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# Norm Inequalities for Fractional Integral Operators on Generalized Weighted Morrey Spaces

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**Abstract.** Considering a class of operators which include fractional integrals related to operators with Gaussian kernel bounds, the fractional integral operators with rough kernels and fractional maximal operators with rough kernels as special cases, we prove that if these operators are bounded on weighted Lebesgue spaces and satisfy some local pointwise control, then these operators and the commutators of these operators with a BMO functions are also bounded on generalized weighted Morrey spaces.

**Key Words**: Fractional integral, rough kernel, Gaussian kernel bound, commutator, generalized weighted Morrey space.

**AMS Subject Classifications**: 42B20, 47G10, 42B35

## 1 Introduction

The classical Morrey spaces were introduced by Morrey [1] in 1938, since then a large number of investigations have been given to them by mathematicians. It is well-known that the classical Morrey spaces and the weighted Lebesgue spaces play important roles in the theory of partial differential equations.

Let $1\le p<\infty$ and $0\le\lambda\le n$, Morrey spaces are defined by

$$L^{p,\lambda}(\mathbb{R}^n)=\Big\{f\in L^p_{loc}(\mathbb{R}^n):\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}<\infty\Big\},$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0} r^{-\lambda/p}\|f\|_{L^p(B(x,r))}. \tag{1.1}$$

Note that $L^{p,0}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)=L^\infty(\mathbb{R}^n)$. If $\lambda<0$ or $\lambda>n$, then $L^{p,\lambda}(\mathbb{R}^n)=\Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

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Let $\Phi(r)$, $r>0$ be a growth function, that is, a positive increasing function in $(0,\infty)$ and satisfies doubling condition

$$\Phi(2r)\leq D\Phi(r) \quad \text{for all } \ r>0,$$

where $D=D(\Phi)\geq 1$ is a doubling constant independent of $r$. In [2], Mizuhara gave a generalization Morrey spaces $L^{p,\Phi}(\mathbb{R}^n)$ considering $\Phi(r)$ instead of $r^\lambda$ in (1.1). He studied also a continuity in $L^{p,\Phi}(\mathbb{R}^n)$ of some classical integral operators.

Komori and Shirai [3] introduced a version of the weighted Morrey space $L^{p,\kappa}(\omega,\mathbb{R}^n)$, which is a natural generalization of the weighted Lebesgue space $L^p(\omega,\mathbb{R}^n)$. Let $1\leq p<\infty$, $0<\kappa<1$ and $\omega$ be a weight function. The spaces $L^{p,\kappa}(\omega,\mathbb{R}^n)$ are defined by

$$L^{p,\kappa}(\omega,\mathbb{R}^n)=\{f\in L^p_{loc}(\omega):\|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}\left(\frac{1}{\omega(B(x,r))^\kappa}\int_{B(x,r)}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}.$$

In order to deal with the fractional order case, we need to consider the weighted Morrey space with two weights, it was also introduced by Komori and Shirai in [3].

Let $1\leq p<\infty$, $0<\kappa<1$. Then for two weights $u$, $v$, the weighted Morrey space is defined by

$$L^{p,\kappa}(u,v)(\mathbb{R}^n)=\{f\in L^p_{loc}(u):\|f\|_{L^{p,\kappa}(u,v)}<\infty\},$$

where

$$\|f\|_{L^{p,\kappa}(u,v)(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}\left(\frac{1}{v(B(x,r))^\kappa}\int_{B(x,r)}|f(y)|^p u(x)dy\right)^{\frac{1}{p}}.$$

Let $1\leq p<\infty$, $\varphi$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and $\omega$ be a non-negative measurable function on $\mathbb{R}^n$. We denote by $M^p_\varphi(\omega,\mathbb{R}^n)$ the generalized weighted Morrey space, the space of all functions $f\in L^p_{loc}(\omega,\mathbb{R}^n)$ with finite norm

$$\|f\|_{M^p_\varphi(w,\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left(\frac{1}{w(B(x,r))}\|f\|^p_{L^p(\omega,B(x,r))}\right)^{1/p},$$

If $\omega=1$ and $\varphi(x,r)=r^{\frac{\lambda-n}{p}}$ with $0\leq\lambda\leq n$, then $M^p_\varphi(\omega,\mathbb{R}^n)=L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space; If $\varphi(x,r)=\omega(B(x,r))^{\frac{\kappa-1}{p}}$, $0<\kappa<1$, then $M^p_\varphi(\omega,\mathbb{R}^n)=L^{p,\kappa}(\omega,\mathbb{R}^n)$ is the weighted Morrey space; If $\omega=1$, $\varphi(x,r)^p=\Phi(r)r^{-n}$, then $M^p_\varphi(\omega,\mathbb{R}^n)=L^{p,\Phi}(\mathbb{R}^n)$.

It has been proved by many authors (see [4–8]) that most of the operators which are bounded on a weighted (unweighted) Lebesgue space are also bounded in an appropriate weighted (unweighted) Morrey space. In this paper, we are going to prove that these results are valid on generalized weighted Morrey space. Our main results can be formulated as follows.

