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Multilinear Commutators of θ-Type Calderón-Zygmund Operators on Non-Homogeneous Metric Measure Spaces

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Abstract. In this paper, the boundedness in Lebesgue spaces of commutators and multilinear commutators generated by θ-type Calderón-Zygmund operators with $RBMO(\mu)$ functions on non-homogeneous metric measure spaces is obtained.

Key Words: Multilinear commutators, θ-type Calderón-Zygmund operators, non-homogeneous metric measure spaces, $RBMO(\mu)$.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

It is well known that non-homogeneous metric measure spaces, which includes both the homogeneous spaces and non-doubling measure spaces, are introduced by Hytönen [8]. From then on, the properties for operators and function spaces in this background are obtained by many researchers. Hytönen et al. [11] and Bui and Duong [1] introduced independently the atomic Hardy space $H^1(\mu)$ and proved that the dual space of $H^1(\mu)$ is $RBMO(\mu)$. Bui and Duong [1] also proved that Calderón-Zygmund operator and commutators are bounded in $L^p(\mu)$ for $1<p<\infty$. Hytönen and Martikainen [9] established Tb theorem on non-homogeneous metric measure spaces. Boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz space is established by Fu et al. [4]. Recently, Morrey spaces and H^p spaces in this settings are also obtained by Cao and Zhou [2] and Fu et al. [3] respectively. We [6,19] obtained the boundedness for commutators of multilinear Calderón-Zygmund operators and multilinear fractional integral operators. For more results, one can refer to [5,7,10–15,22] and the references therein.

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In 1985, Yabuta [21] first introduced θ-type Calderón-Zygmund operator. Later, the properties of this operator are further studied by many researchers. We [20] obtained the boundedness of θ-type Calderón-Zygmund operators and commutators on non-doubling measure spaces. Ri and Zhang [16] researched the boundedness of θ-type Calderón-Zygmund operators on non-homogeneous metric measure spaces. Zheng et al. [23, 24] obtained some properties for bilinear θ-type Calderón-Zygmund operators and maximal bilinear θ-type Calderón-Zygmund operators.

In this paper, the boundedness for commutators and multilinear commutators generated by θ-type Calderón-Zygmund operators with $RBMO(\mu)$ functions on non-homoge -neous metric measure spaces is obtainded. This result includes corresponding results on both the homogeneous spaces and $(\mathbb{R}^n,\mu)$ with non-doubling measure spaces.

Throughout this paper, C always denotes a positive constant independent of the main parameters involved, but it may be different from line to line. And p' is the conjugate index of p, i.e., $1/p+1/p'=1$. Now, let us recall some definitions and notations.

Definition 1.1 (see [8])**.** A metric space (X,d) is geometrically doubling if there exists some $N_0\in\mathbf{N}$ such that, for every ball $B(x,r)\subset X$, there exists a finite ball covering $\{B(x_i,r/2)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most N_0.

Definition 1.2 (see [8])**.** A metric measure space (X,d,μ) is upper doubling if μ is a Borel measure on X and there exists a function $\lambda: X\times(0,+\infty)\to(0,+\infty)$ and a constant $C_\lambda>0$ such that for every $x\in X, r\longmapsto(x,r)$ is non-decreasing, and for any $x\in X, r>0$,

$$\mu(B(x,r))\le\lambda(x,r)\le C_\lambda\lambda(x,r/2). \tag{1.1}$$

Remark 1.1. (i) A homogeneous space is an upper doubling space, if we take $\lambda(x,r)=\mu(B(x,r))$. Also, a non-doubling measure space, satisfying the following polynomial growth condition:

$$\mu(B(x,r))\le Cr^n \tag{1.2}$$

for all $x\in\mathbb{R}^d$ and $r>0$, is also an upper doubling measure space if we take $\lambda(x,r)=Cr^n$.

(ii) It was shown in [11] that there exists another function $\tilde{\lambda}$ such that for any $x,y\in X$, $d(x,y)\le r$,

$$\tilde{\lambda}(x,r)\le\tilde{C}\tilde{\lambda}(y,r). \tag{1.3}$$

Thus, one assumes that λ always satisfies (1.3) in this paper.

(iii) Tan and Li [17] pointed that the upper doubling condition is equivalent to the weak growth condition.

Let $1\le\alpha,\beta\le+\infty$, if $\mu(\alpha B)\le\beta\mu(B)$, then a ball $B\subset X$ is called to be (α,β)-doubling. By Lemma 2.3 of [1], we know that there exist plenties of (α,β)-doubling balls with small radii and with large radii. Unless α and β are specified, otherwise in this paper one means (α,β)-doubling ball is $(6,\beta_0)$-doubling with $\beta_0>\max\{C_\lambda^{3\log_2 6},6^n\}$, where $n=\log_2 N_0$ is the geometric dimension of the space.

Definition 1.3 (see [1])**.** For any two balls $B\subset Q$, let $N_{B,Q}$ be the smallest integer satisfying $6^{N_{B,Q}}l_B\ge l_Q$, denote

$$K_{B,Q}=1+\sum_{k=1}^{N_{B,Q}}\frac{\mu(6^kB)}{\lambda(x_B,6^kl_B)}, \tag{1.4}$$

where x_B and l_B denote the center and semidiameter of the ball B respectively.

Let θ be a nonnegative nondecreasing function on $(0,+\infty)$ such that

$$\int_0^1\frac{\theta(t)}{t}dt<\infty. \tag{1.5}$$

Definition 1.4. A kernel $K(\cdot,\cdot)\in L^1_{loc}(X^2\backslash\{(x,y)|x=y\})$ is called the θ-type Calderón-Zygmund kernel if it satisfies:

$$|K(x,y)|\le\frac{C}{\lambda(x,d(x,y))} \tag{1.6}$$

for all $(x,y)\in X^2$ with $x\neq y$,

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\le\theta\Big(\frac{d(x,x')}{d(x,y)}\Big)\frac{C}{\lambda(x,d(x,y))} \tag{1.7}$$

provided that $Cd(x,x')\le d(x,y)$.

For $f\in L^\infty$ with compact support and $x\notin \mathrm{supp} f$, θ-type Calderón-Zygmund operator with the above kernel K is defined by

$$Tf(x)=\int_X K(x,y)f(y)d\mu(y). \tag{1.8}$$

Definition 1.5 (see [18])**.** Let $\rho>1$. $b\in L^1_{loc}(\mu)$ is $RBMO(\mu)$ function if there exists a constant $C>0$ such that for any ball Q, a number b_Q such that

$$\frac{1}{\mu(\rho Q)}\int_Q|b(x)-b_Q|d\mu(x)\le C, \tag{1.9}$$

and for any two doubling balls $Q\subset R$,

$$|b_Q-b_R|\le CK_{Q,R}. \tag{1.10}$$

The minimal constant C in (1.9) and (1.10) is the $RBMO(\mu)$ norm of b, which is denoted by $\|b\|_*$.

