Weighted Pseudo Almost Periodic Solutions for a Class of Hematopoiesis Model with Time-Varying Delay

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Abstract. In this paper, firstly, a notion of a class of generalized weighted pseudo almost periodic function is introduced, then we investigate some basic and essential properties of the space that consists of these functions. Finally, we study the existence of weighted pseudo almost periodic solutions to hematopoiesis model with time-varying delay.

Key Words: Weighted pseudo almost periodic, hematopoiesis model, time-varying delay.

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1 Introduction

The investigation of almost periodic and pseudo almost periodic differential equations is one of the most interesting topics for many mathematicians [1–8]. In [13], Diagana introduced the concept of weighted pseudo almost periodic functions, which is a natural generalization of the concept of pseudo almost periodic functions. Since then, some interesting and important results concerning with composition theorem, translation invariance and the ergodicity of weighted pseudo almost periodic were obtained [9–14].

In recent years, the hematopoiesis models have been attracting more attentions because of their extensively realistic significance, we refer the reader to [15–20] on this topic. To describe the dynamics of hematopoiesis, Mackey and Glass [15] proposed the following delay differential equation:

\[
h'(t) = -ah(t) + \frac{\beta}{1 + h^n(t-\tau)}, \quad (1.1)
\]

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which is the process of production of all types of blood cells generated by a remarkable self-regulated system. About the details of Eq. (1.1), we can refer [15, 16, 18]. There are some contributions on almost periodic hematopoiesis model [21–24].

Ding et al. [24] considered the following discrete hematopoiesis model:

$$\Delta u(n) = u(n+1) - u(n) = -\alpha(n)u(n) + \frac{\beta(n)}{1 + u(n-\tau)}$$

(1.2)

the authors established the existence of weighted pseudo almost periodic solutions to Eq. (1.2) by a fixed point theorem.

To our best knowledge, there are few investigations of weighted pseudo almost periodic hematopoiesis model with variable time delay. Motivated by the above discussions, the main aim of this paper is to discuss the existence of weighted pseudo almost periodic solutions for the following model of hematopoiesis with time-varying delay:

$$x'(t) = -\alpha(t)x(t) + \frac{\beta(t)}{1 + x(t-\tau(t))}, \quad n > 1.$$  

(1.3)

Under proper assumptions, we will obtain a unique positive weighted pseudo almost periodic solution of Eq. (1.3) by using a fixed point theorem. Let us recall some notions about normal cone and a fixed point theorem.

Let $E$ be a real Banach space and $P$ be a cone in $E$. The semi-order induced by the cone $P$ is denoted by $\leq$. That is, $x \leq y$ if and only if $y-x \in P$ for any $x, y \in P$.

A cone $P$ of $E$ is said to be normal if there exists a positive constant $\delta$ such that $\|x+y\| \geq \delta$ for any $x, y \in P, \|x\| = \|y\| = 1$. In order to obtain the main results, we will state the following fixed point theorem in a cone.

**Theorem 1.1** (see [25, 26]). Let $P$ be a normal cone in a real Banach space $X$, and $u, v \in P$ with $u \leq v$. Suppose that $A: [u, v] \to X$ satisfies:

(a) $A: [u, v] \to X$ is decreasing, i.e., $Ax_1 \leq Ax_2$ for all $x_1, x_2 \in [u, v]$ with $x_1 \geq x_2$;

(b) $A: [u, v] \to X$ is a concave operator, i.e., $A[\alpha x + (1-\alpha)y] \geq \alpha Ax + (1-\alpha)Ay$ for all $x, y \in [u, v]$ and $\alpha \in [0,1]$;

(c) $Av \geq u, Au \leq \frac{1}{2}(v + Av)$. Then there exists a unique point $x^* \in [u, v]$ such that $Ax^* = x^*$.

The rest of the paper is organized as follows. In Section 2, we shall introduce some basic definitions about weighted pseudo almost periodic functions of class $h$, notations and lemmas. Then we will study essential properties of these functions. In Section 3, some conditions for the existence of weighted pseudo almost periodic solutions of Eq. (1.3) are established.

2 Preliminaries

Now let us state the following definitions, notations and lemmas, which will be applied to prove our main results. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ is the set of nonnegative
real numbers, $X$ is a real Banach space with the usual norm $\| \cdot \|$, $BC(\mathbb{R}, X)$ is the set of function $f: \mathbb{R} \rightarrow X$ which is bounded and continuous.