**Theorem 1.1.** *Let $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$, $\omega^{s'}\in A_{p/s',q/s'}$, and $T_\alpha$ be a sublinear operator which satisfies*

$$\sup_{x\in B(x_0,l)}\left|T_\alpha\left(f\chi_{(B(x_0,2l))^c}\right)(x)\right|\le C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{s'}+\alpha}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{1/s'} \tag{1.2}$$

*for any $x_0\in\mathbb{R}^n$ and $l>0$.*

*(i) Suppose $(\varphi_1,\varphi_2)$ satisfies the condition*

$$\int_l^\infty \frac{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x,t)(\omega^p(B(x,t)))^{\frac{1}{p}}}{(\omega^q(B(x,r)))^{\frac{1}{q}}}\frac{dr}{r}\le c_0\varphi_2(x,l), \tag{1.3}$$

*where $c_0$ does not depend on $x$ and $r$. If $T_\alpha$ is bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$, then $T_\alpha$ is also bounded from $M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)$ to $M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)$ and*

$$\|T_\alpha f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\le C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)};$$

*(ii) Suppose $(\varphi_1,\varphi_2)$ satisfies the condition*

$$\int_l^\infty \left(1+\ln\frac{r}{s}\right)\frac{\operatorname*{ess\,inf}_{r<t<\infty}\varphi_1(x,t)(\omega^p(B(x,t)))^{\frac{1}{p}}}{(\omega^q(B(x,r)))^{\frac{1}{q}}}\frac{dr}{r}\le c_0\varphi_2(x,l), \tag{1.4}$$

*where $c_0$ does not depend on $x_0$ and $l$. If $[b,T_\alpha]$ is bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$, then $[b,T_\alpha]$ is also bounded from $M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)$ to $M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)$ and*

$$\|[b,T_\alpha]f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\le C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.$$

**Example 1.1.** Let $\varphi_1(x,t)=(\omega^q(B(x,t)))^{\frac{\kappa}{p}}(\omega^p(B(x,t)))^{-\frac{1}{p}}$, $\varphi_2(x,t)=(\omega^q(B(x,t)))^{\frac{\kappa}{p}-\frac{1}{q}}$, $0<\kappa<p/q$ and $w^q\in A_\infty(\mathbb{R}^n)$. Then $(\varphi_1,\varphi_2)$ satisfies the condition (1.3) and (1.4).

**Example 1.2.** let $\varphi_1(x,t)=(\Phi(t)t^{-n})^{\frac{1}{p}}$, $\varphi_2(x,t)=(\Phi(t))^{\frac{1}{p}}t^{\frac{n}{q}}$, and let $\omega=1$. If $1\le D(\varphi)\le 2^n$, it is easy to prove that $(\varphi_1,\varphi_2)$ satisfies condition (1.3) and (1.4).

Then we have the following corollaries.

**Corollary 1.1.** Let $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$, $\omega^{s'}\in A_{p/s',q/s'}$, and $T_\alpha$ be a sublinear operator which satisfyies (1.2) for any $x_0\in\mathbb{R}^n$ and $l>0$.

(i) If $T_\alpha$ is bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$, then $T_\alpha$ is also bounded from $L^{p,\kappa}(\omega^p,\omega^q)(\mathbb{R}^n)$ to $L^{q,\kappa q/p}(\omega^q,\mathbb{R}^n)$ and

$$\|T_\alpha f\|_{L^{q,\kappa q/p}(\omega^q,\mathbb{R}^n)}\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)(\mathbb{R}^n)}.$$

(ii) If $[b,T_\alpha]$ is bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$, then $[b,T_\alpha]$ is also bounded from $L^{p,\kappa}(\omega^p,\omega^q)(\mathbb{R}^n)$ to $L^{p,\kappa}(\omega^p,\omega^q)(\mathbb{R}^n)$ and

$$\|[b,T_\alpha]f\|_{L^{q,\kappa q/p}(\omega^q,\mathbb{R}^n)}\le C\|b\|_*\|f\|_{L^{p,\kappa}(\omega^p,\omega^q)(\mathbb{R}^n)}.$$

**Corollary 1.2.** Let $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$, $\Phi(r)$ satisfy doubling condition and $1\le D(\Phi)\le 2^n$, $T_\alpha$ be a sublinear operator which satisfies (1.2) for any $x_0\in\mathbb{R}^n$ and $l>0$.

(i) If $T_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $T_\alpha$ is also bounded from $L^{p,\Phi}(\mathbb{R}^n)$ to $L^{q,\Phi^{q/p}}(\mathbb{R}^n)$ and

$$\|T_\alpha f\|_{L^{q,\Phi^{q/p}}(\mathbb{R}^n)}\le C\|f\|_{L^{p,\Phi}(\mathbb{R}^n)}.$$

(ii) If $[b,T_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $[b,T_\alpha]$ is also bounded from $L^{p,\Phi}(\mathbb{R}^n)$ to $L^{q,\Phi^{q/p}}(\mathbb{R}^n)$ and

$$\|[b,T_\alpha]f\|_{L^{q,\Phi^{q/p}}(\mathbb{R}^n)}\le C\|b\|_*\|f\|_{L^{p,\Phi}(\mathbb{R}^n)}.$$

## 2 Some preliminaries

We begin with some properties of $A_p$ weights which play a great role in the proofs of our main results.

