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Parameterized Littlewood-Paley Operators on Weighted Herz Spaces

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Abstract. The strong type and weak type estimates of parameterized Littlewood-Paley operators on the weighted Herz spaces $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ are considered. The boundedness of the commutators generated by BMO functions and parameterized Littlewood-Paley operators are also obtained.

Key Words: Parameterized Littlewood-Paley operator, Herz space, weak Herz space, BMO, commutator, Muckenhoupt weight.

AMS Subject Classifications: 42B20, 30H35, 42B35

1 Introduction

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n (n\geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega\in L^1(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}}\Omega(x')d\sigma(x')=0, \tag{1.1}$$

where $x'=x/|x|$ for any $x\neq 0$. Then for $0<\rho<n$, the area integral $\mu^{\rho}_{\Omega,S}$ and the Littlewood-Paley $\mu^{*,\rho}_{\lambda}$- function are defined respectively by

$$\mu^{\rho}_{\Omega,S}(f)(x)=\left(\iint_{\Gamma(x)}\left|\frac{1}{t^{\rho}}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

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and

$$\mu_\lambda^{*,\rho}(f)(x)=\left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\lambda>1$ and $\Gamma(x)=\{(y,t)\in\mathbb{R}_+^{n+1}:\ |x-y|<t\}$.

Now let us turn to the introductions of the corresponding commutators of the parameterized Littlewood-Paley operators above. Let $b\in L^1_{loc}(\mathbb{R}^n)$, $m\in\mathbb{N}$, the commutators $[b^m,\mu^\rho_{\Omega,S}]$ and $[b^m,\mu_\lambda^{*,\rho}]$ are defined respectively by

$$\begin{aligned}&[b^m,\mu^\rho_{\Omega,S}](f)(x)\\&=\left(\iint_{\Gamma(x)}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&[b^m,\mu_\lambda^{*,\rho}](f)(x)\\&=\left(\iint_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}.\end{aligned}$$

In 1990, Torchinsky and Wang [1] gave the weighted $L^2(\mathbb{R}^n)$ boundedness of $\mu^\rho_{\Omega,S}$ and $\mu_\lambda^{*,\rho}$ for $\rho=1$ and $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$ $(0<\alpha\le 1)$. Here, we say that $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$ if

$$|\Omega(x')-\Omega(y')|\le|x'-y'|^\alpha,\quad x',y'\in\mathbb{S}^{n-1}. \tag{1.2}$$

For general ρ, in 1999, Sakamoto and Yabuta [2] gave $L^p(\mathbb{R}^n)$ boundedness for $\mu^\rho_{\Omega,S}$ and $\mu_\lambda^{*,\rho}$ when $\Omega\in Lip_\alpha(\mathbb{S}^{n-1})$.

Suppose that $\Omega\in L^q(\mathbb{S}^{n-1})$, $q\ge 1$. Then the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta)=\sup_{\|\gamma\|\le\delta}\left(\int_{\mathbb{S}^{n-1}}|\Omega(\gamma x')-\Omega(x')|^q d\sigma(x')\right)^{1/q},$$

where γ denotes a rotation on $\mathbb{S}^{n-1}$ and $\|\gamma\|=\sup_{x'\in\mathbb{S}^{n-1}}|\gamma x'-x'|$.

Recently, Ding and Xue obtained the following weighted results.

Theorem 1.1 (see [3]). *Suppose $\rho>n/2$, $\lambda>2$ and $\Omega\in L^2(\mathbb{S}^{n-1})$ satisfies*

$$\int_0^1\frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma<\infty \tag{1.3}$$

for some $\sigma>1$. If $1<p<\infty$ and $\omega\in A_p$, then both of $\mu^\rho_{\Omega,S}$ and $\mu_\lambda^{*,\rho}$ are bounded on the weighted space $L^p(\mathbb{R}^n,\omega)$.

Theorem 1.2 (see [4]). *Suppose* $\rho>n/2,\lambda>2$ *and* $\Omega\in L^2(\mathbb{S}^{n-1})$ *satisfies* (1.3) *for some* $\sigma>2$. *If* $b\in BMO(\mathbb{R}^n)$ *and* $\omega\in A_p$, *then for any* $1<p<\infty$, *both of* $[b^m,\mu^{\rho}_{\Omega,S}]$ *and* $[b^m,\mu^{*,\rho}_{\lambda}]$ *are bounded on the weighted space* $L^p(\mathbb{R}^n,\omega)$.

From Remark 2 in [5], (1.3) is weaker than (1.2). Then the results of Theorem 1.1 and Theorem 1.2 are extend and improve the results of Torchinsky and Wang in [1] and the results of Sakamoto and Yabuta in [2].