This paper is organized as follows. In Section 2, the boundedness of commutators generated by θ-type Calderón-Zygmund operator with $RBMO(\mu)$ function is proved. The boundedness of multilinear commutators is proved in Section 3.

2 Commutators

In this section, it is shown that if $b\in RBMO(\mu)$ and T is a θ-type Calderón-Zygmund operator bounded on $L^2(\mu)$, then the commutators $[b,T]$ defined by

$$[b,T](f)=bT(f)-T(bf)$$

are bounded on $L^p(\mu)$ $(1<p<\infty)$.

Because the singularity of commutators generated by θ-type Calderón-Zygmund operator with $RBMO(\mu)$ function is stronger than that of θ-type Calderón-Zygmund operator, we need to strengthen the condition assumed on θ in Section 1 as follows:

Let θ be a nonnegative nondecreasing function on $(0,+\infty)$ such that

$$\int_0^1 \frac{\theta(t)}{t}|\log t|dt<\infty. \tag{2.1}$$

Theorem 2.1. *If $b\in RBMO(\mu)$ and θ-type Calderón-Zygmund operator T defined by* (1.8) *with θ as in* (2.1) *is bounded on $L^2(\mu)$, then the commutators $[b,T]$ is bounded on $L^p(\mu)$ $(1<p<\infty)$.*

In order to prove Theorem 2.1, we need to introduce the sharp maximal function

$$M^\sharp f(x)=\sup_{Q\ni x}\frac{1}{\mu(6Q)}\int_Q|f(y)-m_{\widetilde{Q}}(f)|d\mu(y)+\sup_{(Q,R)\in\Delta}\frac{|m_Q(f)-m_R(f)|}{K_{Q,R}},$$

where $\Delta:=\{(Q,R):x\in Q\subset R$ and Q, R are doubling balls$\}$ and $\widetilde{Q}$ is the smallest (α,β)-doubling ball of the form 6^kQ with $k\in\mathbf{N}\bigcup\{0\}$, and

$$m_{\widetilde{Q}}(f)=\frac{1}{\mu(\widetilde{Q})}\int_{\widetilde{Q}}f(x)d\mu(x).$$

The non-centered doubling maximal operator is defined by

$$Nf(x)=\sup_{\substack{Q\ni x,\\ Q\ \text{doubling}}}\frac{1}{\mu(Q)}\int_Q|f(y)|d\mu(y).$$

It is easy to see that for any $f\in L^1_{loc}(\mu)$,

$$|f(x)|\le Nf(x)$$

for $\mu-a.e.$ $x\in X$ by the Lebesgue differential theorem.

For $\eta>1$ and $r>1$, denote

$$M_{r,(\eta)}f(x)=\sup_{Q\ni x}\left(\frac{1}{\mu(\eta Q)}\int_Q|f(y)|^r d\mu(y)\right)^{1/r}.$$

$M_{r,(\eta)}$ is bounded on $L^p(\mu)$ for $p>r$ (see [1]).

Lemma 2.1 (see [1,4]). *Let $f\in L^1_{loc}(\mu)$, with $\int_X f d\mu=0$ if $\|\mu\|<\infty$. If $\inf(1,Nf)\in L^p(\mu)$, then for any $1<p<\infty$,*

$$\|Nf\|_{L^p(\mu)}\le C\|M^\sharp f\|_{L^p(\mu)}.$$

Lemma 2.2 (see [1]). *Let $\rho>1$ and $b\in L^1_{loc}(\mu)$. The following statements are equivalent:*

(i) $b\in RBMO(\mu)$;

(ii) there exists a constant $C>0$ such that for any ball B,

$$\frac{1}{\mu(\rho B)}\int_B|b(x)-m_{\widetilde{B}}(b)|d\mu(x)\le C, \tag{2.2}$$

and for any two doubling balls $B\subset Q$,

$$|m_B(b)-m_Q(b)|\le CK_{B,Q}. \tag{2.3}$$

Moreover, the minimal constant C in (2.2) *and* (2.3) *is equivalent to $\|b\|_*$.*

Lemma 2.3 (see [1,4]). *Let $\rho>1$ and $1<p<\infty$. If $b\in RBMO(\mu)$, then for any ball Q, we have*

$$\frac{1}{\mu(\rho Q)}\int_Q|b(x)-b_Q|^p d\mu(x)\le C\|b\|_*^p,$$
$$\frac{1}{\mu(\rho Q)}\int_Q|b_Q-m_{\tilde{Q}}(b)|^p d\mu(x)\le C\|b\|_*^p.$$

Lemma 2.4 (see [4]). *For any $k\in\mathbb{N}^+$,*

$$|m_{\widetilde{6^k\frac{6}{5}Q}}(b)-m_{\tilde{Q}}(b)|\le Ck\|b\|_*.$$

Lemma 2.5. *If θ-type Calderón-Zygmund operator T defined by* (1.8) *with θ as in* (1.5) *is bounded on $L^2(\mu)$, then T^* is bounded on $L^p(\mu)$ $(1<p<\infty)$, where T^* is defined as*

$$T^*f(x)=\sup_{\epsilon>0}|T_\epsilon f(x)|=\sup_{\epsilon>0}\Big|\int_{d(x,y)>\epsilon}K(x,y)f(y)d\mu(y)\Big|.$$

Remark 2.1. Apply the same method as in [1], we can obtain that T^* satisfies the Cotlar's inequality. Thus Lemma 2.5 holds. Here the details are omitted.

Proof of Theorem 2.1. For all $1<p<\infty$, we first establish the inequality

$$M^\sharp([b,T]f)(x)\le C\|b\|_*\big(M_{p,(6)}(Tf)(x)+M_{p,(5)}f(x)+T^*f(x)\big). \tag{2.4}$$