A weight $\omega$ is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B=B(x_0,r_B)$ denote the ball with the center $x_0$ and radius $r_B$. For a given weight function $\omega$ and a measurable set $E$, we also denote the Lebesgue measure of $E$ by $|E|$ and set weighted measure $\omega(E)=\int_E\omega(x)dx$. For any given weight function $\omega$ on $\mathbb{R}^n$, $X\subseteq\mathbb{R}^n$ and $0<p<\infty$, denote by $L^p(\omega,X)$ the space of all function $f$ satisfying

$$\|f\|_{L^p(\omega,X)}=\left(\int_X|f(x)|^p\omega(x)dx\right)^{1/p}<\infty.$$

A weight $\omega$ is said to belong to $A_p$ for $1<p<\infty$, if there exists a constant

$$\left(\frac{1}{|B|}\int_B\omega(x)dx\right)\left(\frac{1}{|B|}\int_B\omega(x)^{1-p'}dx\right)^{p-1}\le C, \tag{2.1}$$

where $s'$ is the dual of $s$ such that $1/s+1/s'=1$. The class $A_1$ is defined by

$$\frac{1}{|B|}\int_B w(y)dy\le C\cdot\operatorname*{ess\,inf}_{x\in B}w(x)\quad\text{for every ball }\ B\subset\mathbb{R}^n. \tag{2.2}$$

A weight $\omega$ is said to belong to $A_\infty(\mathbb{R}^n)$ if there are positive numbers $C$ and $\delta$ so that

$$\frac{\omega(E)}{\omega(B)}\le C\left(\frac{|E|}{|B|}\right)^\delta$$

for all balls $B$ and all measurable $E\subset B$. It is well known that

$$A_\infty=\bigcup_{1\le p<\infty}A_p.$$

The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$-boundedness of Hardy-Littlewood maximal function in [9].

**Lemma 2.1** (see [9,10]). *Suppose $\omega\in A_p$ and the following statements hold.*

*(i) For any $1\le p<\infty$, there is a positive number $C$ such that*

$$\frac{\omega(B_k)}{\omega(B_j)}\le C2^{np(k-j)}\quad \textit{for}\;\; k>j. \tag{2.3}$$

*(ii) For any $1\le p<\infty$, there is a positive number $C$ and $\delta$ such that*

$$\frac{\omega(B_k)}{\omega(B_j)}\le C2^{\delta(k-j)}\quad \textit{for}\;\; k<j. \tag{2.4}$$

*(iii) For any $1<p<\infty$, one has $\omega^{1-p'}\in A_{p'}$.*

We also need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [11] to studied weighted boundedness of fractional integral operators.

Given $1\le p\le q<\infty$. We say that $\omega\in A_{p,q}$ if there exists a constant $C$ such that for every ball $B\subset\mathbb{R}^n$, the inequality

$$\left(\frac{1}{|B|}\int_B \omega(y)^{-p'}dy\right)^{1/p'}\left(\frac{1}{|B|}\int_B \omega(y)^{q}dy\right)^{1/q}\le C \tag{2.5}$$

holds when $1<p<\infty$, and for every ball $B\subset\mathbb{R}^n$ the inequality

$$\left(\frac{1}{|B|}\int_B \omega(y)^{q}dy\right)^{1/q}\le C\cdot \operatorname*{ess\,inf}_{x\in B} w(x) \tag{2.6}$$

holds when $p=1$.

By (2.5), we have

$$\left(\int_B \omega(y)^{-p'}dy\right)^{1/p'}\left(\int_B \omega(y)^{q}dy\right)^{1/q}\le C|B|^{1/p'+1/q}. \tag{2.7}$$

We summarize some properties about weights $A_{p,q}$ (see [11]).

**Lemma 2.2.** *Given $1\le p\le q<\infty$.*

*(i) $\omega\in A_{p,q}$ if and only if $\omega^q\in A_{1+q/p'}$;*

*(ii) $\omega\in A_{p,q}$ if and only if $\omega^{-p'}\in A_{1+p'/q}$.*

Following [12], a locally integrable function $b$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_{B\subset\mathbb{R}^n}\frac{1}{|B|}\int_B |b(x)-b_B|dx=\|b\|_*<\infty,$$

where

$$b_B=\frac{1}{|B|}\int_B b(y)dy.$$

There is a result concerning $BMO(\mathbb{R}^n)$ functions.

**Lemma 2.3** (see [8]). *Suppose $\omega\in A_\infty$ and $b\in BMO(\mathbb{R}^n)$. Then for any $1\le p<\infty$ and $r_1,r_2>0$, we have*

$$\left(\frac{1}{\omega(B(x_0,r_1))}\int_{B(x_0,r_1)}|b(x)-b_{B(x_0,r_2)}|^p\omega(x)dx\right)^{1/p}\le C\|b\|_*\left(1+\left|\ln\frac{r_1}{r_2}\right|\right). \tag{2.8}$$

**Lemma 2.4** (see [13]). *Let $f$ be a real-valued nonnegative function and measurable on $E$. Then*

$$\left(ess\inf_{x\in E}f(x)\right)^{-1}=ess\sup_{x\in E}\frac{1}{f(x)}. \tag{2.9}$$

# 3 Some auxiliary theorems

**Theorem 3.1.** *Suppose that $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $\omega^{s'}\in A_{p/s',q/s'}$. If $T_\alpha$ is bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$ and satisfies* (1.2)*, then for any $l>0$, there is a constant $C$ independent of $f$ such that*

$$\|T_\alpha(f)\|_{L^q(\omega^q,B(x_0,l))}\le C(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^\infty\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}. \tag{3.1}$$