In 2000, Lu et al. in [6] proved some boundedness results for sublinear operators on weighted Herz spaces with general Muckenhoupt weights. Recently, many authors considered the boundedness of operators on weighted Herz spaces. Wang in [7] proved that the intrinsic square functions are bounded on weighted Herz spaces, Komori and Matsuoka in [8] considered the boundedness of singular integral operators and fractional integral operators on generalized Herz spaces, Wang and Wang in [9] showed the boundedness of commutators of multilinear singular integrals on weighted Herz spaces. More results concerning the boundedness of operators on weighted Herz spaces can be seen in [10, 11].

The main purpose of this paper is to consider the boundedness of parameterized Littlewood-Paley operators on weighted Herz spaces with general Muckenhoupt weights. At the extreme case, we will also prove that parameterized Littlewood-Paley operators are bounded from the weighted Herz spaces to the weighted weak Herz spaces. The boundedness of commutators generated by parameterized Littlewood-Paley operators and BMO functions on weighted Herz spaces are also considered.

Our main results in the paper are formulated as follows.

Theorem 1.3. *Let* $\rho>n/2$, *let* $\lambda>2$, *and let* $\Omega\in L^2(\mathbb{S}^{n-1})$ *satisfies* (1.3) *for some* $\sigma>1$. *Suppose* $0<p<\infty$, $1<q<\infty$, $\omega_1\in A_{q_1}$ *and* $\omega_2\in A_{q_2}$. *Then both of* $\mu^{\rho}_{\Omega,S}$ *and* $\mu^{*,\rho}_{\lambda}$ *are bounded on* $\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)$ *provided that* ω_1 *and* ω_2 *satisfy either of the following*

(i) $\omega_1=\omega_2$, $1\le q_1=q_2\le q$ *and* $-nq_1/q<\alpha q_1<n(1-q_1/q)$;

(ii) $\omega_1\neq\omega_2$, $1\le q_1<\infty$, $1\le q_2\le q$ *and* $0<\alpha q_1<n(1-q_2/q)$.

Theorem 1.4. *Let* $\rho>n/2$, *let* $\lambda>2$, *and let* $\Omega\in L^2(\mathbb{S}^{n-1})$ *satisfies* (1.3) *for some* $\sigma>1$. *Suppose* $0<p<1$, $1<q<\infty$, $\omega_1\in A_{q_1}$ *and* $\omega_2\in A_{q_2}$. *If* $1\le q_1<\infty$, $1\le q_2\le q$ *and* $\alpha q_1=n(1-q_2/q)$, *Then both of* $\mu^{\rho}_{\Omega,S}$ *and* $\mu^{*,\rho}_{\lambda}$ *are bounded from* $\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)$ *to* $W\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)$.

Theorem 1.5. *Let* $\rho>n/2$, *let* $\lambda>2$, *and let* $\Omega\in L^2(\mathbb{S}^{n-1})$ *satisfies* (1.3) *for some* $\sigma>2$. *Suppose* $1<q<\infty$, $\omega_1\in A_{q_1}$, $\omega_2\in A_{q_2}$ *and* $b\in BMO$. *Then* $[b^m,\mu^{\rho}_{\Omega,S}]$ *and* $[b^m,\mu^{*,\rho}_{\lambda}]$ *are bounded on* $\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)$ *provided that* ω_1 *and* ω_2 *satisfy either of the following*

(i) $\omega_1=\omega_2$, $1\le q_1=q_2\le q$ *and* $-nq_1/q<\alpha q_1<n(1-q_1/q)$;

(ii) $\omega_1\neq\omega_2$, $1\le q_1<\infty$, $1\le q_2\le q$ *and* $0<\alpha q_1<n(1-q_2/q)$.

Throughout this paper, we shall use the symbol $A\lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A\le CB$.

2 Some preliminaries

We begin this section with some properties of A_p weights which play important role in the proofs of our main results.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B=B(x_0,r_B)$ denote the ball with the center x_0 and radius r_B. For any ball B and $\lambda>0$, λB denotes the ball concentric with B whose radius is λ times as long. For a given weight function ω and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E)=\int_E\omega(x)dx$.

A weight ω is said to belong to A_p for $1<p<\infty$, if there exists a constant C such that for every ball $B\subset\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B\omega(x)dx\right)\left(\frac{1}{|B|}\int_B\omega(x)^{1-p'}dx\right)^{p-1}\le C, \tag{2.1}$$

where p' is the dual of p such that $1/p+1/p'=1$. The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|}\int_B w(y)dy\lesssim \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B\subset\mathbb{R}^n.$$

A weight ω is said to belong to A_∞ if there are positive numbers C and δ so that

$$\frac{\omega(E)}{\omega(B)}\lesssim\left(\frac{|E|}{|B|}\right)^{\delta}$$

for all balls B and all measurable $E\subset B$. It is well known that

$$A_\infty=\bigcup_{1\le p<\infty}A_p.$$

By (2.1), we have

$$\left(\int_B\omega(x)dx\right)^{\frac{1}{p}}\left(\int_B\omega(x)^{1-p'}dx\right)^{\frac{1}{p'}}\lesssim|B| \tag{2.2}$$

for $1<p<\infty$.

