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# New Characterizations of Operator-Valued Bases on Hilbert Spaces

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**Abstract.** In this paper we study operator valued bases on Hilbert spaces and similar to Schauder bases theory we introduce characterizations of this generalized bases in Hilbert spaces. We redefine the dual basis associated with a generalized basis and prove that the operators of a dual $g$-basis are continuous. Finally we consider the stability of $g$-bases under small perturbations. We generalize two results of Krein-Milman-Rutman and Paley-Wiener [7] to the situation of $g$-basis.

**Key Words**: $g$-bases, dual $g$-bases, $g$-biorthogonal sequence.

**AMS Subject Classifications**: 41A58, 42C15

## 1 Introduction

The frame was first introduced by Duffin and Schaeffer [3] in the study of nonharmonic Fourier series in 1952. Frames are a generalization of the orthonormal bases in Hilbert spaces. Throughout this paper, $\mathcal{H}$, $\mathcal{K}$ are separable Hilbert spaces and $I$, $J$, $J_i$ denote the countable (or finite) index sets and $\pi_W$ denote the orthogonal projection of a closed subspace $W$ of $\mathcal{H}$. We will always use $\mathcal{R}_T$ and $\mathcal{N}_T$ to denote range and the null spaces of an operator $T\in B(\mathcal{H},\mathcal{K})$ respectively. Recall that a family of vectors $\mathcal{F}=\{f_j\}_{j\in J}$ is called a frame for $\mathcal{H}$ if there exist constants $0<A\le B<\infty$ such that

$$A\|f\|^2\le\sum_{j\in J}|\langle f,f_j\rangle|^2\le B\|f\|^2 \quad \text{for all } f\in\mathcal{H}. \tag{1.1}$$

The constants $A$ and $B$ are called frame bounds. If we only have the right-hand inequality of (1.1), $\mathcal{F}$ call a Bessel sequence. The representation space associated with a Bessel sequence $\mathcal{F}=\{f_j\}_{j\in J}$ is $\ell^2(J)$ and the synthesis operator of its is the bounded linear operator $T_{\mathcal{F}}:\ell^2(J)\to\mathcal{H}$ which defines by $T_{\mathcal{F}}(\{c_j\}_{j\in J})=\sum_{j\in J}c_jf_j$. The analysis operator for $\mathcal{F}$

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is $T_{\mathcal{F}}^*:\mathcal{H}\to\ell^2(J)$ which satisfies $T_{\mathcal{F}}^*f=\{\langle f,f_j\rangle\}_{j\in J}$. By composing $T_{\mathcal{F}}$ and $T_{\mathcal{F}}^*$ we obtain the frame operator

$$S_{\mathcal{F}}:\mathcal{H}\to\mathcal{H}\quad S_{\mathcal{F}}f=T_{\mathcal{F}}T_{\mathcal{F}}^*f=\sum_{j\in J}\langle f,f_j\rangle f_j,$$

which is a positive, self-adjoint and invertible operator. A Riesz basis for $\mathcal{H}$ is a family of the form $\{U(e_j)\}_{j\in J}$, where $\{e_j\}_{j\in J}$ is an orthonormal basis for $\mathcal{H}$ and $U:\mathcal{H}\to\mathcal{H}$ is a bounded bijective operator. Nice properties of frames make them very useful in characterization of function spaces and other fields in sciences and engineering including coding theory, filter bank theory, signal and image processing and wireless communications. For more details about the theory and applications of frames and Riesz bases we refer the readers to [1,2,4].

Recently, W. Sun [6] introduced a $g$-frame and a Riesz $g$-basis for a Hilbert space and discussed some properties of them. In this paper we introduce new characterizations of $g$-basis and then we redefined the concepts of orthonormal $g$-basis and Riesz $g$-basis for a Hilbert space. We develop the basis theory to the situation of $g$-basis theory in Hilbert spaces.

The paper is organized as follows: Section 2, contains a new definition of $g$-basis for a Hilbert space. In this section similar to basis theory we first establishes a simple criterion for determining when a complete set of operators is a $g$-basis. Next we define the concepts of $g$-biorthogonal sequence, dual $g$-basis and obtain some characterizations of them. In Section 3, we redefine orthonormal $g$-basis and Riesz $g$-basis for a Hilbert space. We give some characterizations of orthonormal $g$-bases and Riesz $g$-bases. In Section 4, we study the stability of $g$-bases under small perturbations. We also generalize a result of Paley-Wiener [7] to the situation of $g$-basis.

## 2 Generalized Schauder bases

The purpose of this section is to investigate and develop the structure of bases for Hilbert spaces. We first define a $g$-basis for $\mathcal{H}$.

**Definition 2.1.** Let $\{W_j\}_{j\in J}$ be a sequence of closed subspaces of $\mathcal{K}$ and let $\Lambda_j\in B(\mathcal{H},W_j)$ be an onto operator for all $j\in J$. Then the family $\Lambda=\{\Lambda_j\}_{j\in J}$ is called a generalized Schauder basis or simply a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if for any $f\in\mathcal{H}$ there exists an unique sequence $\{g_j: g_j\in W_j\}_{j\in J}$ such that

$$f=\sum_{j\in J}\Lambda_j^*g_j, \tag{2.1}$$

with the convergence being in norm. If the series (2.1) converges unconditionally for each $f\in\mathcal{H}$, we say that $\Lambda$ is an unconditional $g$-basis. $\Lambda$ is called a $g$-basis for $\mathcal{H}$ with respect to $\mathcal{K}$ whenever $W_j=\mathcal{K}$ for all $j\in J$.

**Example 2.1.** For each $N\in\mathbb{N}$, let $\mathcal{H}=\mathbb{C}^N$ and $\mathcal{K}=\mathbb{C}^{N+1}$. Define $W_j\subset\mathcal{K}$ and $\Lambda_j:\mathcal{H}\to W_j$ by

$$W_j=\text{span}\Big\{\sum_{k=1}^{j+1}e_k\Big\},\quad \Lambda_j f=\Big(\sum_{i=1}^{j}z_i\Big)\sum_{k=1}^{j+1}e_k,\quad \forall f=\{z_i\}_{i=1}^N\in\mathcal{H},$$

for all $1\le j\le N$, where $\{e_k\}_{k=1}^{N+1}$ is the standard orthonormal basis for $\mathcal{K}$. Then $\{\Lambda_j\}_{j=1}^N$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j=1}^N$.

**Example 2.2.** Every bounded invertible linear operator $\Lambda:\mathcal{H}\to\mathcal{K}$ is a $g$-basis for $\mathcal{H}$ with respect to $\mathcal{K}$.

**Remark 2.1.** In general if $\{\Lambda_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ may be not a $g$-basis for $\mathcal{H}$ with respect to $\mathcal{K}$. The following is a counterexample.

**Example 2.3.** Let $\mathcal{H}$ and $\{W_j\}_{j=1}^N$ be as in Example 2.1. Then the family $\{\Lambda_j\}_{j=1}^N$ defined by

$$\Lambda_j:\mathcal{H}\to W_j,\quad \Lambda_j f=\frac{z_j}{j+1}\sum_{k=1}^{j+1}e_k,\quad \forall f=\{z_i\}_{i=1}^N\in\mathcal{H},$$

is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j=1}^N$. However the equality (2.1) is also satisfied by sequences $\{(j+1)z_je_{j+1}\}_{j=1}^N$ or $\{z_j\sum_{k=1}^{j+1}e_k\}_{j=1}^N$ for all $f=\{z_j\}_{j=1}^N\in\mathcal{H}$. This shows that the sequence $\{g_j:g_j\in\mathcal{K}\}_{j=1}^N$ in (2.1) is not unique.