**Definition 2.1** (see [5]). A function $f: \mathbb{R} \rightarrow X$ is said to be almost periodic, if for any $\varepsilon > 0$, there is a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$ there exists $\tau$ such that the inequality

$$\|f(t+\tau) - f(t)\| < \varepsilon$$

is satisfied for all $t \in \mathbb{R}$. The number $\tau$ is called an $\varepsilon -$ translation number of $f(t)$. Denote by $AP(\mathbb{R})$ the collections of such functions.

Let $U$ be the collections of weights $\rho: \mathbb{R} \rightarrow \mathbb{R}^+$, which are locally integrable over $\mathbb{R}$ with $\rho > 0$. For a given $r > 0$ and each $\rho \in U$, set

$$m(r, \rho) := \int_{-r}^{r} \rho(t) d(t),$$

$$U_\infty := \{ \rho \in U : \lim_{r \rightarrow \infty} m(r, p) = \infty \},$$

$$U_B := \{ \rho \in U_\infty : \rho \text{ is bounded and } \inf_{x \in \mathbb{R}} \rho(x) > 0 \}.$$

For given $\rho \in U_\infty$ and $h > 0$, we define a new function space:

$$WPAP_0(\mathbb{X}, h, \rho) = \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r \rightarrow \infty} \frac{1}{m(r, p)} \int_{-r}^{r} \left( \sup_{\theta \in [t-h, t]} \|f(\theta)\| \right) \rho(t) dt = 0 \right\}.$$

**Definition 2.2.** For $\rho \in U_\infty$, a function $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be weighted pseudo almost periodic function of class $h$ if it can be expressed as $f = g + \varphi$, where $g \in AP(\mathbb{R})$ and $\varphi \in WPAP_0(\mathbb{X}, h, \rho)$. Denote by $WPAP(\mathbb{R}, h)$ the set of such functions.

Next, let us discuss some basic properties of weighted pseudo almost periodic function of class $h$. By a standard proof [15], the following results are easy to obtain, we omit them here.

**Lemma 2.1** (see [15]). If $f \in WPAP(\mathbb{R}, h)$, the composition $f = g + \varphi$ is unique, where $g \in AP(\mathbb{R}), \varphi \in WPAP_0(\mathbb{X}, h, \rho)$.

**Lemma 2.2** (see [15]). Let $\rho \in U_\infty$. Then the following properties hold:

(a) $f \pm g \in WPAP(\mathbb{R}, h)$ and $f \cdot g \in WPAP(\mathbb{R}, h)$ if $f, g \in WPAP(\mathbb{R}, h)$.

(b) $f / g \in WPAP(\mathbb{R}, h)$ if $f, g \in WPAP(\mathbb{R}, h)$ and $\inf_{t \in \mathbb{R}} |g(t)| > 0$.

(c) $f \circ g \in WPAP(\mathbb{R}, h)$ if $g \in WPAP(\mathbb{R}, h)$ and $f : [\inf_{t \in \mathbb{R}} |g(t)|, \sup_{t \in \mathbb{R}} |g(t)|] \rightarrow \mathbb{R}$ is continuous.

(d) $WPAP(\mathbb{R}, h)$ is Banach space with the supremum norm $\| f \| := \sup_{t \in \mathbb{R}} \| f(t) \|$.
Lemma 2.3 (see [15]). Let \( \rho \in U_{\infty} \). Assume that the following conditions are satisfied:
(i) the function \( u : U \to U \) has second order derivative, for any \( t \in U \), there exists a constant \( 0 < \lambda < 1 \) such that \( 0 \leq u'(t) \leq \lambda \) and \( u''(t) \leq 0 \) hold,
(ii) for each \( t \in \mathbb{R} \), the following hold true:
\[
\limsup_{s \to \infty} \left[ \frac{\rho(s)}{\rho(s-u(s))} \right] < \infty
\]
and
\[
\limsup_{s \to \infty} \left[ \frac{m(r+|u(r)|+|u(-r)|, \rho)}{m(r, \rho)} \right] < \infty,
\]
then for any \( \phi \in \text{PAP}_0(X, h, \rho) \), we have \( t \to \phi(t-u(t)) \in \text{PAP}_0(X, h, \rho) \).

Proof. Since for any \( t \in \mathbb{R} \), \( 0 \leq u'(t) \leq 1 \) and \( u''(t) \leq 0 \), it is easy to know that \( t-u(t) \) and \( u(t) \) are increasing, \( u'(t) \) is decreasing.