By Definition 1.5, let $\{b_Q\}_Q$ be a set of numbers satisfying for any ball Q,

$$\int_Q|b(x)-b_Q|d\mu(x)\le 2\mu(6Q)\|b\|_*$$

and for any two balls $Q\subset R$,

$$|b_Q-b_R|\le 2K_{Q,R}\|b\|_*.$$

Apply the similar method as the proof of Theorem 7.6 in [1] or Theorem 3.10 in [4], we only need to prove that

$$\frac{1}{\mu(6Q)}\int_Q |[b,T]f(y)-h_Q|d\mu(y)\le C\|b\|_*(M_{p,(6)}(Tf)(x)+M_{p,(5)}f(x)) \tag{2.5}$$

holds for any x and Q with $x\in Q$, and

$$|h_Q-h_R|\le CK^2_{Q,R}\|b\|_*(M_{p,(5)}f(x)+T^*f(x)) \tag{2.6}$$

for any two balls $Q\subset R$ with $x\in Q$, where

$$h_Q:=m_Q(T[(b-b_Q)f\chi_{X\setminus\frac{6}{5}Q}]).$$

Let us first prove (2.5). Write $[b,T]f$ as follows:

$$\begin{aligned}[b,T]f(y)&=(b(y)-b_Q)Tf(y)-T((b-b_Q)f)(y)\\&=(b(y)-b_Q)Tf(y)-T((b-b_Q)f_1)(y)-T((b-b_Q)f_2)(y),\end{aligned}$$

where $f_1=f\chi_{\frac{6}{5}Q}$ and $f_2=f\chi_{X\setminus\frac{6}{5}Q}$. Then

$$\begin{aligned}&\frac{1}{\mu(6Q)}\int_Q|[b,T]f(y)-h_Q|d\mu(y)\\ \le&\frac{1}{\mu(6Q)}\int_Q|(b(y)-b_Q)Tf(y)|d\mu(y)+\frac{1}{\mu(6Q)}\int_Q|T((b-b_Q)f_1)(y)|d\mu(y)\\&+\frac{1}{\mu(6Q)}\int_Q|T((b-b_Q)f_2)(y)-h_Q|d\mu(y)\\=:&I+II+III.\end{aligned}$$

For I, by Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}&\frac{1}{\mu(6Q)}\int_Q|(b(y)-b_Q)Tf(y)|d\mu(y)\\ \le&\Big(\frac{1}{\mu(6Q)}\int_Q|b(y)-b_Q|^{p'}d\mu(y)\Big)^{1/p'}\Big(\frac{1}{\mu(6Q)}\int_Q|Tf(y)|^pd\mu(y)\Big)^{1/p}\\ \le&C\|b\|_*M_{p,(6)}(Tf)(x).\end{aligned}$$

For II, take $s=\sqrt{p}$, by Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}&\Big[\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|(b(y)-b_Q)f_1(y)|^sd\mu(y)\Big]^{1/s}\\ \le&\Big(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|b(y)-b_Q|^{ss'}d\mu(y)\Big)^{1/ss'}\Big(\frac{1}{\mu(6Q)}\int_{\frac{6}{5}Q}|f(y)|^pd\mu(y)\Big)^{1/p}\\ \le&C\|b\|_*M_{p,(5)}f(x).\end{aligned}$$

Therefore

$$\begin{aligned} II \le & \frac{\mu(Q)^{1-1/s}}{\mu(6Q)} \|T((b-b_Q)f_1)\|_{L^s(\mu)} \\ \le & C\frac{\mu(Q)^{1-1/s}}{\mu(6Q)^{1-1/s}} \|b\|_* M_{p,(5)} f(x) \\ \le & C\|b\|_* M_{p,(5)} f(x). \end{aligned}$$

For III, we first estimate $|T((b-b_Q)f_2)(y)-b_Q|$. For x, $y\in Q$,

$$\begin{aligned} & |T((b-b_Q)f_2)(y)-T((b-b_Q)f_2)(x)| \\ \le & C\int_{X\setminus\frac{6}{5}Q} \theta\left(\frac{d(y,x)}{d(z,x)}\right) \frac{|b(z)-b_Q\|f(z)|d\mu(z)}{\lambda(z,d(z,x))} \\ \le & C\sum_{k=1}^{\infty}\int_{6^k\frac{6}{5}Q\setminus 6^{k-1}\frac{6}{5}Q} \theta\left(\frac{d(y,x)}{d(z,x)}\right) \frac{1}{\lambda(z,6^{k-1}\frac{6}{5}l_Q)} \\ & \times\left(|b(z)-b_{6^k\frac{6}{5}Q}|+|b_Q-b_{6^k\frac{6}{5}Q}|\right)|f(z)|d\mu(z) \\ \le & C\sum_{k=1}^{\infty}\theta(6^{-k})\frac{1}{\mu(5\times 6^k\frac{6}{5}Q)}\int_{6^k\frac{6}{5}Q}|b(z)-b_{6^k\frac{6}{5}Q}||f(z)|d\mu(z) \\ & +C\sum_{k=1}^{\infty}k\theta(6^{-k})\|b\|_*\frac{1}{\mu(5\times 6^k\frac{6}{5}Q)}\int_{6^k\frac{6}{5}Q}|f(z)|d\mu(z) \\ \le & C\|b\|_* M_{p,(5)}f(x)\sum_{k=1}^{\infty}\theta(6^{-k})+C\|b\|_* M_{p,(5)}f(x)\sum_{k=1}^{\infty}k\theta(6^{-k}) \\ \le & C\|b\|_* M_{p,(5)}f(x)+C\|b\|_* M_{p,(5)}f(x)\int_0^1\frac{\theta(t)}{t}|\log t|dt \\ \le & C\|b\|_* M_{p,(5)}f(x). \end{aligned}$$

Here we have used the following inequality:

$$\int_0^1\frac{\theta(t)}{t}|\log t|dt\ge\sum_{k=1}^{\infty}\int_{6^{-k}}^{6^{1-k}}\frac{\theta(6^{-k})}{6^{1-k}}|\log 6^{-k}|dt\ge C\sum_{k=1}^{\infty}k\theta(6^{-k}).$$

Thus

$$\begin{aligned} III\le & \frac{1}{\mu(6Q)}\int_Q|T((b-b_Q)f_2)(y)-h_Q|d\mu(y) \\ \le & \frac{1}{\mu(6Q)}\int_Q|T((b-b_Q)f_2)(y)-m_Q(T(b-b_Q)f_2)|d\mu(y) \\ \le & C\|b\|_* M_{p,(5)}f(x). \end{aligned}$$

So (2.5) is proved.

Now we turn to prove (2.6). Consider two balls $Q\subset R$ with $x\in Q$. Denote $N=N_{Q,R}+1$, then

$$\begin{aligned}&|m_Q(T((b-b_Q)f\chi_{X\setminus\frac{6}{5}Q}))-m_R(T((b-b_R)f\chi_{X\setminus\frac{6}{5}R}))|\\ \le&|m_Q(T((b-b_Q)f\chi_{6Q\setminus\frac{6}{5}Q}))|+|m_Q(T((b_Q-b_R)f\chi_{X\setminus 6Q}))|\\ &+|m_Q(T((b-b_R)f\chi_{6^NQ\setminus 6Q}))|\\ &+|m_Q(T((b-b_R)f\chi_{X\setminus 6^NQ}))-m_R(T((b-b_R)f\chi_{X\setminus 6^NQ}))|\\ &+|m_R(T((b-b_R)f\chi_{6^NQ\setminus\frac{6}{5}R}))|\\ =:&J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

For $y\in Q$, we have

$$\begin{aligned}&|T((b-b_Q)f\chi_{6Q\setminus\frac{6}{5}Q})(y)|\\ \le&C\int_{6Q}\frac{|b(z)-b_Q||f(z)|}{\lambda(z,d(z,y))}d\mu(z)\\ \le&C\left(\frac{1}{\mu(5\times 6Q)}\int_{6Q}|b(z)-b_Q|^{p'}d\mu(z)\right)^{1/p'}\left(\frac{1}{\mu(5\times 6Q)}\int_{6Q}|f(z)|^pd\mu(z)\right)^{1/p}\\ \le&C\|b\|_*M_{p,(5)}f(x).\end{aligned}$$

Then $J_1\le C\|b\|_*M_{p,(5)}f(x)$.