**Example 2.4.** Let $\mathcal{H}=\mathcal{K}=\ell^2(\mathbb{N})$ and let, $\{\delta_i\}_{i\in\mathbb{N}}$ be the standard basis of $\ell^2(\mathbb{N})$. For each $j\in\mathbb{N}$ define the subspace $W_j\subset\mathcal{K}$ and the operator $\Lambda_j:\mathcal{H}\to W_j$ by

$$W_j=\text{span}\{\delta_{2j-1},\delta_{2j}\},\quad \Lambda_j(\{f_i\}_{i\in\mathbb{N}})=f_{2j-1}\delta_{2j-1}+f_{2j}\delta_{2j}.$$

It is easy to check that $\Lambda_j$ is the orthogonal projection from $\mathcal{K}$ on $W_j$. Also observe if $\{f_i\}_{i\in\mathbb{N}}=\sum_{j\in\mathbb{N}}\Lambda_j^*(\lambda_j\delta_{2j-1}+\mu_j\delta_{2j})$ then $\lambda_j=f_{2j-1}$ and $\mu_j=f_{2j}$. This shows that $\{\Lambda_j\}_{j\in\mathbb{N}}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in\mathbb{N}}$.

**Theorem 2.1.** *Let $\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then*

$$\dim\mathcal{H}=\sum_{j\in J}\dim W_j.$$

*Proof.* Let $\{e_{ij}\}_{i\in J_j}$ be an orthonormal basis for $W_j$ for all $j\in J$. We show that $\{\Lambda_j^*e_{ij}\}_{j\in J,i\in J_j}$ is a basis for $\mathcal{H}$. Since $\{e_{ij}\}_{i\in J_j}$ is an orthonormal basis for $W_j$, hence every $g_j\in W_j$ has a unique expansion of the form

$$g_j=\sum_{i\in J_j}\langle g_j,e_{ij}\rangle e_{ij}.$$

This implies that also every $f\in\mathcal{H}$ has a unique expansion of the form

$$f=\sum_{j\in J}\sum_{i\in J_j}\langle g_j,e_{ij}\rangle\Lambda_j^*e_{ij}.$$

This shows that $\dim\mathcal{H}=\sum_{j\in J}\dim W_j$. ◻

**Corollary 2.1.** Let $\{\Lambda_j\}_{j\in J}$, $\{\Gamma_i\}_{i\in I}$ be $g$-bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, $\{V_i\}_{i\in I}$ respectively. Then

$$\sum_{j\in J}\dim W_j=\sum_{i\in I}\dim V_i.$$

**Definition 2.2.** Let $\Lambda=\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then every $f\in\mathcal{H}$ has a unique expansion of the form

$$f=\sum_{j\in J}\Lambda_j^*g_j.$$

It is clear that each $g_j\in W_j$ is a linear operator of $f$. If we denote this linear operator by $\Gamma_j:\mathcal{H}\to W_j$, then $g_j=\Gamma_jf$, and we have

$$f=\sum_{j\in J}\Lambda_j^*\Gamma_jf.$$

The sequence $\{\Gamma_j\}_{j\in J}$ is called the dual $g$-basis of $\Lambda$.

In the next theorem we show that the operators of a dual $g$-basis are continuous.

**Theorem 2.2.** *Let $\Lambda=\{\Lambda_j\}_{j\in J}$ be a g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, and let $\{\Gamma_j\}_{j\in J}$ be the dual g-basis of $\Lambda$, then $\Gamma_j\in B(\mathcal{H},W_j)$, for all $j\in J$. Moreover, if $\Gamma_j\neq 0$ for some $j\in J$, then*

$$\|\Gamma_j\|\|\Lambda_j\|\geq 1.$$

*Proof.* Define the space

$$\mathcal{A}=\Big\{\{g_j\}_{j\in J}\Big|\,g_j\in W_j,\ \sum_{j\in J}\Lambda_j^*g_j\text{ is convergent}\Big\},$$

with the norm defined by

$$\big\|\{g_j\}_{j\in J}\big\|=\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\Big\|\sum_{i\in F}\Lambda_i^*g_i\Big\|<\infty.$$

It is clear that $\mathcal{A}$ with this norm, is a normed space with respect to the pointwise operations. We show that the space $\mathcal{A}$ is a complete. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}$. If $u_n=\{g_{nj}\}_{j\in J}$, then given any $\varepsilon>0$, there exists a number $N$ such that

$$\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\Big\|\sum_{i\in F}(\Lambda_i^*g_{ni}-\Lambda_i^*g_{mi})\Big\|<\varepsilon \tag{2.2}$$

for all $m,n\geq N$. Now for all $j\in J$ and $m,n\geq N$ we have

$$\|\Lambda_j^*g_{nj}-\Lambda_j^*g_{mj}\|\leq \sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\Big\|\sum_{i\in F}(\Lambda_i^*g_{ni}-\Lambda_i^*g_{mi})\Big\|<\varepsilon.$$

This shows that $\{\Lambda_j^*g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$. Since $\Lambda_j$ is onto hence by Theorem 4.13 of [5] the sequence $\{g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_j$ and thus convergent. Let $g_j\in W_j$ such that $g_j=\lim_{n\to\infty}g_{nj}$ and $u=\{g_j\}_{j\in J}$. From (2.2), by letting $m\to\infty$, we obtain

$$\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\Big\|\sum_{i\in F}(\Lambda_i^*g_{ni}-\Lambda_i^*g_i)\Big\|\leq\varepsilon \tag{2.3}$$

for all $n\geq N$. Since for all finite non-empty subset $F\subset J$ we have

$$\begin{aligned}\Big\|\sum_{i\in F}\Lambda_i^*g_i\Big\|&\leq\Big\|\sum_{i\in F}(\Lambda_i^*g_{Ni}-\Lambda_i^*g_i)\Big\|+\Big\|\sum_{i\in F}\Lambda_i^*g_{Ni}\Big\|\\&\leq\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\Big\|\sum_{i\in F}(\Lambda_i^*g_{Ni}-\Lambda_i^*g_i)\Big\|+\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\Big\|\sum_{i\in F}\Lambda_i^*g_{Ni}\Big\|.\end{aligned}$$

Thus $u\in\mathcal{A}$. Moreover (2.3) implies that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is convergent to $u$ in $\mathcal{A}$. Therefore $\mathcal{A}$ is a Banach space. Define the mapping

$$T:\mathcal{A}\to\mathcal{H},\qquad T(\{g_j\}_{j\in J})=\sum_{j\in J}\Lambda_j^*g_j.$$

Since $\Lambda$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, $T$ is linear, one-to-one and onto. On the other hand, since

$$\|T(\{g_j\}_{j\in J})\|=\Big\|\sum_{j\in J}\Lambda_j^*g_j\Big\|\leq\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\Big\|\sum_{i\in F}\Lambda_i^*g_i\Big\|=\|\{g_j\}_{j\in J}\|.$$

Thus $T$ is continuous and the open mapping theorem then guarantees that $T^{-1}$ is also continuous. This shows that $\mathcal{A}$ and $\mathcal{H}$ are Banach spaces isomorphic. Now suppose that $f=\sum_{j\in J}\Lambda_j^*g_j$ is a fixed, arbitrary element of $\mathcal{H}$ and let $j\in J$. Since $\Lambda_j$ is onto thus by Theorem 4.13 of [5] there is a $m_j>0$ such that $m_j\|g\|\leq\|\Lambda_j^*g\|$ for all $g\in W_j$. Moreover, we have

$$\begin{aligned}\|\Gamma_jf\|=\|g_j\|&\leq\frac{\|\Lambda_j^*g_j\|}{m_j}\leq\frac{\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\|\sum_{i\in F}\Lambda_i^*g_i\|}{m_j}\\&=\frac{\|T^{-1}f\|}{m_j}\leq\frac{\|T^{-1}\|\|f\|}{m_j}.\end{aligned}$$

This shows that each $\Gamma_j$ is continuous and

$$\|\Gamma_j\| \le \frac{\|T^{-1}\|}{m_j}.$$

For the remaining inequality assume that $0 \neq g_j = \Gamma_j f$ for some $f \in \mathcal{H}$, then we have

$$\|g_j\| = \|\Gamma_j \Lambda_j^* g_j\| \le \|\Gamma_j\| \|\Lambda_j\| \|g_j\|,$$

which implies that $\|\Gamma_j\| \|\Lambda_j\| \ge 1$. □

**Definition 2.3.** Let $\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be the dual $g$-basis of $\{\Lambda_j\}_{j\in J}$. Then $F$-partial sum operator of $\{\Lambda_j\}_{j\in J}$ is defined by

$$S_F : \mathcal{H} \to \mathcal{H}, \quad S_F f = \sum_{j\in F} \Lambda_j^* \Gamma_j f,$$

for all finite subset $F \subset J$. By Theorem 2.2, $S_F$ is a bounded operator and

$$1 \le \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \|S_F\| < \infty. \tag{2.4}$$

**Definition 2.4.** A family of operators $\{\Lambda_j \in B(\mathcal{H}, W_j) : j \in J\}$ is called complete set for $\mathcal{H}$, if $\mathcal{H} = \overline{\text{span}}\{\Lambda_j^*(W_j)\}_{j\in J}$. It is easy to check that $\{\Lambda_j\}_{j\in J}$ is a complete set for $\mathcal{H}$, if and only if $\{f : \Lambda_j f = 0, j \in J\} = \{0\}$.