*Proof.* We write $f$ as $f=f_1+f_2$, where $f_1(y)=f(y)\chi_{B(x_0,2l)}(y)$, $\chi_{B(x_0,2l)}$ denotes the characteristic function of $B(x_0,2l)$. Then

$$\|T_\alpha(f)\|_{L^q(\omega^q,B(x_0,l))}\le\|T_\alpha(f_1)\|_{L^q(\omega^q,B(x_0,l))}+\|T_\alpha(f_2)\|_{L^q(\omega^q,B(x_0,l))}.$$

Since $f_1\in L^p(\omega^p,\mathbb{R}^n)$, by the boundedness of $T_\alpha$ from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$ we get

$$\|T_\alpha(f_1)\|_{L^q(\omega^q,B(x_0,l))}\le\|T_\alpha(f_1)\|_{L^q(\omega^q,\mathbb{R}^n)}\le C\|f_1\|_{L^p(\omega^p,\mathbb{R}^n)}=C\|f\|_{L^p(\omega^p,B(x_0,2l))}.$$

By Hölder's inequality,

$$\begin{aligned}1&\le\left(\frac{1}{|B|}\int_B\omega(y)^pdy\right)^{\frac{1}{p}}\left(\frac{1}{|B|}\int_B\omega(y)^{-p'}dy\right)^{\frac{1}{p'}}\\&\le\left(\frac{1}{|B|}\int_B\omega(y)^qdy\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B\omega(y)^{-\frac{s'p}{p-s'}}dy\right)^{\frac{p-s'}{s'p}}.\end{aligned}$$

This means

$$l^{\frac{n}{s'}-\alpha}\le(\omega^q(B(x_0,l)))^{\frac{1}{q}}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,l))}.$$

Hence,

$$
\begin{aligned}
&\|T_\alpha(f_1)\|_{L^q(\omega^q,B(x_0,l))}\\
\le& Cl^{\frac{n}{s'}-\alpha}\|f\|_{L^p(\omega^p,B(x_0,2l))}\int_{2l}^{\infty}\frac{dr}{r^{\frac{n}{s'}-\alpha+1}}\\
\le& C(\omega^q(B(x_0,l))^{\frac{1}{q}}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,l))}\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\frac{dr}{r^{\frac{n}{s'}-\alpha+1}}\\
\le& C(\omega^q(B(x_0,l))^{\frac{1}{q}}\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))}\frac{dr}{r^{\frac{n}{s'}-\alpha+1}}.
\end{aligned}
$$

Since $\omega^{s'}\in A_{p/s',q/s'}$, by (2.7), for all $r>0$ we get

$$
(\omega^q(B(x_0,r)))^{\frac{1}{q}}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))}\le Cr^{\frac{n}{s'}-\alpha}. \tag{3.2}
$$

Therefore,

$$
\begin{aligned}
&\sup_{x\in B(x_0,l)}\|T_\alpha(f_1)\|_{L^q(\omega^q,B(x_0,l))}\\
\le& C(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.
\end{aligned} \tag{3.3}
$$

By (1.2),

$$
|T_\alpha(f_2)(x)|\le C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-\frac{n}{s'}}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}. \tag{3.4}
$$

Since $s'<p$, it follows from Hölder's inequality that

$$
\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}\le C\|f\|_{L^p(\omega^p,B(x_0,2^{j+1}l))}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,2^{j+1}l))}. \tag{3.5}
$$

Thus

$$
\begin{aligned}
&\sup_{x\in B(x_0,l)}|T_\alpha(f_2)(x)|\\
\le& C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-\frac{n}{s'}}\left(\int_{B(x_0,2^{j+1}s)}|f(y)|^{q'}dy\right)^{\frac{1}{q'}}\\
\le& C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-\frac{n}{s'}}\|f\|_{L^p(\omega^p,B(x_0,2^{j+1}l))}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,2^{j+1}l))}\\
\le& C\sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}(2^{j+1}l)^{\alpha-\frac{n}{s'}-1}\|f\|_{L^p(\omega^p,B(x_0,r))}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))}dr\\
\le& C\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))}\frac{dr}{r^{1-\alpha+n/s'}}.
\end{aligned}
$$

From (3.2) we know,

$$\|\omega^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))} \le Cr^{\frac{n}{s'}-\alpha}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}. \tag{3.6}$$

Hence,

$$\sup_{x\in B(x_0,l)} |T_\alpha(f_2)(x)| \le C\int_{2l}^{\infty} \|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}. \tag{3.7}$$

Therefore,

$$\|T_\alpha(f_2)\|_{L^q(\omega^q,B(x_0,l))} \le C(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^{\infty} \|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}. \tag{3.8}$$

Combining (3.3) and (3.8), (3.1) is proved. □

**Theorem 3.2.** *Suppose that $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $\omega^{s'}\in A_{p/s',q/s'}$. If $T_\alpha$ satisfies* (1.2) *and $[b,T_\alpha]$ bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$, then for any $l>0$, there is a constant $C$ independent of $f$ such that*

$$\begin{aligned}&\|[b,T_\alpha](f)\|_{L^q(\omega^q,B(x_0,l))}\\ \le& C(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^{\infty}\Big(1+\ln\frac{r}{l}\Big)\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.\end{aligned} \tag{3.9}$$

*Proof.* We represent $f$ as

$$f(y)=f_1+f_2,\quad f_1(y)=f(y)\chi_{B(x_0,2l)}(y),\quad f_2(y)=f(y)\chi_{(B(x_0,2l))^c}(y).$$