For x, $y\in Q$, we have

$$\begin{aligned}|T(f\chi_{X\setminus 6Q})(y)|\le&T^*f(x)+C\sup_{Q\ni x}\int_{6Q}\frac{|f(z)|}{\lambda(z,d(z,y))}d\mu(z)\\ \le&T^*f(x)+CM_{p,(5)}f(x).\end{aligned}$$

Thus

$$J_2=|(b_Q-b_R)T(f\chi_{X\setminus 6Q})(y)|\le CK_{Q,R}\|b\|_*(T^*f(x)+M_{p,(5)}f(x)).$$

Let us estimate J_3. Let x_Q be the center of Q. For $y\in Q$, we obtain

$$\begin{aligned}&|T((b-b_R)f\chi_{6^NQ\setminus 6Q})(y)|\\ \le&C\sum_{k=1}^{N-1}\int_{6^{k+1}Q\setminus 6^kQ}\frac{|b(z)-b_R||f(z)|}{\lambda(z,6^kl_Q)}d\mu(z)\\ \le&C\sum_{k=1}^{N-1}\frac{1}{\lambda(x_Q,5\times 6^kl_Q)}\left(\int_{6^{k+1}Q}|b(z)-b_R|^{p'}d\mu(z)\right)^{1/p'}\left(\int_{6^{k+1}Q}|f(z)|^pd\mu(z)\right)^{1/p}\\ \le&CK_{Q,R}\|b\|_*\sum_{k=1}^{N-1}\frac{\mu(5\times 6^{k+1}Q)}{\lambda(x_Q,5\times 6^{k+1}l_Q)}\left(\frac{1}{\mu(5\times 6^{k+1}Q)}\int_{6^{k+1}Q}|f(z)|^pd\mu(z)\right)^{1/p}\end{aligned}$$

$$\begin{aligned}&\le CK_{Q,R}\|b\|_*M_{p,(5)}f(x)\sum_{k=1}^{N-1}\frac{\mu(6^{k+2}Q)}{\lambda(x_Q,6^{k+2}l_Q)}\\&\le CK_{Q,R}^2\|b\|_*M_{p,(5)}f(x).\end{aligned}$$

Taking the mean over Q, then

$$J_3\le CK_{Q,R}^2\|b\|_*M_{p,(5)}f(x).$$

Let us estimate J_4. Estimating as in III, for any y, $z\in R$, we have

$$|T((b-b_R)f\chi_{X\setminus 6^NQ})(y)-T((b-b_R)f\chi_{X\setminus 6^NQ})(z)|\le C\|b\|_*M_{p,(5)}f(x).$$

Taking the mean over Q for y and over R for z, then

$$J_4\le C\|b\|_*M_{p,(5)}f(x).$$

For J_5, similar to estimate J_1, we obtain $J_5\le C\|b\|_*M_{p,(5)}f(x)$. Thus (2.6) is proved and (2.4) is obtained.

Next we prove Theorem 2.1. By Lemma 3.11 in [4], we can assume that $b\in L^\infty(\mu)$. Let us consider the following two cases.

Case 1. $\|\mu\|=\infty$. By Lemma 2.1, Lemma 2.5, the boundedness of $M_{p,(\eta)}$ on $L^r(\mu)$ for $r>p$ and (2.4), we have

$$\begin{aligned}\|[b,T]f\|_{L^p(\mu)}&\le\|N([b,T]f)\|_{L^p(\mu)}\\&\le C\|M^\sharp([b,T]f)\|_{L^p(\mu)}\\&\le C\|b\|_*(\|M_{p,(6)}(Tf)\|_{L^p(\mu)}+\|M_{p,(5)}f\|_{L^p(\mu)}+\|T^*f\|_{L^p(\mu)})\\&\le C\|b\|_*\|f\|_{L^p(\mu)}.\end{aligned}$$

Case 2. $\|\mu\|<\infty$. By Lemma 2.3 and Lebesgue dominated convergence theorem, we obtain for all $1<r<\infty$,

$$\left(\frac{1}{\mu(X)}\int_X|b(x)-b_X|^r d\mu(x)\right)^{1/r}\le C\|b\|_*,$$

where

$$b_X:=\frac{1}{\mu(X)}\int_X b(y)d\mu(y).$$

As

$$N([b,T]f)\le N\big([b,T]f-m_X([b,T]f)\big)+|m_X([b,T]f)|.$$

It is easy to see that

$$\int_X\{[b,T]f(x)-m_X([b,T]f)\}d\mu(x)=0,$$

thus

$$\begin{aligned}\|N([b,T]f-m_X([b,T]f))\|_{L^p(\mu)} &\le C\|M^\sharp([b,T]f-m_X([b,T]f))\|_{L^p(\mu)}\\ &=C\|M^\sharp([b,T]f)\|_{L^p(\mu)}\\ &\le C\|b\|_*\|f\|_{L^p(\mu)}.\end{aligned}$$

For $|m_X([b,T]f)|$, write $|[b,T]f|\le|(b-b_X)Tf|+|T(b-b_X)f|$. By Hölder's inequality and the boundedness of T, we have

$$\begin{aligned}|m_X((b-b_X)Tf)| &\le \frac{1}{\mu(X)}\int_X|(b(y)-b_X)(Tf)(y)|d\mu(y)\\ &\le \frac{1}{\mu(X)}\Big(\int_X|b(y)-b_X|^{p'}d\mu(y)\Big)^{1/p'}\Big(\int_X|(Tf)(y)|^p d\mu(y)\Big)^{1/p}\\ &\le \frac{1}{\mu(X)}\Big(\int_X|b(y)-b_X|^{p'}d\mu(y)\Big)^{1/p'}\Big(\int_X|f(y)|^p d\mu(y)\Big)^{1/p}\\ &\le (\mu(X))^{-1/p}\|f\|_{L^p(\mu)}\Big(\frac{1}{\mu(X)}\int_X|b(y)-b_X|^{p'}d\mu(y)\Big)^{1/p'}\\ &\le C(\mu(X))^{-1/p}\|b\|_*\|f\|_{L^p(\mu)}.\end{aligned}$$