**Theorem 2.3.** *Let $\{\Lambda_j \in B(\mathcal{H}, W_j) : j \in J\}$ be a complete set for $\mathcal{H}$. Then $\{\Lambda_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if and only if there exists a constant $M$ such that*

$$\Big\| \sum_{i\in F} \Lambda_i^* g_i \Big\| \le M \Big\| \sum_{i\in G} \Lambda_i^* g_i \Big\|, \tag{2.5}$$

*for all finite subsets $F \subset G \subset J$ and arbitrary vectors $g_j \in W_j$, $j \in G$.*

*Proof.* First suppose that $\{\Lambda_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Put

$$M = \sup_{\substack{0<|F|<\infty \\ F\subseteq J}} \|S_F\|.$$

Then for all finite subsets $F \subset G \subset J$ and arbitrary vectors $g_j \in W_j$ we have

$$\Big\| \sum_{j\in F} \Lambda_j^* g_j \Big\| = \Big\| S_F \Big( \sum_{j\in G} \Lambda_j^* g_j \Big) \Big\| \le M \Big\| \sum_{j\in G} \Lambda_j^* g_j \Big\|.$$

To prove the opposite implication take $f \in \mathcal{H}$. By hypothesis, there exist finite subsets $F_n \subset F_{n+1} \subset J$ and vectors $g_{nj} \in W_j$ for all $n \in \mathbb{N}$, $j \in F_n$ such that

$$f = \lim_{n\to\infty} \sum_{j\in F_n} \Lambda_j^* g_{nj}.$$

For notational convenience, put $g_{nj}=0$ for $j\notin F_n$, then for every $m>n$ and $j\in F_n$ we have

$$\begin{aligned}\|\Lambda_j^* g_{nj}-\Lambda_j^* g_{mj}\| &\le M\Big\|\sum_{i\in F_n}\Lambda_i^*(g_{ni}-g_{mi})\Big\|\\ &\le M^2\Big\|\sum_{i\in F_m}\Lambda_i^*(g_{ni}-g_{mi})\Big\|\\ &=M^2\Big\|\sum_{i\in F_n}\Lambda_i^* g_{ni}-\sum_{i\in F_m}\Lambda_i^* g_{mi}\Big\|\to 0,\quad (n\to\infty).\end{aligned}$$

This shows that $\{\Lambda_j^* g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$. Since $\Lambda_j$ is onto, by Theorem 4.13 [5] the sequence $\{g_{nj}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_j$ and thus convergent. Let $g_j\in W_j$ such that $g_j=\lim_{n\to\infty}g_{nj}$, then $f=\sum_{j\in J}\Lambda_j^* g_j$. Now we show that this representation is unique. If $\sum_{j\in J}\Lambda_j^* g_j=0$, then for any finite subset $F\subset J$ and $j\in F$ we have

$$\|\Lambda_j^* g_j\|\le M\Big\|\sum_{i\in F}\Lambda_i^* g_i\Big\|\to 0.$$

This shows that $\|\Lambda_j^* g_j\|=0$. Since $\Lambda_j^*$ is one-to-one on $W_j$, hence $g_j=0$ which this completes the proof. □

**Corollary 2.2.** Let $\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual $g$-basis $\{\Gamma_j\}_{j\in J}$. Then $\{\Gamma_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and

$$f=\sum_{j\in J}\Gamma_j^*\Lambda_j f,\quad \forall f\in\mathcal{H}.$$

*Proof.* First we prove that $\mathcal{H}=\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in J}$. To see this, let

$$f\perp\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in J}.$$

Then

$$\|\Gamma_j f\|^2=\langle f,\Gamma_j^*\Gamma_j f\rangle=0,$$

which implies that $\Gamma_j f=0$ for all $j\in J$. We also have

$$f=\sum_{j\in J}\Lambda_j^*\Gamma_j f=0.$$

Thus $\mathcal{H}=\overline{\text{span}}\{\Gamma_j^*(W_j)\}_{j\in J}$. We now prove that $\{\Gamma_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. For this, we show that $S_F^* f\to f$ for all $f\in\mathcal{H}$. First assume that $f=\sum_{j\in G}\Gamma_j^* g_j$ for some finite subset $G\subset J$ and let $F\supseteq G$ be a finite arbitrary set. Then by hypothesis for any $i,j\in J$ we have $\Gamma_j\Lambda_i^*\Gamma_i=\delta_{ij}\Gamma_i$ hence $\Gamma_i^*\Lambda_i\Gamma_j^*=\delta_{ij}\Gamma_i^*$. It follows that

$$S_F^* f=\sum_{j\in G}S_F^*\Gamma_j^* g_j=\sum_{j\in G}\sum_{i\in F}\Gamma_i^*\Lambda_i\Gamma_j^* g_j=\sum_{j\in G}\Gamma_j^* g_j=f.$$

Now if $f\in\mathcal{H}$, then given $\varepsilon>0$ we can find $g=\sum_{j\in G}\Gamma_j^* g_j$ such that

$$\|f-g\|<\frac{\varepsilon}{M+1},$$

where

$$M=\sup_{\substack{0<|F|<\infty\\ F\subseteq J}}\|S_F\|.$$

We also have

$$\|S_F^* f-f\|\leq\|S_F^* f-S_F^* g\|+\|g-f\|\leq(\|S_F\|+1)\|f-g\|<\varepsilon.$$

Thus every $f\in\mathcal{H}$ has at least one representation of the form $f=\sum_{j\in J}\Gamma_j^*\Lambda_j f$. We show that this representation is unique. Assume that $\sum_{j\in J}\Gamma_j^* g_j=0$ then by hypothesis for any $i,j\in J$ we have $\Gamma_j\Lambda_i^*\Lambda_i=\delta_{ij}\Lambda_i$ thus $\Lambda_i^*\Lambda_i\Gamma_j^*=\delta_{ij}\Lambda_i^*$. It follows that

$$\Lambda_i^* g_i=\Lambda_i^*\Lambda_i\Big(\sum_{j\in J}\Gamma_j^* g_j\Big)=0.$$

Since $\Lambda_i^*$ is one-to-one on $W_i$, therefore $g_i=0$ for all $i\in J$. This completes the proof. □

**Definition 2.5.** Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be sequences of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

(i) Let $\Lambda_j$ be onto for all $j\in J$, then $\{\Gamma_j\}_{j\in J}$ is called a $g$-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$, if $\Gamma_i\Lambda_j^* g_j=\delta_{ij}g_j$ for all $i,j\in J$, $g_j\in W_j$.

(ii) $\{\Lambda_j\}_{j\in J}$ is called minimal, if for each $j\in J$

$$\Lambda_j^*(W_j)\cap\overline{\text{span}}\{\Lambda_k^*(W_k)\}_{\substack{k\in J,\\ k\neq j}}=\{0\}.$$

(iii) We say that $\{\Lambda_j\}_{j\in J}$ is $\omega$-independent if whenever $\sum_{j\in J}\Lambda_j^* g_j=0$ for some sequence $\{g_j:g_j\in W_j\}_{j\in J}$, then necessarily $g_k=0$ for all $k\in J$.