Then

$$\|[b,T_\alpha](f)\|_{L^q(\omega^q,B(x_0,l))} \le \|[b,T_\alpha](f_1)\|_{L^q(\omega^q,B(x_0,l))} + \|[b,T_\alpha](f_2)\|_{L^q(\omega^q,B(x_0,l))}.$$

By $f_1\in L^p(\omega^p,\mathbb{R}^n)$ and the boundedness of $[b,T_\alpha]$ from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$it follows that

$$\|[b,T_\alpha](f_1)\|_{L^q(\omega^q,B(x_0,l))} \le C\|b\|_*\|f_1\|_{L^p(\omega^p,\mathbb{R}^n)} = C\|b\|_*\|f\|_{L^p(\omega^p,B(x_0,2l))}.$$

Hence,

$$\begin{aligned}&\|[b,T_\alpha](f_1)\|_{L^q(\omega^q,B(x_0,l))}\\ \le& C\|b\|_*(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.\end{aligned} \tag{3.10}$$

We now turn to deal with the term $\|[b,T_\alpha](f_2)\|_{L^q(\omega^q,B(x_0,l))}$. For any given $x\in B(x_0,l)$, we have

$$|[b,T_\alpha](f_2)(x)| \le C|b(x)-b_{B(x_0,l)}||T_\alpha(f_2)(x)| + C|T_\alpha((b-b_{B(x_0,l)})f_2)(x)| = I_1+I_2.$$

From (3.7) we get,

$$I_1 \leq C|b(y)-b_{B(x_0,s)}|\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.$$

By $\omega^{s'}\in A_{p/s',q/s'}$ and Lemma 2.2 we know $\omega^q\in A_\infty$. Then it follows Lemma 2.3 that

$$\|I_1\|_{L^q(\omega^q,B(x_0,l))}\leq C\|b\|_*(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}. \tag{3.11}$$

On the other hand, by (1.2), we have

$$I_2\leq C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-\frac{n}{s'}}\left(\int_{B(x_0,2^{j+1}l)}|(b(y)-b_{B(x_0,l)})f(y)|^{s'}dy\right)^{\frac{1}{s'}}.$$

Applying Hölder's inequality we get

$$\begin{aligned}&\left(\int_{B(x_0,2^{j+1}l)}|b(y)-b_{B(x_0,l)}|^{s'}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}\\ \leq& C\|f\|_{L^p(\omega^p,B(x_0,2^{j+1}l))}\left\|(b(\cdot)-b_{B(x_0,l)})\omega(\cdot)^{-1}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,2^{j+1}l))}.\end{aligned}$$

Consequently,

$$\begin{aligned}I_2\leq&\sum_{j=1}^{\infty}\int_{2^{j+1}l}^{2^{j+2}l}(2^{j+1}l)^{\alpha-\frac{n}{s'}}\|f\|_{L^p(\omega^p,B(x_0,r))}\left\|(b(\cdot)-b_{B(x_0,l)})\omega(\cdot)^{-1}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,r))}dr\\ \leq& C\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}\left\|(b(\cdot)-b_{B(x_0,l)})\omega(\cdot)^{-1}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,r))}\frac{dr}{r^{1-\alpha+n/s'}}.\end{aligned}$$

From $\omega^{s'}\in A_{p/s',q/s'}$ and (ii) of Lemma 2.2, we know $\omega^{-\frac{s'p}{p-s'}}\in A_{1+\frac{ps'}{(p-s')q}}$. Then it follows from (2.8) and (3.6) that

$$\begin{aligned}&\left\|(b(\cdot)-b_{B(x_0,l)})\omega(\cdot)^{-1}\right\|_{L^{\frac{ps'}{p-s'}}(B(x_0,r))}\\ \leq&\left(\int_{B(x_0,r)}|b(y)-b_{B(x_0,l)}|^{\frac{ps'}{p-s'}}\omega^{-\frac{ps'}{p-s'}}(y)dy\right)^{\frac{p-s'}{ps'}}\\ \leq& C\|b\|_*\left(1+\ln\frac{r}{s}\right)(\omega^{-\frac{ps'}{p-s'}}(B(x_0,r)))^{\frac{p-s'}{ps'}}\\ =&C\|b\|_*\left(1+\ln\frac{r}{s}\right)\|\omega^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,r))}\\ \leq& C\|b\|_*\left(1+\ln\frac{r}{s}\right)r^{\frac{n}{s'}-\alpha}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}.\end{aligned}$$

Then,

$$I_2 \le C\|b\|_* \int_{2l}^{\infty} \Big(1+\ln\frac{r}{l}\Big)\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.$$

Hence,

$$\begin{aligned}&\|I_2|\|_{L^q(\omega^q,B(x_0,s))}\\ \le& C(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^{\infty}\Big(1+\ln\frac{r}{l}\Big)\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) we get

$$\begin{aligned}&\|[b,T_\alpha](f_2)\|_{L^p(\omega,B(x_0,l))}\\ \le& C\|b\|_*(\omega^q(B(x_0,l)))^{\frac{1}{q}}\int_{2l}^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.\end{aligned}$$