Take $s=\sqrt{p}$, we have

$$\begin{aligned}|m_X(T((b-b_X)f))| &\le \frac{1}{\mu(X)}\int_X|T((b-b_X)f)(y)|d\mu(y)\\ &\le \frac{(\mu(X))^{1-1/s}}{\mu(X)}\|T((b-b_X)f)\|_{L^s(\mu)}\\ &\le \frac{(\mu(X))^{1-1/s}}{\mu(X)}\|(b-b_X)f\|_{L^s(\mu)}\\ &\le \frac{(\mu(X))^{1-1/s}}{\mu(X)}\Big(\int_X|b(y)-b_X|^{ss'}d\mu(y)\Big)^{1/ss'}\Big(\int_X|f(y)|^p d\mu(y)\Big)^{1/p}\\ &\le (\mu(X))^{-1/p}\|f\|_{L^p(\mu)}\Big(\frac{1}{\mu(X)}\int_X|b(y)-b_X|^{ss'}d\mu(y)\Big)^{1/ss'}\\ &\le C(\mu(X))^{-1/p}\|b\|_*\|f\|_{L^p(\mu)}.\end{aligned}$$

Therefore

$$\|m_X([b,T]f)\|_{L^p(\mu)}\le C\|b\|_*\|f\|_{L^p(\mu)}.$$

The proof of Theorem 2.1 is completed. □

3 Multilinear commutators

To prove $L^p(\mu)$-boundedness of multilinear commutators, we strengthen the assumption on θ as follows.

Let θ be a nonnegative nondecreasing function on $(0,+\infty)$ such that

$$\int_0^1 \frac{\theta(t)}{t}|\log t|^m dt<\infty, \quad m\in\mathbb{N}. \tag{3.1}$$

In fact, one assumes θ satisfies

$$\int_0^1 \frac{\theta(t)}{t}|\log t| dt<\infty$$

for commutators in Section 2, so it is reasonable for multilinear commutators to assume θ satisfies (3.1).

Let T be the θ-type Calderón-Zygmund operator, $m\in\mathbb{N}$ and $b_i\in RBMO(\mu)$, $i=1,2,\cdots,m$. The multilinear commutators $T_{\vec{b}}$ is formally defined by

$$T_{\vec{b}}f(x)=[b_m,[b_{m-1},\cdots,[b_1,T]]]f(x), \tag{3.2}$$

where $\vec{b}=(b_1,b_2,\cdots,b_m)$ and

$$[b_1,T]f(x)=b_1(x)Tf(x)-T(b_1f)(x).$$

If $m=1$ and $\vec{b}=b$, we denote $T_{\vec{b}}f$ simply by T_bf.

For $1\le i\le m$, we denote by C_i^m the family of all finite subsets $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(i)\}$ of $\{1,2,\cdots,m\}$ with i different elements. For any $\sigma\in C_i^m$, $\sigma'=\{1,2,\cdots,m\}\backslash\sigma$. Let $\vec{b}=(b_1,b_2,\cdots,b_m)$ be a finite set of locally integrable functions. For all $1\le i\le m$ and $\sigma=\{\sigma(1),\cdots,\sigma(i)\}\in C_i^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\cdots,b_{\sigma(i)})$, $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(i)}$, $\|\vec{b}_\sigma\|_*=\|b_{\sigma(1)}\|_*\cdots\|b_{\sigma(i)}\|_*$ and $\|\vec{b}\|_*=\|b_1\|_*\cdots\|b_m\|_*$, see [20].

For any ball $Q\in X$ and $y,z\in X$, denote

$$\begin{aligned}
&[b(y)-b(z)]_\sigma=[b_{\sigma(1)}(y)-b_{\sigma(1)}(z)]\cdots[b_{\sigma(i)}(y)-b_{\sigma(i)}(z)],\\
&[m_{\widetilde{Q}}(b)-b(y)]_\sigma=[m_{\widetilde{Q}}(b_{\sigma(1)})-b_{\sigma(1)}(y)]\cdots[m_{\widetilde{Q}}(b_{\sigma(i)})-b_{\sigma(i)}(y)].
\end{aligned}$$

For any $\sigma\in C_i^m$, set

$$T_{\vec{b}_\sigma}f(x)=[b_{\sigma(i)},[b_{\sigma(i-1)},\cdots,[b_{\sigma(1)},T]]]f(x).$$

When $\sigma=\{1,\cdots,m\}$, denote $T_{\vec{b}_\sigma}$ simply by $T_{\vec{b}}$, see [20].

Theorem 3.1. *Let $m\in\mathbb{N}$ and for $i=1,2,\cdots,m$, $b_i\in RBMO(\mu)$. Let $T_{\vec{b}}$ be defined by* (3.2) *with θ as in* (3.1). *Assume that T is bounded on $L^2(\mu)$, then multilinear commutators $T_{\vec{b}}$ is bounded on $L^p(\mu)$ $(1<p<\infty)$.*

Proof. The proof of the case $\|\mu\|<\infty$ is similar to $\|\mu\|=\infty$. Without loss of generality, we only prove the case $\|\mu\|=\infty$. The theorem is proved by induction on m. If $m=1$, the result of Theorem 2.1 asserts that T_b is bounded on $L^p(\mu)$ for any $p\in(1,\infty)$.

Now we assume that $m\geq 2$ is an integer and that for any $1\leq i\leq m-1$ and any subset $\sigma=\{\sigma(1),\cdots,\sigma(i)\}$ of $\{1,\cdots,m\}$, $T_{\vec{b}_\sigma}$ is bounded on $L^p(\mu)$ for any $p\in(1,\infty)$. Let us first claim that for any $1<r<\infty$, $T_{\vec{b}}$ satisfies the following sharp maximal function estimate

$$\begin{aligned}&M^\sharp(T_{\vec{b}}f)(x)\\ \leq& C\|\vec{b}\|_*\{M_{r,(6)}(Tf)(x)+M_{r,(5)}f(x)\}\\ &+C\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|_*M_{r,(6)}(T_{\vec{b}_{\sigma'}}f)(x).\end{aligned}\tag{3.3}$$