Since $\Lambda_j^*\Lambda_j\Gamma_i^*=\delta_{ij}\Lambda_j^*$ for all $i,j\in J$ and $\Lambda_j^*$ is one-to-one on $W_j$ hence if $\{\Gamma_j\}_{j\in J}$ is a $g$-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ is also a $g$-biorthogonal sequence of $\{\Gamma_j\}_{j\in J}$.

**Proposition 2.1.** Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $\Lambda_j$ be onto for all $j\in J$, then $\{\Lambda_j\}_{j\in J}$ is minimal if and only if it is $\omega$-independent.

*Proof.* First assume that $\{\Lambda_j\}_{j\in J}$ is not $\omega$-independent, then there is a sequence $\{g_j:g_j\in W_j\}_{j\in J}$ with $g_k\neq 0$ for some $k\in J$, such that $\sum_{j\in J}\Lambda_j^* g_j=0$. It follows

$$\Lambda_k^* g_k=\sum_{\substack{j\in J,\\ j\neq k}}\Lambda_j^*(-g_j),$$

which implies that

$$\Lambda_k^* g_k \in \overline{\text{span}}\{\Lambda_j^*(W_j)\}_{\substack{j\in J,\\ j\neq k}}.$$

That is, $\{\Lambda_j\}_{j\in J}$ is not minimal. The other implication is obvious. □

**Proposition 2.2.** Every $g$-basis for a Hilbert space possesses a unique $g$-biorthogonal sequence.

*Proof.* By definition, the dual $g$-basis of a $g$-basis is a $g$-biorthogonal sequence of it. Moreover, if $\{\Gamma_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ be $g$-biorthogonal sequences of $g$-basis $\{\Lambda_j\}_{j\in J}$, then for all $f\in\mathcal{H}$ and $i,j\in J$ we have $\Psi_i\Lambda_j^*\Gamma_j f=\delta_{ij}\Gamma_j f$, which implies that

$$\Lambda_i^*\Psi_i f=\sum_{j\in J}\Lambda_i^*\Psi_i\Lambda_j^*\Gamma_j f=\sum_{j\in J}\delta_{ij}\Lambda_i^*\Gamma_j f=\Lambda_i^*\Gamma_i f.$$

Since $\Lambda_i^*$ is one-to-one on $W_i$, hence $\Gamma_i=\Psi_i$. □

**Proposition 2.3.** Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, and let $\Lambda_j$ be onto for all $j\in J$. Then

(i) $\{\Lambda_j\}_{j\in J}$ has a $g$-biorthogonal sequence, if and only if $\{\Lambda_j\}_{j\in J}$ is minimal.

(ii) The $g$-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$ is unique if and only if $\{\Lambda_j\}_{j\in J}$ is complete.

*Proof.* For the proof of (i) suppose that $\{\Gamma_j\}_{j\in J}$ is a $g$-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$, and let

$$f\in\Lambda_k^*(W_k)\cap\overline{\text{span}}\{\Lambda_j^*(W_j)\}_{\substack{j\in J,\\ j\neq k}}$$

for any given $k\in J$. Then there exists a sequence $\{g_j: g_j\in W_j\}_{j\in J}$ such that

$$f=\Lambda_k^* g_k=\sum_{\substack{j\in J,\\ j\neq k}}\Lambda_j^* g_j.$$

We also have

$$g_k=\Gamma_k\Lambda_k^* g_k=\sum_{\substack{j\in J,\\ j\neq k}}\Gamma_k\Lambda_j^* g_j=\sum_{\substack{j\in J,\\ j\neq k}}\delta_{kj}g_j=0,$$

which implies that $f=0$. That is, $\{\Lambda_j\}_{j\in J}$ is minimal. For the opposite implication in (i), suppose that $\{\Lambda_j\}_{j\in J}$ is minimal, and let $\mathcal{H}_0=\overline{\text{span}}\{\Lambda_j^*(W_j)\}_{j\in J}$. Proposition 2.1 follows that $\{\Lambda_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}_0$ with respect to $\{W_j\}_{j\in J}$. Let $\{\Gamma_j'\}_{j\in J}$ be dual $g$-basis of $\{\Lambda_j\}_{j\in J}$. If we define $\Gamma_j=\Gamma_j'P$ for all $j\in J$, where $P$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$. Then $\{\Gamma_j\}_{j\in J}$ is a $g$-biorthogonal sequence for $\{\Lambda_j\}_{j\in J}$.

(ii) Let $\{\Gamma_j\}_{j\in J}$ be a $g$-biorthogonal sequence of $\{\Lambda_j\}_{j\in J}$. If $\{\Lambda_j\}_{j\in J}$ is not complete, then the sequence $\{\Psi_j\}_{j\in J}$ defined by $\Psi_j=\Gamma_j+\Lambda_j(Id_{\mathcal{H}}-P)$ for all $j\in J$ is a $g$-biorthogonal sequence for $\{\Lambda_j\}_{j\in J}$. For the other implication in (ii), assume that $\{\Lambda_j\}_{j\in J}$ is complete. If $\sum_{j\in J}\Lambda_j^* g_j=0$ for any given sequence $\{g_j: g_j\in W_j\}_{j\in J}$, then for every $k\in J$ we have

$$g_k=\sum_{j\in J}\delta_{kj}g_j=\sum_{j\in J}\Gamma_k\Lambda_j^* g_j=\Gamma_k\Big(\sum_{j\in J}\Lambda_j^* g_j\Big)=0.$$

This shows that $\{\Lambda_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Now the conclusion follows from Proposition 2.2. □

**Theorem 2.4.** *Let $\{\Lambda_j\}_{j\in J}$ be a g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $T:\mathcal{H}\to\mathcal{U}$ be a bounded linear operator such that $\Gamma_j=\Lambda_j T^*$ for all $j\in J$. Then $\{\Gamma_j\}_{j\in J}$ is a g-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ if and only if $T$ is invertible.*

*Proof.* Let $T$ be invertible and let $g\in\mathcal{U}$, then we can write $g=Tf$ for some $f\in\mathcal{H}$. Since $\{\Lambda_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ hence $f\in\mathcal{H}$ has an unique expansion of the form $f=\sum_{j\in J}\Lambda_j^* g_j$ where $g_j\in W_j$ for all $j\in J$. It follows that

$$g=Tf=\sum_{j\in J}T\Lambda_j^* g_j=\sum_{j\in J}\Gamma_j^* g_j,$$

which implies that $\{\Gamma_j\}_{j\in J}$ is a $g$-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$. Now we assume $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are $g$-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ respectively. Since for every sequence $\{g_j: g_j\in W_j\}_{j\in J}$ we have

$$T\Big(\sum_{j\in J}\Lambda_j^* g_j\Big)=\sum_{j\in J}\Gamma_j^* g_j.$$

Therefore $T$ is invertible. □

**Definition 2.6.** Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be $g$-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ respectively. Then $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are said to be equivalent if for any given sequence $\{g_j:\ g_j\in W_j\}_{j\in J}$ the series $\sum_{j\in J}\Lambda_j^* g_j$ is convergent if and only if the series $\sum_{j\in J}\Gamma_j^* g_j$ is convergent.