This completes the proof of Theorem 3.2. □

# 4 Proof of Theorem 1.1

*Proof* of Theorem 1.1. Since $f\in M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)$, then from (2.9) and the fact $\|f\|_{L^p(\omega^p,B(x_0,r))}$ is a non-decreasing function of $r$, we get

$$\begin{aligned}&\frac{\|f\|_{L^p(\omega^p,B(x_0,r))}}{\operatorname*{essinf}_{0<r<t<\infty}\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}\\ \le& \operatorname*{esssup}_{0<r<t<\infty}\frac{\|f\|_{L^p(\omega^p,B(x_0,r))}}{\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}\\ \le& \sup_{t>0,x_0\in\mathbb{R}^n}\frac{\|f\|_{L^p(\omega^p,B(x_0,t))}}{\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}\\ \le&\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.\end{aligned}$$

Since $(\varphi_1,\varphi_2)$ satisfies (1.3), we have

$$\begin{aligned}&\int_l^{\infty}\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}\\ \le&\int_l^{\infty}\frac{\|f\|_{L^p(\omega^p,B(x_0,r))}}{\operatorname*{essinf}_{r<t<\infty}\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}\frac{\operatorname*{essinf}_{r<t<\infty}\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}{(\omega^q(B(x_0,r)))^{\frac{1}{q}}}\frac{dr}{r}\\ \le& C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}\int_l^{\infty}\frac{\operatorname*{essinf}_{r<t<\infty}\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}{(\omega^q(B(x_0,r)))^{\frac{1}{q}}}\frac{dr}{r}\\ \le& C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}\varphi_2(x_0,l).\end{aligned}$$

Then by (3.1), we get

$$
\begin{aligned}
&\|T_\alpha(f)\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\\
\le& C\sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2(x_0,l)}\Big(\frac{1}{\omega^q(B(x_0,l)}\int_{B(x_0,l)}|T_\alpha(f)(y)|^q\omega^q(y)dy\Big)^{1/q}\\
\le& C\sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2(x_0,l)}\int_l^\infty\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}\\
\le& C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.
\end{aligned}
$$

If $(\varphi_1,\varphi_2)$ satisfies (1.4), and $f\in M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)$, we have

$$
\begin{aligned}
&\int_l^\infty\Big(1+\ln\frac{r}{l}\Big)\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}\\
\le&\int_l^\infty\frac{\|f\|_{L^p(\omega^p,B(x_0,r))}}{\underset{r<t<\infty}{\operatorname{essinf}}\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}\Big(1+\ln\frac{r}{l}\Big)\frac{\underset{r<t<\infty}{\operatorname{essinf}}\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}{(\omega^q(B(x_0,r)))^{\frac{1}{q}}}\frac{dr}{r}\\
\le& C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}\int_l^\infty\Big(1+\ln\frac{r}{l}\Big)\frac{\underset{r<t<\infty}{\operatorname{essinf}}\varphi_1(x_0,t)(\omega^p(B(x_0,t)))^{\frac{1}{p}}}{(\omega^q(B(x_0,r)))^{\frac{1}{q}}}\frac{dr}{r}\\
\le& C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}\varphi_2(x_0,l).
\end{aligned}
$$

Then by (3.9) we get

$$
\begin{aligned}
&\|[b,T_\alpha](f)\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\\
\le& C\sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2(x_0,l)}\Big(\frac{1}{\omega^q(B(x_0,l)}\int_{B(x_0,l)}|[b,T_\alpha](f)(y)|^q\omega^q(y)dy\Big)^{1/q}\\
\le& C\sup_{x_0\in\mathbb{R}^n,l>0}\frac{1}{\varphi_2(x_0,l)}\int_l^\infty\Big(1+\ln\frac{r}{l}\Big)\|f\|_{L^p(\omega^p,B(x_0,r))}(\omega^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}\\
\le& C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.
\end{aligned}
$$

This completes the proof of Theorem 1.1. □

# 5 Some applications

In this section, we shall apply Theorem 1.1 to several particular operators such as the fractional integrals associated to operator with Gaussian kernel bounds, the fractional integral operators with rough kernels and the fractional maximal operators with rough kernels.

### 5.1 Fractional integrals related to operators with Gaussian kernel bounds

Let $L$ be a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup $e^{-tL}$ with a kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \leq \frac{C}{t^{n/2}} e^{-c\frac{|x-y|^2}{t}}, \tag{5.1}$$

for $x,y \in \mathbb{R}^n$ and all $t>0$. This property is satisfied by a large class of differential operators [14]. For $0<\alpha<n$, the fractional power $L^{-\alpha/2}$ of the operator $L$ is defined by

$$L^{-\alpha/2}(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f)(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L=-\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential $I_\alpha$. Let $b$ be a locally integrable function on $\mathbb{R}^n$, the commutator of $b$ and $L^{-\alpha/2}$ is defined by

$$[b,L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x),$$

When $b \in BMO(\mathbb{R}^n)$, Duong and Yan [16] proved that the commutator $[b,L^{-\alpha/2}]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $0<\alpha<n$, $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Auscher and Martell [15] were concerned with the weighted estimates of $L^{-\alpha/2}$ and its commutator. They showed that if $\omega \in A_{p,q}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$, then both the operator $L^{-\alpha/2}$ and the commutator $[b,L^{-\alpha/2}]$ are bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$ for $0<\alpha<n$ and for $1<p<n/\alpha$, $1/q=1/p-\alpha/n$. More results about weighted estimates for $[b,L^{-\alpha/2}]$ can be found in [16,17].