For each \( \phi \in \text{PAP}_0(X, h, \rho) \), we have
\[
\frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-h, t]} \| \phi(\theta - u(\theta)) \| \right) \rho(t) dt
\]
\[
= \frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-u(t)-h, t-u(t)]} \| \phi(\theta) \| \right) \rho(t) dt
\]
\[
\leq \frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-u(t)-h, t-u(t)]} \| \phi(\theta) \| \right) \rho(t) dt
\]
\[
= \frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-u(t)-h, t-u(t)]} \| \phi(\theta) \| \right) \frac{\rho(t)}{\rho(t-u(t))} dt
\]
\[
\leq \frac{1}{1-u'(-r)} \frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-h, t]} \| \phi(\theta) \| \right) \rho(t) dt
\]
\[
\leq \frac{1}{1-u'(-r)} \frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-h, t]} \| \phi(\theta) \| \right) \rho(t) dt
\]
\[
\leq \frac{m(r+|u(r)|+|u(-r)|, \rho)}{m(r, \rho)} \frac{1}{1-u'(-r)} \frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-h, t]} \| \phi(\theta) \| \right) \rho(t) dt,
\]
where \( \xi_u \in (-r-|u(-r)|-u(r), r+|u(-r)|+|u(r)|) \).

From \( \phi \in \text{PAP}_0(X, h, \rho) \), we obtain that
\[
\lim_{r \to \infty} \frac{1}{m(r+|u(-r)|+|u(r)|, \rho)} \int_{-r}^{r} \left( \sup_{\theta \in [t-h, t]} \| \phi(\theta) \| \right) \rho(t) dt = 0.
\]
Thus, combining with (ii), we get
\[ \lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \left( \sup_{t \in [-r, r]} \| \phi(t - u(t)) \| \right) \rho(t) dt = 0, \]
which implies that
\[ \phi(t - u(t)) \in \text{PAP}_0(\mathfrak{X}, h, \rho), \quad t \in \mathbb{R}. \]
The proof is completed. \qed

3 Main results

In this section, we mainly discuss the existence of weighted pseudo almost periodic solutions to Eq. (1.3). For convenience, we first list the following notations:
\[ h^+ = \sup_{t \in \mathbb{R}} h(t), \quad h^- = \inf_{t \in \mathbb{R}} h(t). \]

Theorem 3.1. Assume that \( \rho \in \mathcal{U}_\infty \) satisfies the assumptions of Lemma 2.3 and the following conditions hold:

(H1) \( \alpha \in \text{AP}(\mathbb{R}) \) with \( \alpha^- > 0 \),

(H2) \( \beta, \tau \in \text{WPAP}(\mathbb{R}, h) \) with \( 0 < \beta^- < \frac{n+1}{n+2} \sqrt{\frac{n+1}{n+2}}, \)

(H3) \( \tau: \mathbb{R} \to \mathbb{R} \) has second order derivative \( \tau''(t) \leq 0 \), there exists a constant \( 0 < \lambda < 1 \) such that \( 0 \leq u'(t) \leq \lambda \).

Then Eq. (1.3) has a positive weighted pseudo almost periodic solution.

Proof. Let
\[ P = \{ x \in \text{WPAP}(\mathbb{R}, h) : x(t) \geq 0, \ t \in \mathbb{R} \}. \]
Then, one can figure out that \( P \) is a normal cone in \( \text{WPAP}(\mathbb{R}, h) \).

For each \( \phi \in P \), consider the following auxiliary equation:
\[ x_\phi(t) = \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u) du} \frac{\beta(s)}{1 + \varphi^n(s - \tau(s))} ds. \] (3.1)

Let us define the operator \( \Psi \) by
\[ (\Psi \phi)(t) = \int_{-\infty}^{t} e^{-\int_{s}^{t} \alpha(u) du} \frac{\beta(s)}{1 + \varphi^n(s - \tau(s))} ds. \] (3.2)