**Theorem 2.5.** *Two g-bases $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ are equivalent if and only if there exists a bounded linear invertible operator $T:\mathcal{H}\to\mathcal{U}$ such that $\Gamma_j=\Lambda_j T^*$.*

*Proof.* Assume that $T:\mathcal{H}\to\mathcal{U}$ be the bounded linear invertible operator such that $\Gamma_j=\Lambda_j T^*$ for all $j\in J$. Then the sufficiency follows from the fact that for every sequence $\{g_j: g_j\in W_j\}_{j\in j}$ we have

$$\sum_{j\in J}\Gamma_j^* g_j=T\Big(\sum_{j\in J}\Lambda_j^* g_j\Big)\quad\text{and}\quad\sum_{j\in J}\Lambda_j^* g_j=T^{-1}\Big(\sum_{j\in J}\Gamma_j^* g_j\Big).$$

Now suppose that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are equivalent $g$-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$. If $f\in\mathcal{H}$ with unique expansion $f=\sum_{j\in J}\Lambda_j^* g_j$, then the series $\sum_{j\in J}\Gamma_j^* g_j$ converges to an element $Tf \in \mathcal{U}$. Therefore, $Tf$ is well defined. Since $\Lambda_j^*$ is one-to-one on $W_j$ for all $j \in J$, hence it is easy to check that $T$ is linear, bijective and $\Gamma_j = \Lambda_j T^*$. To show that $T$ is a bounded invertible operator, we define operators $T_F$ by $T_F f = \sum_{j\in F}\Gamma_j^* g_j$ for every non-empty finite subset $F \subset J$. Then $Tf = \lim_F T_F f$ for every $f \in \mathcal{H}$. By Theorem 2.4 each $T_F$ is bounded thus the Banach-Steinhaus Theorem implies that $T$ is bounded. Moreover the open mapping Theorem guarantees that $T$ is invertible. □

**Theorem 2.6.** *The g-biorthogonal sequences associated with equivalent g-bases are equivalent.*

*Proof.* Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be equivalent $g$-bases for $\mathcal{H}$ and $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ and let, $\{\Psi_j\}_{j\in J}$ and $\{\Phi_j\}_{j\in J}$ be $g$-biorthogonal sequences for them respectively. By assumption there exists a bounded invertible operator $T: \mathcal{H} \to \mathcal{U}$ such that $\Gamma_j = \Lambda_j T^*$. For any $f \in \mathcal{H}$ we have

$$f = T^{-1}Tf = T^{-1}\Big(\sum_{j\in J}\Gamma_j^*\Phi_j Tf\Big) = T^{-1}\Big(\sum_{j\in J}T\Lambda_j^*\Phi_j Tf\Big) = \sum_{j\in J}\Lambda_j^*\Phi_j Tf.$$

By Proposition 2.2 it follows that $\Psi_j = \Phi_j T$ for all $j \in J$. that is $\{\Psi_j\}_{j\in J}$ and $\{\Phi_j\}_{j\in J}$ are equivalent. □

**Definition 2.7.** For each sequence $\{W_j\}_{j\in J}$ of closed subspaces of $\mathcal{K}$, we define the Hilbert space associated with $\{W_j\}_{j\in J}$ by

$$\Big(\sum_{j\in J}\oplus W_j\Big)_{\ell^2} = \Big\{\{g_j\}_{j\in J} | g_j \in W_j \quad \text{and} \quad \sum_{j\in J}\|g_j\|^2 < \infty\Big\}. \tag{2.6}$$

with inner product given by

$$\langle \{f_k\}_{k\in J}, \{g_k\}_{k\in J}\rangle = \sum_{j\in J}\langle f_j, g_j\rangle. \tag{2.7}$$

## 3 Orthonormal $g$-bases and Riesz $g$-bases

In this section we give some characterizations of orthonormal $g$-bases and Riesz $g$-bases in Hilbert spaces.

**Definition 3.1.** Let $\{\Xi_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then

(i) $\{\Xi_j\}_{j\in J}$ is called an orthonormal $g$-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, if:

$$\Xi_i \Xi_j^* g_j = \delta_{ij} g_j, \quad \forall i,j \in J, \quad g_j \in W_j.$$

(ii) $\{\Xi_j\}_{j\in J}$ is called an orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if it is a complete orthonormal $g$-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

**Corollary 3.1.** Let $\{\Xi_j\}_{j\in J}$ be an orthonormal $g$-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then $\Xi_j$ is onto and $\|\Xi_j\|=1$ for all $j\in J$.

*Proof.* For any $j\in J$ and $g\in W_j$, we have $\Xi_j\Xi_j^*g=g$ which implies that $\Xi_j$ is onto. We further have $\Xi_j\Xi_j^*\Xi_j=\Xi_j$. This shows that $\Xi_j^*\Xi_j$ is an orthogonal projection from $\mathcal{H}$ onto $\mathcal{R}_{\Xi_j^*}$ and hence $\|\Xi_j^*\Xi_j\|=1$. This yields

$$\|\Xi_j\|^2=\sup_{\|f\|=1}\|\Xi_j f\|^2=\sup_{\|f\|=1}\langle \Xi_j f,\Xi_j f\rangle=\sup_{\|f\|=1}\|\Xi_j^*\Xi_j f\|^2=1.$$

Thus, we complete the proof. □

**Example 3.1.** Let $\mathcal{H}=\mathcal{K}=\mathbb{C}^{N+1}$ and let $\{e_k\}_{k=1}^{N+1}$ be the standard basis of $\mathbb{C}^{N+1}$. For each $1\le j\le N+1$ define the subspace $W_j\subset\mathcal{K}$ and the operator $\Xi_j:\mathcal{H}\to W_j$ by

$$W_j=\operatorname{span}\Big\{\sum_{\substack{k=1\\k\neq j}}^{N+1}e_k\Big\},\quad \Xi_j(\{z_i\}_{i=1}^{N+1})=\frac{z_j}{\sqrt{N}}\sum_{\substack{k=1\\k\neq j}}^{N+1}e_k.$$

Then $\{\Xi_j\}_{j=1}^{N+1}$ is an orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j=1}^{N+1}$.

**Corollary 3.2.** Orthonormal $g$-systems are $\omega$-independent.

*Proof.* This follows immediately from the definition. □

**Theorem 3.1.** *Let $\{\Xi_j\}_{j\in J}$ be an orthonormal $g$-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then the series $\sum_{j\in J}\Xi_j^*g_j$ converges if and only if $\{g_j\}_{j\in J}\in\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$ and in that case*

$$\Big\|\sum_{j\in J}\Xi_j^*g_j\Big\|^2=\sum_{j\in J}\|g_j\|^2.$$

*Proof.* For any finite subset $F\subset J$ we have

$$\Big\|\sum_{j\in F}\Xi_j^*g_j\Big\|^2=\sum_{j\in F}\|g_j\|^2.$$

From this the result follows. □

**Theorem 3.2** (Bessel's inequality). *Let $\{\Xi_j\}_{j\in J}$ be an orthonormal $g$-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then*

$$\sum_{j\in J}\|\Xi_j f\|^2\le\|f\|^2$$

*for all $f\in\mathcal{H}$.*

*Proof.* Let $f\in\mathcal{H}$. Fix $F\subset J$ with $|F|<\infty$. Then By Theorem 3.1 we have

$$\begin{aligned}\Big\|f-\sum_{j\in F}\Xi_j^*g_j\Big\|^2&=\|f\|^2-\sum_{j\in F}\langle\Xi_jf,g_j\rangle-\sum_{j\in F}\langle g_j,\Xi_jf\rangle+\sum_{j\in F}\|g_j\|^2\\&=\|f\|^2-\sum_{j\in F}\|\Xi_jf\|^2+\sum_{j\in F}\|\Xi_jf-g_j\|^2\end{aligned}$$

for arbitrary vectors $\{g_j:g_j\in W_j\}_{j\in F}$. In particular, if $g_j=\Xi_jf$, then

$$\Big\|f-\sum_{j\in F}\Xi_j^*\Xi_jf\Big\|^2=\|f\|^2-\sum_{j\in F}\|\Xi_jf\|^2.$$

From this we have

$$\sum_{j\in F}\|\Xi_jf\|^2\le\|f\|^2,$$

which implies that

$$\sum_{j\in J}\|\Xi_jf\|^2\le\|f\|^2.$$

Thus, we complete the proof. □

**Corollary 3.3.** Let $\{\Xi_j\}_{j\in J}$ be an orthonormal $g$-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then for all $f\in\mathcal{H}$ the series $\sum_{j\in J}\Xi_j^*\Xi_jf$ convergent and

$$\|f-\sum_{j\in J}\Xi_j^*\Xi_jf\|^2\le\|f-\sum_{j\in J}\Xi_j^*g_j\|^2$$

for every $\{g_j\}_{j\in J}\in\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$.