With the similar method as proving Theorem 1.9 in [4], it suffices to show that

$$\begin{aligned}&\frac{1}{\mu(6Q)}\int_Q|T_{\vec{b}}f(y)-h_Q|d\mu(y)\\ \leq& C\|\vec{b}\|_*\{M_{r,(6)}(Tf)(x)+M_{r,(5)}f(x)\}\\ &+C\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|_*M_{r,(6)}(T_{\vec{b}_{\sigma'}}f)(x)\end{aligned}\tag{3.4}$$

holds for any x and Q with $x\in Q$, and

$$\begin{aligned}&|h_Q-h_R|\\ \leq& CK_{Q,R}^{m+1}\|\vec{b}\|_*\{M_{r,(6)}(Tf)(x)+M_{r,(5)}f(x)\}\\ &+CK_{Q,R}^{m+1}\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|_*M_{r,(6)}(T_{\vec{b}_{\sigma'}}f)(x)\end{aligned}\tag{3.5}$$

holds for any two balls $x\in Q\subset R$, where Q is an arbitrary ball, R is a doubling ball,

$$h_Q=m_Q\Big(T\Big[\Big(m_{\tilde{Q}}(b_1)-b_1\Big)\cdots\Big(m_{\tilde{Q}}(b_m)-b_m\Big)f\chi_{X\setminus\frac{6}{5}Q}\Big]\Big),$$

and

$$h_R=m_R\Big(T\Big[(m_R(b_1)-b_1)\cdots(m_R(b_m)-b_m)f\chi_{X\setminus\frac{6}{5}R}\Big]\Big).$$

We first estimate (3.4). It is easy to see that for $y,z\in X$,

$$\prod_{i=1}^m\Big[m_{\tilde{Q}}(b_i)-b_i(z)\Big]=\sum_{i=0}^m\sum_{\sigma\in C_i^m}[b(y)-b(z)]_{\sigma'}[m_{\tilde{Q}}(b)-b(y)]_\sigma,$$

where if $i=0$, then $\sigma=\emptyset$ and $\sigma'=\{1,2\cdots m\}$. Then

$$T_{\vec{b}}f(y)=T\left(\prod_{i=1}^m\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\right)(y)-\sum_{i=1}^m\sum_{\sigma\in C_i^m}\Big[m_{\tilde{Q}}(b)-b(y)\Big]_\sigma T_{\vec{b}_{\sigma'}}f(y),$$

where if $i=m$, we denote $T_{\vec{b}_{\sigma'}}f(y)$ by $Tf(y)$. Thus

$$\begin{aligned}
&\frac{1}{\mu(6Q)}\int_Q |T_{\vec{b}}f(y)-h_Q|d\mu(y)\\
\leq&\frac{1}{\mu(6Q)}\int_Q\left|T\left(\prod_{i=1}^m\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{\frac{6}{5}Q}\right)(y)\right|d\mu(y)\\
&+\sum_{i=1}^m\sum_{\sigma\in C_i^m}\frac{1}{\mu(6Q)}\int_Q\left|\Big[m_{\tilde{Q}}(b)-b(y)\Big]_\sigma\right|\left|T_{\vec{b}_{\sigma'}}f(y)\right|d\mu(y)\\
&+\frac{1}{\mu(6Q)}\int_Q\left|T\left(\prod_{i=1}^m\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{X\setminus\frac{6}{5}Q}\right)(y)-h_Q\right|d\mu(y)\\
=:&L_1+L_2+L_3.
\end{aligned}$$

Take $s=\sqrt{r}$ and write

$$b_i(y)-m_{\tilde{Q}}(b_i)=b_i(y)-m_{\widetilde{\frac{6}{5}Q}}(b_i)+m_{\widetilde{\frac{6}{5}Q}}(b_i)-m_{\tilde{Q}}(b_i)$$

for $i=1,\cdots,m$. By the boundedness of T on $L^s(\mu)$ for $1<s<\infty$, Hölder's inequality and Lemma 2.3,

$$\begin{aligned}
L_1\leq&\frac{\mu(Q)^{1-1/s}}{\mu(6Q)}\left\|T\left(\prod_{i=1}^m\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{\frac{6}{5}Q}\right)\right\|_{L^s(\mu)}\\
\leq&\frac{C\mu(Q)^{1-1/s}}{\mu(6Q)}\left\|\prod_{i=1}^m\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{\frac{6}{5}Q}\right\|_{L^s(\mu)}\\
\leq&\frac{C}{\mu(6Q)^{1/s}}\left(\int_{\frac{6}{5}Q}\prod_{i=1}^m\left|b_i(y)-m_{\tilde{Q}}(b_i)\right|^{ss'}d\mu(y)\right)^{1/ss'}\times\left(\int_{\frac{6}{5}Q}|f(y)|^r d\mu(y)\right)^{1/r}\\
\leq&C\|\vec{b}\|_*M_{r,(5)}f(x).
\end{aligned}$$

From Hölder's inequality, then

$$\begin{aligned}
L_2\leq&\sum_{i=1}^m\sum_{\sigma\in C_i^m}\left(\frac{1}{\mu(6Q)}\int_Q\left|\Big[b(y)-m_{\tilde{Q}}(b)\Big]_\sigma\right|^{r'}d\mu(y)\right)^{1/r'}\\
&\times\left(\frac{1}{\mu(6Q)}\int_Q|T_{\vec{b}_{\sigma'}}f(y)|^r d\mu(y)\right)^{1/r}\\
\leq&C\sum_{i=1}^m\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|_*M_{r,(6)}(T_{\vec{b}_{\sigma'}}f)(x).
\end{aligned}$$

To estimate L_3, take y, $y_1 \in Q$, then by the condition (1.7), Hölder's inequality and Lemma 2.4,