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^q boundedness of Hardy-Littlewood maximal function in [12].

Lemma 2.1 (see [12,13]). *Suppose $\omega\in A_p(\mathbb{R}^n)$ and the following statements hold.*

(i) For any $1\le p<\infty$, there are positive numbers C and δ such that

$$\frac{\omega(B_k)}{\omega(B_j)}\lesssim 2^{np(k-j)} \quad \textit{for } k>j, \tag{2.3}$$

and

$$\frac{\omega(B_k)}{\omega(B_j)} \lesssim 2^{\delta(k-j)} \quad for \ \ k<j, \tag{2.4}$$

(ii) $A_{p_1}(\mathbb{R}^n) \subset A_{p_2}(\mathbb{R}^n)$ *for any* $1 \le p_1 < p_2 \le \infty$,

(iii) For any $1<p<\infty$, *one has* $\omega^{1-p'} \in A_{p'}(\mathbb{R}^n)$.

A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |b(x)-b_B|dx = \|b\|_* < \infty,$$

where $b_B = |B|^{-1}\int_B b(y)dy$.

Lemma 2.2 (see [14]). *Suppose* $\omega \in A_\infty$ *and* $b \in BMO$. *Then for any* $1\le q<\infty$ *and* $r_1,r_2>0$, *we have*

$$\left(\frac{1}{\omega(B(x_0,r_1))}\int_{B(x_0,r_1)} |b(x)-b_{B(x_0,r_2)}|^q \omega(x)dx\right)^{1/q} \lesssim \|b\|_* \left(1+\left|\ln\frac{r_1}{r_2}\right|\right). \tag{2.5}$$

Finally, let us recall the definition of weighted Herz space and weighted weak Herz space.

Let $B_k = \{x\in\mathbb{R}^n : |x| \le 2^k\}$ and $C_k = B_k\setminus B_{k-1}$ for any $k\in\mathbb{Z}$. Denote $\chi_k = \chi_{B_k} - \chi_{B_{k-1}}$ for $k\in\mathbb{Z}$, where χ_{B_k} is the characteristic function of the set B_k. The following weighted Herz space was introduced by Lu and Yang in [15].

Let $\alpha\in\mathbb{R}$, $0<p,q<\infty$ and ω_1, ω_2 be two weight functions on $\mathbb{R}^n$. The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n) = \left\{ f\in L^q_{loc}(\mathbb{R}^n\setminus\{0\},\omega_2) : \ \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\alpha p/n} \|f\chi_k\|^p_{L^q_{\omega_2}(\mathbb{R}^n)}\right)^{1/p}.$$

For any $k\in\mathbb{Z}$, $\lambda>0$ and any measurable function f on $\mathbb{R}^n$, we write $E_k(\lambda,f) = \{x\in C_k : |f(x)|>\lambda\}$. The weighted weak Herz space was introduced by Lu, Yabuta and Yang in [6].

Let $\alpha\in\mathbb{R}$, $0<p,q<\infty$ and ω_1, ω_2 be two weight function on $\mathbb{R}^n$. A measurable function f on $\mathbb{R}^n$ is said to belong to the homogeneous weighted weak Herz space $W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \sup_{\lambda>0}\lambda\left(\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\alpha p/n}[\omega_2(E_k(\lambda,f))]^{p/q}\right)^{1/p} < \infty.$$

Obviously, if $\alpha=0$, then $\dot{K}_p^{0,p}(\omega_1,\omega_2) = L^p(\omega_2)$ and $W\dot{K}_p^{0,p}(\omega_1,\omega_2) = WL^p(\omega_2)$ for any $0<p<\infty$. Thus, weighted Herz spaces are generalizations of the weighted Lebesgue spaces and weighted weak Herz spaces are generalizations of the weighted weak Lebesgue spaces.

3 Proof of theorems

It is easy to cheak that

$$\mu^{\rho}_{\Omega,S}(f)(x)\lesssim 2^{n\lambda}\mu^{*,\rho}_{\lambda}(f)(x),$$

and

$$[b^m,\mu^{\rho}_{\Omega,S}](f)(x)\lesssim 2^{n\lambda}[b^m,\mu^{*,\rho}_{\lambda}](f)(x).$$

Therefore, it is enough to consider the operators $\mu^{*,\rho}_{\lambda}$ and $[b^m,\mu^{*,\rho}_{\lambda}]$ in the proofs of our results.