**Theorem 3.3.** *Let $\Xi=\{\Xi_j\}_{j\in J}$ be an orthonormal g-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the following conditions are equivalent:*

*(i)* $\Xi$ *is an orthonormal g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*(ii)* $f=\sum_{j\in J}\Xi_j^*\Xi_jf$, $\forall f\in\mathcal{H}$.

*(iii)* $\|f\|^2=\sum_{j\in J}\|\Xi_j^*\Xi_jf\|^2$, $\forall f\in\mathcal{H}$.

*(iv)* $\|f\|^2=\sum_{j\in J}\|\Xi_jf\|^2$, $\forall f\in\mathcal{H}$.

*(v)* $\langle f,g\rangle=\sum_{j\in J}\langle\Xi_jf,\Xi_jg\rangle$, $\forall f,g\in\mathcal{H}$.

*(vi)* *If $\Xi_jf=0$ for all $j\in J$, then $f=0$.*

*Proof.* The implication (i)⇒ (ii) follows immediately from Corollary 3.3. To prove (ii) ⇒ (iii) assume that $f \in \mathcal{H}$. Since $\Xi$ is an orthonormal $g$-system, hence $(\Xi_j^*\Xi_j)^2 f = \Xi_j^*\Xi_j f$ for all $j\in J$. This yields

$$\|f\|^2 = \Big\langle \sum_{j\in J} \Xi_j^*\Xi_j f, f \Big\rangle = \sum_{j\in J} \|\Xi_j^*\Xi_j f\|^2,$$

which implies (iii).

The implications (iii)⇒ (iv) ⇒ (v) are clear. To prove (v) ⇒ (vi) assume that $\Xi_j f = 0$ for all $j\in J$, then we have $\|f\|^2 = \sum_{j\in J}\|\Xi_j f\|^2 = 0$. It follows that $f=0$. To prove (vi) ⇒ (i) suppose that $f \perp \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j\in J}$, then for every $j\in J$ we have $\|\Xi_j f\|^2 = \langle f, \Xi_j^*\Xi_j f\rangle = 0$, which implies that $f=0$. Therefore $\mathcal{H} = \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j\in J}$. □

**Theorem 3.4.** *Let $\{\Xi_j\}_{j\in J}$ be an orthonormal g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $T:\mathcal{H}\to\mathcal{U}$ be a bounded linear operator such that $\Xi_j' = \Xi_j T^*$ for all $j\in J$. Then $\{\Xi_j'\}_{j\in J}$ is an orthonormal g-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ if and only if $T$ is unitary.*

*Proof.* First suppose that $\{\Xi_j'\}_{j\in J}$ is an orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then by Theorem 3.3 for every $g\in\mathcal{U}$ we have

$$\|T^*g\|^2 = \sum_{j\in J}\|\Xi_j T^* g\|^2 = \sum_{j\in J}\|\Xi_j' g\|^2 = \|g\|^2.$$

Hence $T$ is co-isometry. We also see from Theorem 2.5 that $T$ is unitary. Now if $T$ is unitary then we have

$$\|g\|^2 = \|T^*g\|^2 = \sum_{j\in J}\|\Xi_j T^* g\|^2 = \sum_{j\in J}\|\Xi_j' g\|^2$$

for all $g\in\mathcal{U}$. From this follows that $\{\Xi_j'\}_{j\in J}$ is an orthonormal $g$-basis for $\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$. □

**Corollary 3.4.** Let $\{\Xi_j\}_{j\in J}$ be an orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the orthonormal $g$-bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ are precisely the sets $\{\Xi_j T\}_{j\in J}$, where $T:\mathcal{H}\to\mathcal{H}$ is an unitary operator.

**Corollary 3.5.** Let $\{W_j\}_{j\in J}$ be a family of closed subspaces of $\mathcal{H}$ such that

$$\sum_{j\in J}\|\pi_{W_j} f\|^2 = \|f\|^2, \quad \forall f\in\mathcal{H},$$

where $\pi_{W_j}$ is the orthogonal projections from $\mathcal{H}$ onto $W_j$. Then $\{\pi_{W_j}\}_{j\in J}$ is an orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

*Proof.* For each $j\in J$ and $g_j\in W_j$ we have

$$\|g_j\|^2=\sum_{i\in J}\|\pi_{W_i}g_j\|^2=\|g_j\|^2+\sum_{\substack{i\in J\\ i\neq j}}\|\pi_{W_i}g_j\|^2,$$

which shows that $\pi_{W_i}g_j=\delta_{ij}g_j$. It follows that $\{\pi_{W_j}\}_{j\in J}$ is an orthonormal $g$-system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Now the result follows from the Theorem 3.3. $\square$

In the following, we give some characterizations of Riesz $g$-bases in Hilbert spaces.

**Definition 3.2.** A sequence of operators $\{\Lambda_j\in B(\mathcal{H},W_j):j\in J\}$ is called a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if there is an orthonormal $g$-basis $\{\Xi_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and a bounded invertible linear operator $T$ on $\mathcal{H}$ such that $\Lambda_j=\Xi_jT^*$ for all $j\in J$.

**Corollary 3.6.** If $\{\Lambda_j\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then

$$0<\inf_{j\in J}\|\Lambda_j\|\leq\sup_{j\in J}\|\Lambda_j\|<\infty.$$

*Proof.* According to the definition we can write $\{\Lambda_j\}_{j\in J}=\{\Xi_jT^*\}_{j\in J}$, where $T$ is a bounded bijective operator and $\{\Xi_j\}_{j\in J}$ is an orthonormal $g$-basis. By Corollary 3.1 for every $j\in J$ we have

$$\|T^{-1}\|^{-1}\leq\|\Lambda_j\|\leq\|T\|.$$

From this the result follows. $\square$

**Theorem 3.5.** *If $\{\Lambda_j\}_{j\in J}=\{\Xi_jT^*\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then $\{\frac{\Lambda_j}{\|\Lambda_j\|}\}_{j\in J}$ is also a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*Proof.* Define a mapping $S:\mathcal{H}\to\mathcal{H}$ by $Sf=\sum_{j\in J}\frac{\Xi_j^*\Xi_jf}{\|\Lambda_j\|}$. By Theorem 3.3 and Corollary 3.6 we have

$$\|T\|^{-1}\|f\|\leq\|Sf\|\leq\|T^{-1}\|\|f\|,$$

which implies that $S$ is bounded and injective. Since $S$ is self-adjoint hence $S$ is invertible. Moreover, the operator $\Theta=TS$ is also bounded, invertible and we have

$$\begin{aligned}\Xi_j\Theta^*&=\Xi_jST^*=\Big(\sum_{i\in J}\frac{\Xi_j\Xi_i^*\Xi_i}{\|\Lambda_j\|}\Big)T^*\\&=\Big(\sum_{i\in J}\frac{\delta_{ji}\Xi_i}{\|\Lambda_j\|}\Big)T^*=\frac{\Xi_jT^*}{\|\Lambda_j\|}=\frac{\Lambda_j}{\|\Lambda_j\|},\end{aligned}$$

for any $j\in J$. Consequently $\{\frac{\Lambda_j}{\|\Lambda_j\|}\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. $\square$

**Corollary 3.7.** Let $\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual $g$-basis $\{\Gamma_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if and only if $\{\Gamma_j\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

*Proof.* This follows immediately from the definition and Theorem 2.6. $\square$

To check Riesz $g$-baseness of a family of operators $\{\Lambda_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, we derive the following useful characterization.