$$\begin{aligned}
&\left|T\left(\prod_{i=1}^{m}\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{X\backslash\frac{6}{5}Q}\right)(y)-T\left(\prod_{i=1}^{m}\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{X\backslash\frac{6}{5}Q}\right)(y_1)\right|\\
\le& C\int_{X\backslash\frac{6}{5}Q}\theta\left(\frac{d(y,y_1)}{d(z,y)}\right)\frac{1}{\lambda(z,d(z,y))}\prod_{i=1}^{m}\Big|b_i(z)-m_{\tilde{Q}}(b_i)\Big|\Big|f(z)|d\mu(z)\\
\le& C\sum_{j=1}^{\infty}\int_{6^j\frac{6}{5}Q\backslash 6^{j-1}\frac{6}{5}Q}\theta\left(\frac{d(y,y_1)}{d(z,y)}\right)\frac{1}{\lambda(z,6^{j-1}\frac{6}{5}l_Q)}\\
&\quad\times\prod_{i=1}^{m}\left(\Big|b_i(z)-m_{\widetilde{6^j\frac{6}{5}Q}}(b_i)\Big|+\Big|m_{\widetilde{6^j\frac{6}{5}Q}}(b_i)-m_{\tilde{Q}}(b_i)\Big|\right)|f(z)|d\mu(z)\\
\le& C\sum_{j=1}^{\infty}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\theta(6^{-j})j^{m-i}\|\vec{b}_{\sigma'}\|_*\\
&\quad\times\frac{1}{\mu(5\times 6^j\frac{6}{5}l_Q)}\int_{6^j\frac{6}{5}Q}\left|\Big[b(z)-m_{\widetilde{6^j\frac{6}{5}Q}}\Big]_{\sigma}\right|\Big|f(z)|d\mu(z)\\
\le& C\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\sum_{j=1}^{\infty}\theta(6^{-j})j^{m-i}\|\vec{b}_{\sigma}\|_*\|\vec{b}_{\sigma'}\|_*M_{r,(5)}f(x)\\
\le& C\|\vec{b}\|_*M_{r,(5)}f(x)\int_0^1\frac{\theta(t)}{t}|\log t|^m dt\\
\le& C\|\vec{b}\|_*M_{r,(5)}f(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|T\left(\prod_{i=1}^{m}\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{X\backslash\frac{6}{5}Q}\right)(y)-h_Q\right|\\
=&\left|T\left(\prod_{i=1}^{m}\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{X\backslash\frac{6}{5}Q}\right)(y)-m_Q\left[T\left(\prod_{i=1}^{m}\Big[m_{\tilde{Q}}(b_i)-b_i\Big]f\chi_{X\backslash\frac{6}{5}Q}\right)\right]\right|\\
\le& C\|\vec{b}\|_*M_{r,(5)}f(x).
\end{aligned}$$

Thus

$$L_3\le C\|\vec{b}\|_*M_{r,(5)}f(x).$$

(3.4) is proved by the estimate for L_1, L_2 and L_3.

Now we estimate (3.5). For any two balls $Q\subset R$ with $x\in Q$ and R being a doubling ball, let $N:=N_{Q,R}+1$, then

$$
\begin{aligned}
&\left|m_Q\left[T\left(\prod_{i=1}^m\left[m_{\tilde{Q}}(b_i)-b_i\right]f\chi_{X\setminus\frac{6}{5}Q}\right)\right]-m_R\left[T\left(\prod_{i=1}^m[m_R(b_i)-b_i]f\chi_{X\setminus\frac{6}{5}R}\right)\right]\right|\\
\le&\left|m_Q\left[T\left(\prod_{i=1}^m\left[m_{\tilde{Q}}(b_i)-b_i\right]f\chi_{X\setminus 6^NQ}\right)\right]-m_R\left[T\left(\prod_{i=1}^m\left[m_{\tilde{Q}}(b_i)-b_i\right]f\chi_{X\setminus 6^NQ}\right)\right]\right|\\
&+\left|m_R\left[T\left(\prod_{i=1}^m[m_R(b_i)-b_i]f\chi_{X\setminus 6^NQ}\right)\right]-m_R\left[T\left(\prod_{i=1}^m\left[m_{\tilde{Q}}(b_i)-b_i\right]f\chi_{X\setminus 6^NQ}\right)\right]\right|\\
&+\left|m_Q\left[T\left(\prod_{i=1}^m\left[m_{\tilde{Q}}(b_i)-b_i\right]f\chi_{6^NQ\setminus\frac{6}{5}Q}\right)\right]\right|+\left|m_R\left[T\left(\prod_{i=1}^m[m_R(b_i)-b_i]f\chi_{6^NQ\setminus\frac{6}{5}R}\right)\right]\right|\\
=:&I_1+I_2+I_3+I_4.
\end{aligned}
$$

By the similar estimate for L_3, we have

$$
I_1\le CK_{Q,R}^m\|\vec{b}\|_*M_{r,(5)}f(x).
$$

For I_2, write

$$
\begin{aligned}
&\left|T\left(\prod_{i=1}^m[m_R(b_i)-b_i]f\chi_{X\setminus 6^NQ}\right)(y)-T\left(\prod_{i=1}^m\left[m_{\tilde{Q}}(b_i)-b_i\right]f\chi_{X\setminus 6^NQ}\right)(y)\right|\\
=&\left|T\left(\prod_{i=1}^m[m_R(b_i)-b_i]f\chi_{X\setminus 2^NQ}\right)(y)\right.\\
&\left.-\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left[m_{\tilde{Q}}(b)-m_R(b)\right]_{\sigma'}T\left([m_R(b)-b]_\sigma f\chi_{X\setminus 6^NQ}\right)(y)\right|\\
\le&C\sum_{i=0}^{m-1}\sum_{\sigma\in C_i^m}K_{Q,R}^{m-i}\|\vec{b}_{\sigma'}\|_*\left|T\left([m_R(b)-b]_\sigma f\chi_{X\setminus 6^NQ}\right)(y)\right|\\
\le&C\sum_{i=0}^{m-1}\sum_{\sigma\in C_i^m}K_{Q,R}^{m-i}\|\vec{b}_{\sigma'}\|_*\left\{T\left([m_R(b)-b]_\sigma f\right)(y)+T\left([m_R(b)-b]_\sigma f\chi_{6^NQ}\right)(y)\right\}\\
\le&C\sum_{i=0}^{m-1}\sum_{\sigma\in C_i^m}K_{Q,R}^{m-i}\|\vec{b}_{\sigma'}\|_*\left\{\sum_{j=0}^{i}\sum_{\eta\in C_j^i}|[m_R(b)-b(y)]_{\eta'}\left|T_{\vec{b}_\eta}f(y)\right|\right.\\
&\left.+\left|T\left([m_R(b)-b]_\sigma f\chi_{6^NQ\setminus\frac{6}{5}R}\right)(y)\right|+\left|T\left([m_R(b)-b]_\sigma f\chi_{\frac{6}{5}R}\right)(y)\right|\right\}.
\end{aligned}
$$

From the fact that R is a doubling cube and Hölder's inequality, then

$$
\begin{aligned}
&\frac{1}{\mu(R)}\int_R|[b(y)-m_R(b)]_{\eta'}\|T_{\vec{b}_\eta}f(y)|d\mu(y)\\
\le&C\|\vec{b}_{\eta'}\|_*M_{r,(6)}(T_{\vec{b}_\eta}f)(x).
\end{aligned}
$$

By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}
&\Big|T\Big([m_R(b)-b]_\sigma f\chi_{6^NQ\setminus\frac{6}{5}R}\Big)(y)\Big|\\
\le&\int_{6^NQ\setminus\frac{6}{5}R}|K(y,z)||[m_R(b)-b(z)]_\sigma||f(z)|d\mu(z)\\
\le&C\int_{6^NQ}\frac{1}{\lambda(z,d(y,z))}|[b(z)-m_R(b)]_\sigma||f(z)|d\mu(z)\\
\le&C\left(\frac{1}{\mu(5\times6^NQ)}\int_{6^NQ}|[b(z)-m_R(b)]_\sigma|^{r'}d\mu(z)\right)^{1/r'}\\
&\quad\times\left(\frac{1}{\mu(5\times6^NQ)}\int_{6^NQ}|f(z)|^rd\mu(z)\right)^{1/r}\\
\le&C\|\vec{b}_\sigma\|_*M_{r,(5)}f(x).
\end{aligned}$$