Proof of Theorem 1.3. Using Minkowski's inequality we get

$$\begin{aligned}\mu^{*,\rho}_{\lambda}(f\chi_k)(x)=&\left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^{\rho}}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(f\chi_k)(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ \le&\int_{C_k}|f(z)|\left(\int_0^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz\\ \le&\int_{C_k}|f(z)|\left(\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz\\ &+\int_{C_k}|f(z)|\left(\int_{|x-z|}^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+1+2\rho}}\right)^{1/2}dz.\end{aligned}$$

By the coordinate transformation, we have

$$\begin{aligned}&\int_{|x-z|}^\infty\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+1+2\rho}}\\ \le&\int_{|x-z|}^\infty\int_0^t r^{2\rho-n-1}dr\int_{\mathbb{S}^{n-1}}|\Omega(y')|^2d\sigma(y')\frac{dt}{t^{2\rho+n+1}}\\ \lesssim&\int_{|x-z|}^\infty t^{-2n-1}dt\lesssim|x-z|^{-2n}.\end{aligned}$$

Since $t+|x-y|>|y-z|+|x-y|\ge|x-z|$, $\lambda>2$, we see that

$$\begin{aligned}&\int_0^{|x-z|}\int_{|y-z|<t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+1+2\rho}}\\ \lesssim&|x-z|^{-\lambda n}\int_0^{|x-z|}\int_0^t r^{2\rho-n-1}dr\int_{\mathbb{S}^{n-1}}|\Omega(y')|^2d\sigma(y')\frac{dt}{t^{n+1+2\rho-\lambda n}}\\ \lesssim&\|\Omega\|^2_{L^2(\mathbb{S}^{n-1})}|x-z|^{-\lambda n}\int_0^{|x-z|}t^{\lambda n-2n-1}dt\\ \lesssim&|x-z|^{-2n}.\end{aligned}$$

Thus

$$\mu_\lambda^{*,\rho}(f\chi_k)(x)\lesssim |x-z|^{-n}\|f_k\|_{L^1(\mathbb{R}^n)}. \tag{3.1}$$

Let $f\in\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. Then we have

$$\begin{aligned}
&\|\mu_\lambda^{*,\rho}(f)\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}\\
\lesssim&\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\big\|(\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\big\|_{L^q(\omega_2)}\right)^p\\
&+\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\big\|(\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\big\|_{L^q(\omega_2)}\right)^p\\
&+\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=j+2}^{\infty}\big\|(\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\big\|_{L^q(\omega_2)}\right)^p\\
=&E_1+E_2+E_3.
\end{aligned}$$

By the fact that $\mu_\lambda^{*,\rho}$ is a bounded operator on $L^q(\omega_2)$, we get

$$\begin{aligned}
E_2\lesssim&\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\big\|(\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\big\|_{L^q(\omega_2)}\right)^p\\
\lesssim&\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\|f\chi_k\|_{L^q(\omega_2)}\right)^p\\
\lesssim&\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.
\end{aligned}$$

When $j\geq k+2$, by (3.1) we get

$$\big|\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\big|\lesssim 2^{-jn}\|f\chi_k\|_{L^1}. \tag{3.2}$$

Then

$$\Big\|\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\Big\|_{L^q(\omega_2)}\lesssim 2^{-jn}\omega_2(B_j)^{1/q}\|f\chi_k\|_{L^1}.$$

By Hölder's inequality,

$$\|f\chi_k\|_{L^1}\lesssim\|f\chi_k\|_{L^q(\omega_2)}\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}. \tag{3.3}$$

Since $\omega_2\in A_{q_2}(\mathbb{R}^n)\subset A_q(\mathbb{R}^n)$, by (2.2) and (2.3) we get,

$$\begin{aligned}&\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q}\\&=\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_k}\omega_2(x)dx\Big)^{1/q}\Big(\frac{\omega_2(B_j)}{\omega_2(B_k)}\Big)^{1/q}\\&\lesssim 2^{kn+(j-k)nq_2/q}.\end{aligned}\tag{3.4}$$

Then, for $j\geq k+2$ we have

$$\Big\|\mu_\lambda^{*,\rho}(f_k)\chi_j(x)\Big\|_{L^q(\omega_2)}\lesssim\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}.$$

Thus

$$\begin{aligned}E_1&\lesssim\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\Big(\sum_{k=-\infty}^{j-2}\big\|\mu_\lambda^{*,\rho}(f_k)\chi_j(x)\big\|_{L^q(\omega_2)}\Big)^p\\&\lesssim\|b\|_*^p\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j-2}\big(\omega_1(B_j)\big)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)n(q_2/q-1)}\Big)^p\\&\lesssim\|b\|_*^p\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j-2}\big(\omega_1(B_k)\big)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)(\alpha q_1+q_2n/q-n)}\Big)^p.\end{aligned}$$

When $0<p\leq 1$, we get

$$\begin{aligned}E_1&\lesssim\|b\|_*^p\sum_{k=-\infty}^{\infty}\big(\omega_1(B_k)\big)^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p\sum_{j=-\infty}^{k+2}2^{(j-k)p(\alpha q_1+q_2n/q-n)}\\&\lesssim\|b\|_*^p\sum_{k=-\infty}^{\infty}\big(\omega_1(B_k)\big)^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p\\&\lesssim\|b\|_*^p\|f\|^p_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)}.\end{aligned}$$

