{mi_2}+\cdots+a_{mi_k})-(a_{li_1}+a_{li_2}+\cdots+a_{li_k})][h(p_n)-h(p_1)]\\ =&[1-(a_{mi_{k+1}}+a_{mi_{k+2}}+\cdots+a_{mi_n}+a_{li_1}+a_{li_2}+\cdots+a_{li_k})][h(p_n)-h(p_1)]\\ =&[1-\sum_j\min\{a_{lj},a_{mj}\}][h(p_n)-h(p_1)]\\ \leq&[1-\min_{i_1,i_2}\sum_j\min\{a_{i_1j},a_{i_2j}\}][h(p_n)-h(p_1)]=\lambda_1[h(p_n)-h(p_1)].\end{aligned}$$

Similarly we have

$$h(F_1p_m)-h(F_1p_l)\leq\lambda_1\cdot Osc(h,K).$$

Thus

$$|h(F_1p_l)-h(F_1p_m)|\leq\lambda_1\cdot Osc(h,K).$$

So

$$Osc(h,F_1K)=\max_{1\leq j<k\leq n}|h(F_1p_j)-h(F_1p_k)|\leq\lambda_1\cdot Osc(h,K)\quad\text{for}\quad 1\leq i\leq N.$$

Then (3.2) can be derived recursively.

By (3.2), we have

$$\begin{aligned}|h(x)-h(y)|\leq&\max_{x,y\in F_w(K)}|h(x)-h(y)|=Osc(h,F_wK)\\ \leq&\lambda_w\cdot Osc(h,K)\leq\lambda_w\cdot 2\|h\|_\infty=2\lambda_w\cdot\|h\|_\infty.\end{aligned}$$

Thus we finished the proof of Theorem 3.1. □

Remark 3.1. Note that for a stochastic matrix $M=(M_{i,j})$, if there exists $m_{ij}=1\in M$, then $\delta(M)=\lambda(M)$. See the appendix for the proof. Note that for general n-gasket with $4\nmid n$ with its boundary consisting of exactly n vertices, every harmonic extension matrix A_i, $i=1,2,\cdots,n$ always has an element $a_{ii}=1$. For level n Sierpinski gasket, the harmonic extension matrices are 3×3 matrices, so $\delta(M)=\lambda(M)$. This is just Theorem 4.1 in [9].

Theorem 3.2. *Let μ be a self-similar measure. For every function f on $dom(\Delta_\mu)$*

$$|f(x)-f(y)|\leq c\lambda_w,\quad\text{if}\ \ x,y\in F_wK,\tag{3.4}$$

where the constant c may be a multiple of $\|h\|_\infty$, and h is a harmonic function on K satisfying $h|_{V_0}=f|_{V_0}$.

Proof. Replacing r_w by λ_w in the proof of Theorem 8.4 in [6] will prove the result. □

Appendix

For a stochastic matrix $M=(M_{i,j})$, if there exists $M_{ij}=1\in M$, then $\delta(M)=\lambda(M)$.

Without loss of generality, we assume $M_{11}=1$. Then $M_{1j}=0$ for $j=2,3,\cdots,n$.

(1) When

$$\delta(M)=\max_{i_1,i_2}|M_{i_1,1}-M_{i_2,1}|.$$

There exists k such that

$$M_{k,1}=\min_{i}\{M_{i,1}\}.$$

Then

$$|M_{i_1,1}-M_{i_2,1}|\le|M_{1,1}-M_{k,1}|=1-M_{k,1},$$

so $1-M_{k,1}=\delta(M)$, $M_{k,1}=1-\delta(M)$.

For any i_1, i_2, we have

$$\begin{aligned}
&\min\{M_{1,j},M_{k,j}\}\le\min\{M_{i_1,j},M_{i_2,j}\},\\
&\sum_j\min\{M_{1,j},M_{k,j}\}\le\sum_j\min\{M_{i_1,j},M_{i_2,j}\},\\
&\sum_j\min\{M_{1,j},M_{k,j}\}=\min_{i_1,i_2}\sum_j\min\{M_{i_1,j},M_{i_2,j}\},\\
&1-\sum_j\min\{M_{1,j},M_{k,j}\}=1-\min_{i_1,i_2}\sum_j\min\{M_{i_1,j},M_{i_2,j}\}=\lambda(M),\\
&\lambda(M)=1-(1-\delta(M))=\delta(M).
\end{aligned}$$

(2) When

$$\delta(M)=\max_{i_1,i_2}|M_{i_1,h}-M_{i_2,h}|,\quad h\neq 1.$$

There exists k such that

$$M_{k,h}=\max_{i}\{M_{i,h}\}.$$

Then

$$|M_{i_1,h}-M_{i_2,h}|\le|M_{k,h}-M_{1,h}|=M_{k,h},$$

so $M_{k,h}=\delta(M)$.

If there exists i such that $M_{i,1}<1-\delta(M)$, then

$$|M_{1,1}-M_{i,1}|=M_{1,1}-M_{i,1}>1-(1-\delta(M))=\delta(M),$$

which is contradict with the definition of $\delta(M)$. That means $M_{i,1}\ge 1-\delta(M)$. Since

$$1=M_{k,1}+M_{k,2}+\cdots+M_{k,n}\ge M_{k,1}+M_{k,h}\ge 1-\delta(M)+\delta(M)=1,$$

thus $M_{k,1}=1-\delta(M)$.

For any i_1, i_2, we have

$$\begin{aligned}
&\min\{M_{1,j},M_{k,j}\}\le\min\{M_{i_1,j},M_{i_2,j}\},\\
&\sum_j\min\{M_{1,j},M_{k,j}\}\le\sum_j\min\{M_{i_1,j},M_{i_2,j}\},\\
&\sum_j\min\{M_{1,j},M_{k,j}\}=\min_{i_1,i_2}\sum_j\min\{M_{i_1,j},M_{i_2,j}\},\\
&1-\sum_j\min\{M_{1,j},M_{k,j}\}=1-\min_{i_1,i_2}\sum_j\min\{M_{i_1,j},M_{i_2,j}\}=\lambda(M),\\
&\lambda(M)=1-(1-\delta(M))=\delta(M).
\end{aligned}$$

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