Taking the mean over $y\in R$,

$$m_R\left[\left|T\Big([m_R(b)-b]_\sigma f\chi_{6^NQ\setminus\frac{6}{5}R}\Big)\right|\right]\le C\|\vec{b}_\sigma\|_*M_{r,(5)}f(x).$$

Also, we have

$$m_R\left[\left|T\Big([m_R(b)-b]_\sigma f\chi_{\frac{6}{5}R}\Big)\right|\right]\le C\|\vec{b}_\sigma\|_*M_{r,(5)}f(x).$$

Thus

$$I_2\le CK^m_{Q,R}\Big\{\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|\vec{b}_{\sigma'}\|_*M_{r,(6)}(T_{\vec{b}_\sigma}f)(x)+\|\vec{b}\|_*M_{r,(6)}(Tf)(x)+\|\vec{b}\|_*M_{r,(5)}f(x)\Big\}.$$

Next we estimate I_3. For $y\in Q$,

$$\begin{aligned}
&\left|T(\prod_{i=1}^m[m_{\tilde{Q}}(b_i)-b_i]f\chi_{6^NQ\setminus\frac{6}{5}Q})(y)\right|\\
\le&C\sum_{j=1}^{N-1}\frac{1}{\lambda(x_Q,6^jl_Q)}\int_{6^{j+1}Q\setminus6^jQ}\prod_{i=1}^m|b_i(z)-m_{\tilde{Q}}(b_i)||f(z)|d\mu(z)\\
&\quad+\frac{C}{\lambda(x_Q,l_Q)}\int_{6Q\setminus\frac{6}{6}Q}\prod_{i=1}^m|b_i(z)-m_{\tilde{Q}}(b_i)||f(z)|d\mu(z)\\
\le&C\sum_{j=1}^{N-1}\frac{1}{\lambda(x_Q,6^jl_Q)}\left\{\int_{6^{j+1}Q}\prod_{i=1}^m|b_i(z)-m_{\tilde{Q}}(b_i)|^{r'}d\mu(z)\right\}^{1/r'}\times\left\{\int_{6^{j+1}Q}|f(z)|^rd\mu(z)\right\}^{1/r}\\
&\quad+\frac{C}{\mu(5\times6Q)}\left\{\int_{6Q}\prod_{i=1}^m|b_i(z)-m_{\tilde{Q}}(b_i)|^{r'}d\mu(z)\right\}^{1/r'}\times\left\{\int_{6Q}|f(z)|^rd\mu(z)\right\}^{1/r}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{N-1}\frac{1}{\lambda(x_Q,6^jl_Q)}\Big\{\int_{6^{j+1}Q}\prod_{i=1}^m(|b_i(z)-m_{\widetilde{6^{j+1}Q}}(b_i)|+|m_{\widetilde{6^{j+1}Q}}(b_i)-m_{\widetilde{Q}}(b_i)|)^{r'}d\mu(z)\Big\}^{1/r'}\\
&\quad\times\Big\{\int_{6^{j+1}Q}|f(z)|^rd\mu(z)\Big\}^{1/r}+C\|\vec{b}\|_*\Big\{\frac{1}{\mu(5\times 6Q)}\int_{6Q}|f(z)|^rd\mu(z)\Big\}^{1/r}\\
&\le CK_{Q,R}^m\|\vec{b}\|_*\sum_{j=1}^{N-1}\frac{\mu(5\times 6^{j+1}Q)}{\lambda(x_Q,6^jl_Q)}\\
&\quad\times\Big\{\frac{1}{\mu(5\times 6^{j+1}Q)}\int_{6^{j+1}Q}|f(z)|^rd\mu(z)\Big\}^{1/r}+C\|\vec{b}\|_*M_{r,(5)}f(x)\\
&\le CK_{Q,R}^{m+1}\|\vec{b}\|_*M_{r,(5)}f(x).
\end{aligned}
$$

Taking the mean over $y\in Q$,

$$I_3\le CK_{Q,R}^{m+1}\|\vec{b}\|_*M_{r,(5)}f(x).$$

For I_4, we have

$$
\begin{aligned}
&\Big|T\Big(\prod_{i=1}^m[m_R(b_i)-b_i]f\chi_{6^NQ\setminus\frac{6}{5}R}\Big)(y)\Big|\\
\le& C\int_{6^NQ\setminus\frac{6}{5}R}\frac{1}{\lambda(y,l(R))}\prod_{i=1}^m|b_i(y)-m_R(b_i)||f(y)|d\mu(y)\\
\le& C\Big\{\frac{1}{\lambda(6^Nl(Q))}\int_{6^NQ}\prod_{i=1}^m|b_i(y)-m_R(b_i)|^{r'}d\mu(y)\Big\}^{1/r'}\\
&\quad\times\Big\{\frac{1}{\lambda(6^Nl(Q))}\int_{6^NQ}|f(y)|^rd\mu(y)\Big\}^{1/r}\\
\le& C\|\vec{b}\|_*M_{r,(5)}f(x).
\end{aligned}
$$

Thus

$$I_4\le C\|\vec{b}\|_*M_{r,(5)}f(x).$$

Then (3.5) is proved by the estimate for I_1, I_2, I_3 and I_4.

Now we prove Theorem 3.1. By Lemma 3.11 in [4], one may assume that $b_i\in L^\infty(\mu)$ for $i\in\{1,2,\cdots,m\}$. Choosing r such that $1<r<p<\infty$, by Lemma 2.1, the boundedness of $M_{r,(\eta)}$ on $L^p(\mu)$ for $p>r$ and (3.3), we have

$$
\begin{aligned}
\|T_{\vec{b}}f\|_{L^p(\mu)}&\le C\|N(T_{\vec{b}}f)\|_{L^p(\mu)}\le C\|M^\sharp(T_{\vec{b}}f)\|_{L^p(\mu)}\\
&\le C\|\vec{b}\|_*\{\|M_{r,(6)}(Tf)\|_{L^p(\mu)}+\|M_{r,(5)}f\|_{L^p(\mu)}\}\\
&\qquad+C\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|_*\|M_{r,(6)}(T_{\vec{b}_{\sigma'}}f)\|_{L^p(\mu)}\\
&\le C\|\vec{b}\|_*\|f\|_{L^p(\mu)}.
\end{aligned}
$$

Thus the proof of Theorem 3.1 is finished. □

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