When $p>1$, by Hölder's inequality we get

$$\begin{aligned}E_1&\lesssim\|b\|_*^p\sum_{j=-\infty}^{\infty}\Big(\sum_{k=-\infty}^{j-2}\big(\omega_1(B_k)\big)^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p2^{(j-k)(\alpha q_1+q_2n/q-n)}\Big)\\&\quad\times\Big(\sum_{k=-\infty}^{j-2}2^{(j-k)(\alpha q_1+q_2n/q-n)}\Big)^{p/p'}\\&\lesssim\|b\|_*^p\sum_{k=-\infty}^{\infty}\big(\omega_1(B_k)\big)^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p\\&\lesssim\|b\|_*^p\|f\|^p_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)(\mathbb{R}^n)}.\end{aligned}$$

Let us now turn to estimate the last term E_3. In the case $k\geq j+2$, by (3.1) we have

$$\left|\mu_\lambda^{*,\rho}(f_k)\chi_j(x)\right|\lesssim 2^{-kn}\|f\chi_k\|_{L^1}. \tag{3.5}$$

So,

$$\left\|\mu_\lambda^{*,\rho}(f_k)\chi_j(x)\right\|_{L^q(\omega_2)}\lesssim 2^{-kn}\omega_2(B_j)^{1/q}\|f\chi_k\|_{L^1}.$$

By (2.2) and (2.4),

$$\begin{aligned}
&\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q}\\
=&\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_k}\omega_2(x)dx\Big)^{1/q}\Big(\frac{\omega_2(B_j)}{\omega_2(B_k)}\Big)^{1/q}\\
\lesssim&2^{kn+(j-k)\delta_2 n/q}.
\end{aligned}\tag{3.6}$$

Combining with (3.3), (3.5) and (3.6), we have

$$\left\|\mu_\lambda^{*,\rho}(f_k)\chi_j(x)\right\|_{L^q(\omega_2)}\lesssim\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)\delta_2 n/q}$$

for $k\geq j+2$.

When $0<p\leq 1$, we get

$$\begin{aligned}
E_3\lesssim&\|b\|_*^p\sum_{j=-\infty}^{\infty}\omega_1(B_j)^{\frac{\alpha p}{n}}\sum_{k=j+2}^{\infty}\|f\chi_k\|_{L^q(\omega_2)}^p2^{(j-k)p\delta_2 n/q}\\
\lesssim&\|b\|_*^p\sum_{k=-\infty}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p\sum_{j=-\infty}^{k-2}\Big(\frac{\omega_1(B_j)}{\omega_1(B_k)}\Big)^{\frac{\alpha p}{n}}2^{(j-k)p\delta_2 n/q}\\
\lesssim&\|b\|_*^p\sum_{k=-\infty}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p\sum_{j=-\infty}^{k-2}2^{(j-k)p(\delta_1\alpha+\delta_2 n/q)}\\
\lesssim&\|b\|_*^p\|f\|_{\dot K_q^{\alpha,p}(\omega_1,\omega_2)}^p.
\end{aligned}$$

When $q>1$, by Hölder's inequality we get

$$\begin{aligned}
E_3\lesssim&\|b\|_*^p\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j+2}^{\infty}(\omega_1(B_k))^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)(\delta_1\alpha+\delta_2 n/q)}\Big)^p\\
\lesssim&\|b\|_*^p\sum_{j=-\infty}^{\infty}\Big(\sum_{k=j+2}^{\infty}(\omega_1(B_k))^{\frac{\alpha p}{n}}\|f\chi_k\|_{L^q(\omega_2)}^p2^{(j-k)(\delta_1\alpha+\delta_2 n/q)}\Big)\\
&\times\Big(\sum_{k=j+2}^{\infty}2^{(j-k)(\delta_1\alpha+\delta_2 n/q)}\Big)^{p/p\prime}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|b\|_*^p \sum_{k=-\infty}^{\infty} (\omega_1(B_k))^{\frac{\alpha p}{n}} \|f\chi_k\|_{L^q(\omega_2)}^p \sum_{j=-\infty}^{k-2} 2^{(j-k)(\delta_1\alpha+\delta_2 n/q)} \\
&\lesssim \|b\|_*^p \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p.
\end{aligned}
$$

Combining the above estimates for E_1, E_2 and E_3, we complete the proof. $\square$