**Theorem 3.6.** *Let $\{\Lambda_j\}_{j\in J}$ be a g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual g-basis $\{\Gamma_j\}_{j\in J}$. Then the following conditions are equivalent:*

*(i) The sequence $\{\Lambda_j\}_{j\in J}$ is a Riesz g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*(ii) There is an equivalent inner product on $\mathcal{H}$, with respect to which the sequence $\{\Gamma_j\}_{j\in J}$ becomes an orthonormal g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*Proof.* (i)$\Rightarrow$(ii) Assume that $\{\Lambda_j\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$, and write it in the form $\{\Xi_j T^*\}_{j\in J}$ as in the definition. Define a new inner product $\langle\cdot,\cdot\rangle_T$ on $\mathcal{H}$ by

$$\langle f,g\rangle_T = \langle T^*f, T^*g\rangle, \quad \forall f,g\in\mathcal{H}.$$

If $\|\cdot\|_T$ is the norm generated by this inner product, then for all $f\in\mathcal{H}$ we have

$$\|T^{-1}\|^{-1}\|f\| \le \|f\|_T \le \|T\|\|f\|,$$

which implies that the new inner product is equivalent to the original one. By Theorem 2.6 for any $g\in\mathcal{K}$ and arbitrary vector $g_j\in W_j$, $i,j\in J$ we have

$$\begin{aligned}\langle \Gamma_i\Gamma_j^* g_j, g\rangle &= \langle \Gamma_j^* g_j, \Gamma_i^* g\rangle_T = \langle T^*\Gamma_j^* g_j, T^*\Gamma_i^* g\rangle \\ &= \langle \Xi_j^* g_j, \Xi_i^* g\rangle = \langle \Xi_i\Xi_j^* g_j, g\rangle = \langle \delta_{ij} g_j, g\rangle.\end{aligned}$$

Now the Corollary 3.7 follows that $\{\Gamma_j\}_{j\in J}$ is an orthonormal $g$-basis for $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle_T$ with respect to $\{W_j\}_{j\in J}$. (ii)$\Rightarrow$(i) Suppose that $\langle\cdot,\cdot\rangle_1$ is an equivalent inner product on $\mathcal{H}$ with respect to which $\{\Gamma_j\}_{j\in J}$ is an orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Therefore there exist positive constants $m,M$ such that

$$m\|f\| \le \|f\|_1 \le M\|f\|, \quad \forall f\in\mathcal{H}.$$

By Theorem 3.1, we obtain

$$\begin{aligned}\frac{1}{M^2}\sum_{j\in F}\|g_j\|^2 &= \frac{1}{M^2}\Big\|\sum_{j\in F}\Gamma_j^* g_j\Big\|_1^2 \le \Big\|\sum_{j\in F}\Gamma_j^* g_j\Big\|^2 \\ &\le \frac{1}{m^2}\Big\|\sum_{j\in F}\Gamma_j^* g_j\Big\|_1^2 = \frac{1}{m^2}\sum_{j\in F}\|g_j\|^2,\end{aligned}$$

for any finite subset $F\subset J$ and arbitrary vectors $g_j\in W_j$. Now let $\{\Xi_j\}_{j\in J}$ be an arbitrary orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and define the mapping

$$T:\mathcal{H}\to\mathcal{H}, \quad \text{with } T\Xi_j^* g_j=\Gamma_j^* g_j, \quad \forall g_j\in W_j, \quad j\in J.$$

Let $f\in\mathcal{H}$ with $f=\sum_{j\in J}\Xi_j^* g_j$, then we have

$$\frac{1}{M^2}\|f\|^2=\frac{1}{M^2}\sum_{j\in J}\|g_j\|^2\le\|T(f)\|^2\le\frac{1}{m^2}\sum_{j\in J}\|g_j\|^2=\frac{1}{m^2}\|f\|^2.$$

It follows that $T$ is invertible and $T\Xi_j^*\Xi_j=\Gamma_j^*\Xi_j$, which from this $\Xi_j T^*=\Gamma_j$ holds for all $j\in J$. Thus $\{\Gamma_j\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. From this the result follows at once. ☐

The next theorem was proved by Sun in [6] we prove this theorem with another way.

**Theorem 3.7.** *Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then the following conditions are equivalent:*

*(i) The sequence $\{\Lambda_j\}_{j\in J}$ is a Riesz g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*(ii) The family $\{\Lambda_j\}_{j\in J}$ is a complete set for $\mathcal{H}$ and there exist positive constants $A,B$ such that for any finite subset $F\subset J$ and arbitrary vectors $g_j\in W_j$, we have*

$$A\sum_{j\in F}\|g_j\|^2\le\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2\le B\sum_{j\in F}\|g_j\|^2.$$

*Proof.* (i) ⇒ (ii) Assume that $\{\Lambda_j\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$, and write it in the form $\{\Xi_j T^*\}_{j\in J}$ as in the definition. Then for any finite subset $F\subset J$ and arbitrary vectors $g_j\in W_j$ we have

$$\frac{1}{\|T^{-1}\|^2}\sum_{j\in F}\|g_j\|^2=\frac{1}{\|T^{-1}\|^2}\Big\|\sum_{j\in F}\Xi_j^* g_j\Big\|^2\le\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|^2\le\|T\|^2\sum_{j\in F}\|g_j\|^2.$$

(ii) ⇒(i) Let $\{\Xi_j\}_{j\in J}$ be an arbitrary orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and define the mapping

$$T:\mathcal{H}\to\mathcal{H}, \quad \text{with } T\Xi_j^* g_j=\Lambda_j^* g_j, \quad \forall g_j\in W_j, \quad j\in J.$$

Suppose that $f\in\mathcal{H}$ with $f=\sum_{j\in J}\Xi_j^* g_j$, then we have

$$A\|f\|^2=A\sum_{j\in J}\|g_j\|^2\le\|T(f)\|^2\le B\sum_{j\in J}\|g_j\|^2=B\|f\|^2.$$

From this and completeness of $\{\Lambda_j\}_{j\in J}$ follows that $T$ is invertible and $T\Xi_j^*\Xi_j=\Lambda^*\Xi_j$, which implies that $\Xi_j T^*=\Lambda_j$ for all $j\in J$. ☐

Let $\Lambda=\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. If $f=\sum_{j\in J}\Lambda_j^* g_j$, then the coordinate representation of $f\in\mathcal{H}$ relative to the $g$-basis $\Lambda$ is $[f]_\Lambda=\{g_j\}_{j\in J}$.

Let $\Xi=\{\Xi_j\}_{j\in J}$, $\Xi'=\{\Xi'_i\}_{i\in I}$ be orthonormal $g$-bases for $\mathcal{H}$ and $\mathcal{U}$ respectively. Then the matrix representation of the linear map $T:\mathcal{H}\to\mathcal{U}$ relative to the orthonormal $g$-bases $\Xi,\Xi'$ is the matrix $[T]=\{T_{ij}\}_{i\in I,j\in J}$ whose $(i,j)$ entry is $T_{ij}=\Xi'_i T\Xi_j^*$ for all $i\in I,j\in J$. For any $f\in\mathcal{H}$ we also have

$$[Tf]_{\Xi'}=[T][f]_\Xi.$$

Moreover, if $S,T$ are linear maps on $\mathcal{H}$ represented by matrices $[S],[T]$ respectively, then $S+T$ and $ST$ is represented by the matrices $[S]+[T]$ and $[S][T]$ respectively. Further $T$ is a invertible operator if and only if $[T]$ is invertible.

Let $\Lambda=\{\Lambda_j\}_{j\in J}=\{\Xi_j T^*\}_{j\in J}$ be a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the analysis operator $\Theta_\Lambda$ of $\Lambda$ is defined by

$$\Theta_\Lambda:\mathcal{H}\to\Big(\sum_{j\in J}\oplus W_j\Big)_{\ell^2}\quad\text{with }\ \Theta_\Lambda f=\{\Lambda_j f\}_{j\in J},\quad \forall f\in\mathcal{H}.$$

It can easily be shown that $\Theta_\Lambda$ is linear, bounded and $\|\Theta_\Lambda\|\le\|T\|$. The synthesis operator $\Theta_\Lambda^*$, which is the adjoint operator of $\Theta_\Lambda$ is given by

$$\Theta_\Lambda^*:\Big(\sum_{j\in J}\oplus W_j\Big)_{\ell^2}\to\mathcal{H}\quad\text{with }\ \Theta_\Lambda^* g=\sum_{j\in J}\Lambda_j^* g_j,\quad \forall g=\{g_j\}_{j\in J}\in\Big(\sum_{j\in J}\oplus W_j\Big)_{\ell^2}.$$

**Example 3.2.** For every sequence of closed subspaces $\{W_j\}_{j\in J}$ of $\mathcal{K}$ the sequence $\{\Xi_j\}_{j\in J}$ defined by

$$\Xi_j g=g_j,\quad \forall j\in J,\quad g=\{g_j\}_{j\in J}\in\Big(\sum_{j\in J}\oplus W_j\Big)_{\ell^2},$$

is an orthonormal $g$-basis for $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$ with respect to $\{W_j\}_{j\in J}$, which is called the standard orthonormal $g$-basis of it.