**Theorem 5.1.** *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $\omega \in A_{p,q}$ and $b \in BMO(\mathbb{R}^n)$. If $(\varphi_1,\varphi_2)$ satisfies the condition* (1.3)*, then there is a constant $C>0$ independent of $f$ such that*

$$\|L^{-\alpha/2}f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)} \leq C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.$$

*If $(\varphi_1,\varphi_2)$ satisfies the condition* (1.4)*, then there is a constant $C>0$ independent of $f$ such that*

$$\|[b,L^{-\alpha/2}]f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)} \leq C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.$$

*Proof.* We only need to prove $L^{-\alpha/2}$ satisfies (1.2). Denote the kernel of $L^{-\alpha/2}$ by $K_\alpha(x,y)$, then for any $x \in B(x_0,l)$, we write

$$L^{-\alpha/2}\left(f\chi_{(B(x_0,2l))^c}\right)(x) = \int_{(B(x_0,2l))^c} K_\alpha(x,y)f(y)dy.$$

Since the kernel of $e^{-tL}$ is $p_t(x,y)$, it follows immediately from (5.2) that

$$K_\alpha(x,y) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p_t(x,y) \frac{dt}{t^{-\alpha/2+1}}. \tag{5.2}$$

Thus, by using the Gaussian upper bound (5.1) and the expression (5.2), we can deduce

$$\begin{aligned}|K_\alpha(x,y)| &\le C\int_0^\infty |p_t(x,y)|t^{\alpha/2-1}dt\\ &\le C\int_0^\infty e^{-c\frac{|x-y|^2}{t}}t^{\alpha/2-n/2-1}dt \le \frac{C}{|x-y|^{n-\alpha}}.\end{aligned}$$

Then

$$\begin{aligned}|L^{-\alpha/2}\Big(f\chi_{(B(x_0,2l))^c}\Big)(x)| &\le \int_{(B(x_0,2l))^c}|K_\alpha(x,y)||f(y)|dy\\ &\le C\sum_{j=1}^\infty\int_{(B(x_0,2^{j+1}l))\setminus B(x_0,2^jl)}\frac{1}{|x-y|^{n-\alpha}}|f(y)|dy\\ &\le C\sum_{j=1}^\infty(2^{j+1}l)^{-n+\alpha}\int_{B(x_0,2^{j+1}l)}|f(y)|dy.\end{aligned}$$

So, we complete the proof. □

### 5.2 Fractional integrals operators with rough kernels

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n (n\ge 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega\in L^s(\mathbb{S}^{n-1})$ with $1<s\le\infty$ be homogeneous of degree zero. For any $0<\alpha<n$, then the fractional integral operator with rough kernel $\Omega$ is defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(y)}{|y|^{n-\alpha}}f(x-y)dy.$$

Let $b$ be a locally integrable function on $\mathbb{R}^n$, then for $0<\alpha<n$, the commutators generated by fractional integral operators with rough kernels and $b$ is defined by

$$[b,T_{\Omega,\alpha}]f(x)=b(x)T_{\Omega,\alpha}f(x)-T_{\Omega,\alpha}(bf)(x).$$

Ding and Lu [18,19] considered the weighted norm inequalities for $T_{\Omega,\alpha}$ and $[b,T_{\Omega,\alpha}]$ with general weights, they proved the following results:

Let $0<\alpha<n$, $1\le s'<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Suppose that $\Omega\in L^s(\mathbb{S}^{n-1})$, $b\in BMO(\mathbb{R}^n)$ and $\omega^{s'}\in A_{p/s',q/s'}$. Then the both $T_{\Omega,\alpha}$ and $[b,T_{\Omega,\alpha}]$ are bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$.

**Theorem 5.2.** *Let $0<\alpha<n$, $1\le s'<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Suppose that $\Omega\in L^s(\mathbb{S}^{n-1})$, $\omega^{s'}\in A_{p/s',q/s'}$. If $(\varphi_1,\varphi_2)$ satisfies the condition* (1.3), *then there is a constant $C>0$ independent of $f$ such that*

$$\|T_{\Omega,\alpha}f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\le C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.$$

*If $(\varphi_1,\varphi_2)$ satisfies the condition* (1.4) *and $b\in BMO(\mathbb{R}^n)$, then there is a constant $C>0$ independent of $f$ such that*

$$\|[b,T_{\Omega,\alpha}]f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\le C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.$$

*Proof.* We first deduce from Hölder's inequality that

$$\begin{aligned}&|T_{\Omega,\alpha}(f\chi_{(B(x_0,2l))^c}(x)|\le\left|\int_{B(x_0,2l)^c}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy\right|\\ \le&\sum_{j=1}^{\infty}\Big(\int_{B(x_0,2^{j+1}l)\backslash B(x_0,2^jl)}|\Omega(x-y)|^sdy\Big)^{\frac{1}{s}}\Big(\int_{B(x_0,2^{j+1}l)\backslash B(x_0,2^jl)}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\Big)^{\frac{1}{s'}}.\end{aligned}$$

When $x\in B(x_0,l)$ and $y\in B(x_0,2^{j+1}l)\backslash B(x_0,2^jl)$, we can easily see that $2^{j-1}l\le|y-x|<2^{j+1}l$. Then, by a simple computation, we deduce

$$\Big(\int_{B(x_0,2^{j+1}l)\backslash B(x_0,2^jl)}|\Omega(x-y)|^sdy\Big)^{\frac{1}{s}}\le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}|B(x_0,2^{j+1}l)|^{\frac{1}{s}}. \tag{5.3}$$