Proof of Theorem 1.4. Let $f \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. Then

$$
\begin{aligned}
&\lambda^p \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left(\omega_2(\{x\in C_j : |\mu_\lambda^{*,\rho}(f)(x)|>\lambda\})\right)^{p/q} \\
\le & \lambda^p \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left(\omega_2\left(\left\{x\in C_j : \sum_{k=-\infty}^{j-2} |\mu_\lambda^{*,\rho}(f\chi_k)(x)|>\lambda/3\right\}\right)\right)^{p/q} \\
&+\lambda^p \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left(\omega_2\left(\left\{x\in C_j : \sum_{k=j-1}^{j+1} |\mu_\lambda^{*,\rho}(f\chi_k)(x)|>\lambda/3\right\}\right)\right)^{p/q} \\
&+\lambda^p \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left(\omega_2\left(\left\{x\in C_j : \sum_{k=j-2}^{\infty} |\mu_\lambda^{*,\rho}(f\chi_k)(x)|>\lambda/3\right\}\right)\right)^{p/q} \\
= & F_1+F_2+F_3.
\end{aligned}
$$

Applying Chebyshev's inequality and Theorem 1.1, we obtain

$$
\begin{aligned}
F_2 &\lesssim \lambda^p \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left(\frac{1}{\lambda^q} \sum_{k=j-1}^{j+1} \|\mu_\lambda^{*,\rho}(f\chi_k)\|_{L^q(\omega_2)}^q\right)^{p/q} \\
&\lesssim \sum_{j=-\infty}^{\infty} (\omega_1(B_j))^{\frac{\alpha p}{n}} \left(\sum_{k=j-1}^{j+1} \|f\chi_k\|_{L^q(\omega_2)}\right)^p \\
&\lesssim \|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}^p.
\end{aligned}
$$

For $j \ge k+2$, by the inequalities (3.2), (3.3) and (3.4) we have

$$
\begin{aligned}
\left|\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\right| &\lesssim 2^{-jn} \|f\chi_k\|_{L^1} \\
&\lesssim 2^{-jn} \|f\chi_k\|_{L^q(\omega_2)} \left(\int_{B_k} \omega_2(x)^{1-q'} dx\right)^{1/q'} \\
&\lesssim 2^{-jn} \|f\chi_k\|_{L^q(\omega_2)} (\omega_2(B_j))^{-1/q} (\omega_2(B_j))^{1/q} \left(\int_{B_k} \omega_2(x)^{1-q'} dx\right)^{1/q'} \\
&\lesssim (\omega_2(B_j))^{-1/q} 2^{(k-j)n(1-q_2/q)} \|f\chi_k\|_{L^q(\omega_2)}.
\end{aligned}
$$

Noting the fact $\alpha q_1 = n(1-q_2/q)$, then

$$
\left|\mu_\lambda^{*,\rho}(f\chi_k)\chi_j(x)\right| \lesssim (\omega_2(B_j))^{-1/q} 2^{(k-j)\alpha q_1} \|f\chi_k\|_{L^q(\omega_2)}.
$$

Moreover, since $0<p\leq 1$, for any $x\in C_j$, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{j-2}\left|\mu_\lambda^{*,\rho}(f\chi_k)(x)\right| \\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\sum_{k=-\infty}^{j-2}2^{(k-j)\alpha q_1}\big(\omega_1(B_k)\big)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)}\left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\frac{\alpha}{n}} \\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\sum_{k=-\infty}^{j-2}\big(\omega_1(B_k)\big)^{\frac{\alpha}{n}}\|f\chi_k\|_{L^q(\omega_2)} \\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\left(\sum_{k=-\infty}^{j-2}\big(\omega_1(B_k)\big)^{\frac{\alpha p}{n}}\|f\chi_k\|^p_{L^q(\omega_2)}\right)^{1/p} \\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.
\end{aligned}$$

If

$$\left\{x\in C_j:\sum_{k=-\infty}^{j-2}|\mu_\lambda^{*,\rho}(f\chi_k)(x)|>\lambda/3\right\}=\emptyset,$$

then

$$F_1\lesssim\|f\|^p_{\dot{K}_p^{\alpha,p}(\omega_1,\omega_2)}$$

holds is trivially. Now we suppose

$$\left\{x\in C_j:\sum_{k=-\infty}^{j-2}|\mu_\lambda^{*,\rho}(f\chi_k)(x)|>\lambda/3\right\}\neq\emptyset.$$

Let

$$S_j=\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}.$$

Since $\alpha>0$, it is easy to see that

$$\lim_{j\to\infty}S_j=0.$$

Then for any $\lambda>0$, we can find a maximal positive integer j_λ such that

$$\lambda/3\lesssim S_{j_\lambda}\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