Let $\Lambda=\{\Lambda_j\}_{j\in J}$ be a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the matrix representing of the linear operator $\Theta_\Lambda\Theta_\Lambda^*$ relative to the standard orthonormal $g$-basis of $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$ is the matrix $[\Theta_\Lambda\Theta_\Lambda^*]=\{\Lambda_i\Lambda_j^*\}_{i\in I,j\in J}$ which is called the Gram matrix associated with $\Lambda$.

**Theorem 3.8.** *Let $\{\Lambda_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then the following conditions are equivalent:*

*(i) The sequence $\{\Lambda_j\}_{j\in J}$ is a Riesz g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*(ii) The family $\{\Lambda_j\}_{j\in J}$ is complete set for $\mathcal{H}$ and its Gram matrix $\{\Lambda_i\Lambda_j^*\}_{i\in I,j\in J}$ defines a bounded, invertible operator on $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$.*

*Proof.* (i)⇒(ii) Assume that $\{\Lambda_j\}_{j\in J}=\{\Xi_j T^*\}_{j\in J}$ is a Riesz $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. If $G=\{G_{ij}\}_{i,j\in J}$ denotes the matrix of the invertible operator $T^*T$ relative to $\{\Xi_j\}_{j\in J}$, then

$$G_{ij}=\Xi_i T^* T\Xi_j^*=\Lambda_i\Lambda_j^*.$$

Therefore the Gram matrix of $\{\Lambda_j\}_{j\in J}$ is $G$.

(ii)⇒(i) Suppose that Gram matrix of $\{\Lambda_j\}_{j\in J}$ defines a bounded, invertible operator on $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$. Let $\{\Xi_j\}_{j\in J}$ be an arbitrary orthonormal $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and define the mapping

$$T:\mathcal{H}\to\mathcal{H}\quad \text{with}\ \ T\Xi_j^*g_j=\sum_{i\in J}\Xi_i^*\Lambda_i\Lambda_j^*g_j,\quad \forall g_j\in W_j,\quad j\in J.$$

It is straightforward that $T$ is linear, bounded and invertible. Suppose that $f\in\mathcal{H}$ with $f=\sum_{j\in J}\Xi_j^*g_j$, then we have

$$\begin{aligned}\langle Tf,f\rangle&=\sum_{j\in J}\sum_{i\in J}\langle T\Xi_j^*g_j,\Xi_i^*g_i\rangle=\sum_{j\in J}\sum_{i\in J}\sum_{k\in J}\langle \Xi_i\Xi_k^*\Lambda_k\Lambda_j^*g_j,g_i\rangle\\&=\sum_{j\in J}\sum_{i\in J}\langle\Lambda_i\Lambda_j^*g_j,g_i\rangle=\Big\|\sum_{j\in J}\Lambda_j^*g_j\Big\|^2.\end{aligned}$$

Thus $T$ is positive and self-adjoint. Since $T$ is positive, it has a unique positive square-root. Let $P$ denote the square-root of $T$, then the above calculation follows that

$$\frac{1}{\|T^{-1}\|}\sum_{j\in J}\|g_j\|^2\le\Big\|\sum_{j\in J}\Lambda_j^*g_j\Big\|^2=\Big\|P\Big(\sum_{j\in J}\Xi_j^*g_j\Big)\Big\|^2\le\|T\|^2\sum_{j\in J}\|g_j\|^2.$$

Now the result follows from Theorem 3.7. □

## 4 The Stability of $g$-bases under perturbations

The stability of bases is important in practice and is therefore studied widely by many authors, e.g., see [7]. In this section we study the stability of $g$-bases for a Hilbert space $\mathcal{H}$. First we generalized a result of Paley-Wiener [7] to the situation of $g$-basis.

**Theorem 4.1.** *Let $\{\Lambda_j\}_{j\in J}$ be a g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ such that*

$$\Big\|\sum_{j\in F}(\Lambda_j^*g_j-\Gamma_j^*g_j)\Big\|\le\lambda\Big\|\sum_{j\in F}\Lambda_j^*g_j\Big\|$$

*for some constant $0\le\lambda<1$ and each finite subset $F\subset J$ and arbitrary vectors $g_j\in W_j$. Then $\{\Gamma_j\}_{j\in J}$ is a g-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*Proof.* By assumption the series $\sum_{j\in J}(\Lambda_j^* g_j - \Gamma_j^* g_j)$ is convergent whenever the series $\sum_{j\in J}\Lambda_j^* g_j$ is convergent for all arbitrary vectors $g_j \in W_j$. If we define the mapping

$$T:\mathcal{H}\to\mathcal{H}, \quad \text{with } \ T\Lambda_j^* g_j = \Lambda_j^* g_j - \Gamma_j^* g_j, \quad \forall g_j \in W_j, \quad j\in J.$$

Then $T$ is a bounded operator and $\|T\|\le\lambda<1$. Thus the operator $Id_{\mathcal{H}}-T$ is invertible and we have $(Id_{\mathcal{H}}-T)\Lambda_j^*\Lambda_j = \Gamma_j^*\Lambda_j$, consequently $\Lambda_j^*\Lambda_j(Id_{\mathcal{H}}-T^*) = \Lambda_j^*\Gamma_j$. Since $\Lambda_j^*$ is one-to-one on $W_j$, thus $\Lambda_j(Id_{\mathcal{H}}-T^*)=\Gamma_j$. Now the conclusion follows from Theorem 2.4. ☐

**Corollary 4.1.** Let $\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, with dual $g$-basis $\{\Psi_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ such that

$$\sum_{j\in J}\|\Lambda_j-\Gamma_j\|\|\Psi_j\|<1.$$

Then $\{\Gamma_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

*Proof.* If $\lambda=\sum_{j\in J}\|\Lambda_j-\Gamma_j\|\|\Psi_j\|$, then $0\le\lambda<1$. Fix $F\subset J$ with $|F|<\infty$ and let $f=\sum_{j\in F}\Lambda_j^* g_j$ for arbitrary vectors $g_j\in W_j$. Then we compute

$$\begin{aligned}\Big\|\sum_{j\in F}(\Lambda_j^* g_j-\Gamma_j^* g_j)\Big\| &= \Big\|\sum_{j\in F}(\Lambda_j^*-\Gamma_j^*)\Psi_j f\Big\| \\ &\le \sum_{j\in F}\|(\Lambda_j^*-\Gamma_j^*)\Psi_j f\| \\ &\le \sum_{j\in J}\|\Lambda_j-\Gamma_j\|\|\Psi_j\|\|f\| = \lambda\Big\|\sum_{j\in F}\Lambda_j^* g_j\Big\|.\end{aligned}$$

From this the result follows by Theorem 4.1. ☐

In the following we generalized a result of Krein-Milman-Rutman [7] to the situation of $g$-basis.

**Theorem 4.2.** *Let $\{\Lambda_j\}_{j\in J}$ be a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and let $\{\Gamma_j\}_{j\in J}$ be a sequence of operators for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. If there exists a sequence $\{\varepsilon_j\}_{j\in J}$ of positive numbers, such that $\|\Lambda_j-\Gamma_j\|<\varepsilon_j$ for all $j\in J$. Then $\{\Gamma_j\}_{j\in J}$ is a $g$-basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.*

*Proof.* If $\{\Psi_j\}_{j\in J}$ is the dual $g$-basis of $\{\Lambda_j\}_{j\in J}$. Then the result follows from Corollary 4.1, to choose $\varepsilon_j$ small enough such that $\sum_{j\in J}\varepsilon_j\|\Psi_j\|<1$. ☐


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