We also note that if $x\in B(x_0,l),y\in B(x_0,2l)^c$, then $|y-x|\sim|y-x_0|$. Consequently

$$\Big(\int_{B(x_0,2^{j+1}l)\backslash B(x_0,2^jl)}\frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\Big)^{\frac{1}{s'}}\le\frac{1}{|B(x_0,2^{j+1}l)|^{1-\alpha/n}}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}.$$

Then

$$\begin{aligned}&\sup_{x\in B(x_0,l)}|T_{\Omega,\alpha}(f\chi_{(B(x_0,2l))^c}(x)|\\ \le&C\sum_{j=1}^{\infty}\frac{1}{|B(x_0,2^{j+1}l)|^{1-\alpha/n-1/s}}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}\\ =&C\sum_{j=1}^{\infty}(2^{j+1}l)^{-\frac{n}{s'}+\alpha}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}dy\right)^{1/s'}.\end{aligned}$$

Thus, we complete the proof. ◻

### 5.3 Fractional maximal operators with rough kernel

Suppose that $0<\alpha<n$, $\Omega$ is homogeneous of degree zero, and $\Omega\in L^s(\mathbb{S}^{n-1})$, where $\mathbb{S}^{n-1}$ denotes the sphere of $\mathbb{R}^n$ and $s>1$. We consider the fractional maximal operator with rough kernel $\Omega$ defined by

$$M_{\Omega,\alpha}f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)||f(y)|dy.$$

Let $b$ be a locally integrable function on $\mathbb{R}^n$, the commutator generated by fractional maximal operator with rough kernel and $b$ is defined by

$$[b,M_{\Omega,\alpha}]f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)||b(x)-b(y)||f(y)|dy.$$

Ding and Lu [18,19] proved the following results:

Let $0<\alpha<n$, $1\le s'<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Suppose that $\Omega\in L^s(\mathbb{S}^{n-1}),b\in BMO(\mathbb{R}^n)$ and $\omega^{s'}\in A_{p/s',q/s'}$. Then both $M_{\Omega,\alpha}$ and $[b,M_{\Omega,\alpha}]$ are bounded from $L^p(\omega^p,\mathbb{R}^n)$ to $L^q(\omega^q,\mathbb{R}^n)$.

**Theorem 5.3.** *Let $0<\alpha<n$, $1\le s'<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Suppose that $\Omega\in L^s(\mathbb{S}^{n-1})$, $\omega^{s'}\in A_{p/s',q/s'}$. If $(\varphi_1,\varphi_2)$ satisfy the condition* (1.3), *then there is a constant $C>0$ independent of $f$ such that*

$$\|M_{\Omega,\alpha}f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\le C\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.$$

*If $(\varphi_1,\varphi_2)$ satisfy the condition* (1.4) *and $b\in BMO(\mathbb{R}^n)$, then there is a constant $C>0$ independent of $f$ such that*

$$\|[b,M_{\Omega,\alpha}]f\|_{M^q_{\varphi_2}(\omega^q,\mathbb{R}^n)}\le C\|b\|_*\|f\|_{M^p_{\varphi_1}(\omega^p,\mathbb{R}^n)}.$$

*Proof.* Note that, if

$$\{y:|x-y|<r\}\cap(B(x_0,2^{j+1}l)\setminus B(x_0,2^jl))\neq\phi,$$

then

$$r>|x-y|\ge|y-x_0|-|x-x_0|\ge 2^{j+1}l-l\ge C2^{j+1}l.$$

Hence

$$r^{\alpha-n}\le C(2^{j+1}l)^{\alpha-n}.$$

Therefore, for any $r>0$,

$$\begin{aligned}&r^{\alpha-n}\int_{\{y:|x-y|<r\}\cap(B(x_0,2l))^c}|\Omega(x-y)||f(y)|dy\\ \le&\sum_{j=1}^{\infty}r^{\alpha-n}\int_{\{y:|x-y|<r\}\cap(B(x_0,2^{j+1}l)\setminus B(x_0,2^jl))}|\Omega(x-y)||f(y)|dy\\ \le&C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-n}\int_{(B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)}|\Omega(x-y)||f(y)|dy.\end{aligned}$$

By Hölder's inequality, the above expression is majorized by

$$C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-n}\Big(\int_{B(x_0,2^{j+1}l)\setminus B(x_0,2^jl)}|\Omega(x-y)|^sdy\Big)^{\frac{1}{s}}\Big(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}\Big)^{1/s'}.$$

Applying (5.3) we get

$$\begin{aligned}&r^{\alpha-n}\int_{\{y:|x-y|<r\}\cap(B(x_0,2l))^c}|\Omega(x-y)||f(y)|dy\\ \leq& C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-n/s'}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}\right)^{1/s'}\end{aligned}$$

for any $r>0$. This means

$$M_{\Omega,\alpha}(f\chi_{(B(x_0,2l))^c}(x)\leq C\sum_{j=1}^{\infty}(2^{j+1}l)^{\alpha-n/s'}\left(\int_{B(x_0,2^{j+1}l)}|f(y)|^{s'}\right)^{1/s'}$$

holds for any $x_0\in\mathbb{R}^n$ and $l>0$. □

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