By (3.1), it is not difficult to find that Eq. (1.3) has one unique positive weighted pseudo almost periodic solution if and only if the operator \( \Psi \) has exactly one fixed point.
We first show that \((\Psi \varphi) \in WPAP(\mathbb{R}, h)\), for any \(\varphi \in WPAP(\mathbb{R}, h)\). From (H2)-(H3), Lemma 2.2 and Lemma 2.3, for any \(t \in \mathbb{R}\), we can obtain
\[
\frac{\beta(s)}{1 + \varphi^\prime(s - \tau(s))} \in WPAP(\mathbb{R}, h).
\]
According to Lemma 2.1, we set
\[
\frac{\beta(s)}{1 + \varphi^\prime(s - \tau(s))} = \beta_1 + \beta_2, \quad t \in \mathbb{R},
\]
where \(\beta_1 \in AP(\mathbb{R})\) and \(\beta_2 \in PAP_0(\mathbb{X}, h, \rho)\). By [5], it follows that
\[
\int_{-\infty}^{t} e^{-\int_{\tau(s)}^{u-a(s)} \beta_1(u) du} ds \in AP(\mathbb{R}).
\]
It remains to verify that
\[
\int_{-\infty}^{t} e^{-\int_{\tau(s)}^{u-a(s)} \beta_2(u) du} ds \in PAP_0(\mathbb{X}, h, \rho).
\]
Actually, since
\[
\int_{-\infty}^{t} e^{-\int_{\tau(s)}^{u-a(s)} \beta_2(u) du} ds \leq \int_{0}^{+\infty} e^{-a-s} \beta_2(t-s) ds.
\]
Noting that
\[
\sup_{\theta \in [1-h, h]} \left( \int_{0}^{+\infty} e^{-a-s} \beta_2(\theta - s) ds \right) \leq \int_{0}^{+\infty} e^{-a-s} \sup_{\theta \in [1-h, h]} \beta_2(\theta - s) ds,
\]
by \(\beta_2 \in PAP_0(\mathbb{X}, h, \rho)\), we easily conclude that
\[
\int_{-\infty}^{t} e^{-\int_{\tau(s)}^{u-a(s)} \beta_2(u) du} ds \in PAP_0(\mathbb{X}, h, \rho).
\]
This proves that \((\Psi \varphi) \in WPAP(\mathbb{R}, h)\).

Now, by (H2), we can find a small enough positive constant \(\varepsilon\) such that
\[
\frac{\beta^-}{\alpha^+} \leq \varepsilon \quad \left( \frac{n+1}{2n} \right).
\]
and
\[
\frac{\alpha^-}{1 + \varepsilon^a} \leq \alpha^- \frac{n-1}{2} \sqrt{\frac{n-1}{n+1}}.
\]
Define the operator
\[
\Psi : [\varepsilon^\sqrt{\frac{n-1}{n+1}}] \to WPAP(\mathbb{R}, h)
\]
by
\[(\Psi \phi)(t) = \int_{-\infty}^{t} e^{-\int_{s}^{t} a(u)du} \frac{\beta(s)}{1 + \phi^\alpha(s - \tau(s))} ds, \quad t \in \mathbb{R}. \quad (3.5)\]

It is easy to see that \(\Psi: [\varepsilon, \frac{\sqrt{n-1}}{n+1}] \rightarrow WPAP(\mathbb{R}, h)\) is decreasing and concave. For any \(t \in \mathbb{R}\), by (3.3), we get
\[(\Psi \sqrt{\frac{n-1}{n+1}})(t) \geq \int_{-\infty}^{t} e^{-\alpha^+(t-s)} \frac{\beta^-}{1 + \frac{\sqrt{n-1}}{n+1}} = \frac{\beta^-}{\alpha^+} \cdot \frac{n+1}{2n} \geq \varepsilon\]
and by (3.4), we have
\[(\Psi \varepsilon)(t) \leq \int_{-\infty}^{t} e^{-\alpha^-(t-s)} \frac{\beta^+}{1 + \varepsilon^\alpha} ds = \frac{\beta^+}{\alpha^-} \cdot \frac{1}{\alpha^-} \leq \frac{\varepsilon}{\sqrt{n+1}} + \Psi \sqrt{\frac{n-1}{n+1}}.\]

Thus, the conditions of Theorem 1.1 are satisfied. Therefore there exists a fixed point \(\phi \in [\varepsilon, \frac{\sqrt{n-1}}{n+1}]\) such that \(\Psi \phi = \phi\), that is,
\[\phi(t) = \int_{-\infty}^{t} e^{-\int_{s}^{t} a(u)du} \frac{\beta(s)}{1 + \phi^\alpha(s - \tau(s))} ds, \quad t \in \mathbb{R},\]
which is just the unique weighted pseudo almost periodic solution. The proof is completed. \(\square\)

**Remark 3.1.** It is easy to find that the time delay of system (1.3) is variable delay, the results in the paper improve the known results of hematopoiesis model. And from the conditions of time delay \(\tau\), we can see that time delay also has key effect on the weighted pseudo almost periodic solution.

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