So

$$\begin{aligned}
F_1\leq&\lambda^p\sum_{j=-\infty}^{j_\lambda}\big(\omega_2(B_j)\big)^{p/q}\big(\omega_1(B_j)\big)^{\alpha p/n} \\
\lesssim&\|f\|^p_{\dot{K}_p^{\alpha,p}(\omega_1,\omega_2)}\sum_{j=-\infty}^{j_\lambda}\left(\frac{\omega_1(B_j)}{\omega_1(B_{j_\lambda})}\right)^{\frac{\alpha p}{n}}\left(\frac{\omega_2(B_j)}{\omega_2(B_{j_\lambda})}\right)^{\frac{p}{q}}
\end{aligned}$$

$$\begin{aligned}&\lesssim\|f\|^p_{\dot K^{\alpha,p}_p(\omega_1,\omega_2)}\sum_{j=-\infty}^{j_\lambda}2^{(j-j_\lambda)(\delta_1\alpha p+\delta_2 pn/q)}\\&\lesssim\|f\|^p_{\dot K^{\alpha,p}_q(\omega_1,\omega_2)}.\end{aligned}$$

For $k\geq j+2$, from (3.3), (3.5) and (3.6) we have

$$\left|\mu^{*,\rho}_\lambda(f\chi_k)\chi_j(x)\right|\lesssim\big(\omega_2(B_j)\big)^{-1/q}2^{(j-k)\delta_2 n/q}\|f\chi_k\|_{L^q(\omega_2)}.$$

So

$$\begin{aligned}&\sum_{k=j-2}^{\infty}|\mu^{*,\rho}_\lambda(f\chi_k)(x)|\\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\sum_{k=j-2}^{\infty}\big(\omega_1(B_k)\big)^{\alpha/n}\|f\chi_k\|_{L^q(\omega_2)}2^{(j-k)\delta_2 n/q}\left(\frac{\omega_1(B_j)}{\omega_1(B_k)}\right)^{\alpha/n}\\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\sum_{k=j-2}^{\infty}\big(\omega_1(B_k)\big)^{\alpha/n}\|f\chi_k\|_{L^q}2^{(j-k)(\delta_2 n/q+\alpha\delta_1)}\\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\sum_{k=j-2}^{\infty}\big(\omega_1(B_k)\big)^{\alpha/n}\|f\chi_k\|_{L^q}.\end{aligned}$$

Note that $0<p\leq 1$, we have

$$\begin{aligned}&\sum_{k=j-2}^{\infty}|\mu^{*,\rho}_\lambda(f\chi_k)(x)|\\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\left(\sum_{k=j-2}^{\infty}|\big(\omega_1(B_k)\big)^{\alpha p/n}\|f\chi_k\|^p_{L^q}\right)^{1/p}\\
\lesssim&\big(\omega_2(B_j)\big)^{-1/q}\big(\omega_1(B_j)\big)^{-\alpha/n}\|f\|_{\dot K^{\alpha,p}_q(\omega_1,\omega_2)}.\end{aligned}$$

Repeating the arguments used for the term F_1, we can also obtain

$$F_3\lesssim\|f\|^p_{\dot K^{\alpha,p}_q(\omega_1,\omega_2)}.$$

Combining the estimates for F_1, F_2 and F_3, and taking the supremum for all $\lambda>0$, the proof of Theorem 1.4 is finished. □

Proof of Theorem 1.5. Let $f\in\dot K^{\alpha,p}_q(\omega_1,\omega_2)$, then

$$\begin{aligned}&\big\|[b^m,\mu^{*,\rho}_\lambda](f)\big\|^p_{\dot K^{\alpha,p}_q(\omega_1,\omega_2)}\\
\leq&C\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=-\infty}^{j-2}\big\|[b^m,\mu^{*,\rho}_\lambda](f\chi_k)\chi_j(x)\big\|_{L^q(\omega_2)}\right)^p\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\big\|[b^m,\mu_\lambda^{*,\rho}](f\chi_k)\chi_j(x)\big\|_{L^q(\omega_2)}\right)^p\\
&+C\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=j+2}^{\infty}\big\|[b^m,\mu_\lambda^{*,\rho}](f\chi_k)\chi_j(x)\big\|_{L^q(\omega_2)}\right)^p\\
=&G_1+G_2+G_3.
\end{aligned}
$$

By the fact that $[b^m,\mu_\lambda^{*,\rho}]$ is a bounded operator on $L^q(\omega_2)$, we obtain

$$
G_2\lesssim\|b\|_*^{mp}\sum_{j=-\infty}^{\infty}\big(\omega_1(B_j)\big)^{\frac{\alpha p}{n}}\left(\sum_{k=j-1}^{j+1}\big\|f\chi_k\big\|_{L^q(\omega_2)}\right)^p\lesssim\|b\|_*^{mp}\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.
$$

From the binomial theorem and the definition of $[b^m,\mu_\lambda^{*,\rho}]$, we have

$$
\begin{aligned}
&\Big\|[b^m,\mu_\lambda^{*,\rho}](f\chi_k)\chi_j(x)\Big\|_{L^q(\omega_2)}\\
\lesssim&\sum_{l=0}^{m}\left(\int_{C_j}\Big|(b(x)-b_{B_k})^{m-l}\mu_\lambda^{*,\rho}((b(\cdot)-b_{B_k})^l f\chi_k)(x)\Big|^q\omega_2(x)dx\right)^{1/q}.
\end{aligned}
$$

When $j\geq k+2$, by (3.2) and Hölder's inequality we have

$$
\begin{aligned}
&|\mu_\lambda^{*,\rho}((b(\cdot)-b_{B_k})^l f\chi_k)\chi_j(x)|\\
\lesssim&2^{-jn}\|(b(\cdot)-b_{B_k})^l f\chi_k)\|_{L^1}\\
\lesssim&2^{-jn}\int_{B_k}|b(z)-b_{B_k}|^l|f(z)|dz\\
\lesssim&2^{-jn}\left(\int_{B_k}|b(z)-b_{B_k}|^{lq'}\omega_2(z)^{1-q'}dz\right)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}.
\end{aligned}
$$

Since $\omega_2\in A_{q_2}$, by (iii) of Lemma 2.1 we know that $\omega_2^{1-q_2'}\in A_{q_2'}$. Therefore by Lemma 2.2 we have

$$
\left(\int_{B_k}|b(x)-b_{B_k}|^{lq'}\omega_2(x)^{1-q'}dx\right)^{1/q'}\lesssim\|b\|_*^l\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}. \tag{3.7}
$$

Then, using Lemma 2.2 again, we get

$$
\begin{aligned}
&\left(\int_{C_j}\Big|(b(x)-b_{B_k})^{m-l}\mu_\lambda^{*,\rho}((b(\cdot)-b_{B_k})^l f\chi_k)(x)\Big|^q\omega_2(x)dx\right)^{1/q}\\
\lesssim&2^{-jn}\|b\|_*^l\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\left(\int_{C_j}|b(x)-b_{B_k}|^{m-l}\omega_2(x)dx\right)^{1/q}\|f\chi_k\|_{L^q(\omega_2)}\\
\lesssim&2^{-jn}(j-k)\|b\|_*^m\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q}\|f\chi_k\|_{L^q(\omega_2)}.
\end{aligned}
$$

Hence, by (3.4) we obtain

$$\Big\|[b^m,\mu_\lambda^{*,\rho}](f\chi_k)\chi_j(x)\Big\|_{L^q(\omega_2)}\lesssim\|b\|_*^m\|f\chi_k\|_{L^q(\omega_2)}(j-k)2^{(j-k)n(q_2/q-1)}. \tag{3.8}$$

Using (3.8) and repeating the estimates for E_1, we obtain

$$G_1\lesssim\|b\|_*^{mp}\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}$$

for $0<p<\infty$.

Finally, let us estimate G_3. Since $k\geq j+2$, by (3.5), Hölder's inequality and (3.7) we have

$$\begin{aligned}
&|\mu_\lambda^{*,\rho}((b(\cdot)-b_{B_k})^l f\chi_k)\chi_j(x)|\\
\lesssim&2^{-kn}\|(b(\cdot)-b_{B_k})^l f\chi_k)\|_{L^1}\\
\lesssim&2^{-kn}\left(\int_{B_k}|b(z)-b_{B_k}|^{lq'}\omega_2(z)^{1-q'}dz\right)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}\\
\lesssim&2^{-kn}\|b\|_*^l\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\|f\chi_k\|_{L^q(\omega_2)}.
\end{aligned}$$

Then, using Lemma 2.2 again we get

$$\begin{aligned}
&\left(\int_{C_j}\left|(b(x)-b_{B_k})^{m-l}\mu_\lambda^{*,\rho}((b(\cdot)-b_{B_k})^l f\chi_k)(x)\right|^q\omega_2(x)dx\right)^{1/q}\\
\lesssim&2^{-kn}\|b\|_*^l\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\left(\int_{C_j}|b(x)-b_{B_k}|^{m-l}\omega_2(x)dx\right)^{1/q}\|f\chi_k\|_{L^q(\omega_2)}\\
\lesssim&2^{-kn}(k-j)\|b\|_*^m\Big(\int_{B_k}\omega_2(x)^{1-q'}dx\Big)^{1/q'}\Big(\int_{B_j}\omega_2(x)dx\Big)^{1/q}\|f\chi_k\|_{L^q(\omega_2)}.
\end{aligned}$$

Hence, by (3.6) we get

$$\Big\|[b^m,\mu_\lambda^{*,\rho}](f\chi_k)\chi_j(x)\Big\|_{L^q(\omega_2)}\lesssim\|b\|_*^m\|f\chi_k\|_{L^q(\omega_2)}(k-j)2^{(j-k)n(q_2/q-1)}$$

for $k\geq j+2$. Repeating the estimation process of E_3, we obtain

$$G_3\lesssim\|b\|_*^{mp}\|f\|^p_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}$$

for $0<p<\infty$. Summing up the estimates of G_1, G_2 and G_3, it completes the proof of Theorem 1.5. □

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