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Some Results on the Growth of Polynomials

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Abstract. In this paper, we have studied the Lacunary type of polynomials and proved a result which generalizes as well as refines some well-known polynomial inequalities regarding the growth of polynomials not vanishing inside a circle. Further the paper corrects the proofs of some already known results.

Key Words: Lacunary polynomial, Growth, Bernstien-inequality.

AMS Subject Classifications: 30A10, 30C15, 26D10

1 Introduction and statement of results

For an entire function $f(z)$, let $M(f,r)=\max_{|z|=r}|f(z)|$ and $m(f,r)=\min_{|z|=r}|f(z)|$. If $P(z)$ is a polynomial of degree n and $P'(z)$ denote its derivative, then according to a famous result of Bernstien [3],

$$M(P',1)\leq nM(P,1). \tag{1.1}$$

Inequality (1.1) is best possible and equality holds for the polynomial having all its zeros at the origin.

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then

$$M(P',1)\leq \frac{n}{2}M(P,1). \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and later verified by Lax [8] and is best possible for the polynomials having all its zeros on $|z|=1$.

Aziz and Dawood [1] under the same hypothesis refined the inequality (1.2) and proved

$$M(P',1)\leq \frac{n}{2}\Big\{M(P,1)-m(P,1)\Big\}. \tag{1.3}$$

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This result is sharp and equality holds for the polynomial $P(z)=(z+1)^n$.

As an extension of (1.2), Malik [9] proved that if $P(z)$ does not vanish in $|z|<k$, $k\geq 1$, then

$$M(P',1)\leq \frac{n}{1+k}M(P,1), \tag{1.4}$$

whereas Govil [7] under the same hypothesis proved that

$$M(P',1)\leq \frac{n}{1+k}\Big\{M(P,1)-m(P,k)\Big\}. \tag{1.5}$$

In the literature there exist several generalizations and extensions of the above inequalities, for example see [2,5,6,12–15]. As a generalization of inequality (1.4), Dewan and Bidkham [5] proved that if $P(z)$ has no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq \rho\leq k$,

$$M(P',\rho)\leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}M(P,r). \tag{1.6}$$

Recently Dewan and Mir [6] generalized inequality (1.6) and proved that if $P(z)=\sum_{j=0}^n a_j z^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq \rho\leq k$,

$$M(P',\rho)\leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\frac{\rho-r}{k+\rho}\Big)\Big(\frac{k+r}{k+\rho}\Big)^{n-1}\right\}M(P,r). \tag{1.7}$$

Further, Aziz and Zargar [2] obtained a generalization of (1.7) by proving the following result.

Theorem 1.1. *If $P(z)=\sum_{j=0}^n a_j z^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq \rho\leq k$,*

$$\begin{aligned}M(P',\rho)\leq \frac{n}{\rho+k}\Bigg[&\Big(\frac{\rho+k}{k+r}\Big)^n\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\frac{\rho-r}{k+\rho}\Big)\Big(\frac{k+r}{k+\rho}\Big)^{n-1}\right\}M(P,r)\\&-\left\{\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\bigg(\Big(\Big(\frac{\rho+k}{r+k}\Big)^n-1\Big)-n(\rho-r)\bigg)\right\}m(P,k)\Bigg].\end{aligned} \tag{1.8}$$

The result is best possible and equality holds in (1.8) for $P(z)=(z+k)^n$.

Recently Zireh [12] obtained the following result which is a refinement of (1.8) and proved.

Theorem 1.2. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq\rho\leq k$,*

$$\begin{aligned}M(P',\rho)\leq&\frac{n(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\\&\times\Bigg[\Big(\frac{\rho+k}{k+r}\Big)^n\Big\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\frac{\rho-r}{k+\rho}\Big)\Big(\frac{k+r}{k+\rho}\Big)^{n-1}\Big\}M(P,r)\\&-\Big\{\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\Big(\Big(\frac{\rho+k}{r+k}\Big)^n-1\Big)-n(\rho-r)\Big)\Big\}m(P,k)\Bigg].\end{aligned}\tag{1.9}$$

The result is best possible and equality holds in (1.9) for $P(z)=(z+k)^n$.

More recently Mir, Baba and Pukhta [10] proved the following result

Theorem 1.3. *If $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, then for $0\leq r\leq\rho\leq k$,*

$$\begin{aligned}&M(P',\rho)\\\leq&\frac{n\rho^{\mu-1}}{\rho^\mu+k^\mu}\Bigg[\Big(\frac{\rho^\mu+k^\mu}{k^\mu+r^\mu}\Big)^{\frac{n}{\mu}}\Big\{1-\frac{k^\mu(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)n}{\mu\Big(|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)\Big)}\\&\times\Big(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\Big)\Big(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\Big)^{\frac{n}{\mu}-1}\Big\}M(P,r)\\&-\Big\{\frac{(n|a_0|\rho+\mu k^{\mu+1}|a_\mu|)(r+k)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\Big(\Big(\Big(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\Big)^{\frac{n}{\mu}}-1\Big)-\frac{n}{\mu}\Big(\frac{\rho^\mu-r^\mu}{r^\mu+k^\mu}\Big)\Big)\Big\}m(P,k)\Bigg].\end{aligned}\tag{1.10}$$

The result is best possible and equality holds in (1.10) for $P(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

In the proof of Theorem 1.1, one can find a miscalculation in one of the lemmas used to prove it due to which the result does not come out to be more refined as it must come out to be. The same lemma was used to prove Theorem 1.2, hence Theorem 1.2 faces the same thing as well.

2 Lemmas

In the proof we shall make use of the following lemmas.

Lemma 2.1. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then*

for $0\le r\le\rho\le k$,

$$\begin{aligned}M(P,\rho)\le&\Big(\frac{\rho+k}{k+r}\Big)^n\left[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\frac{\rho-r}{k+\rho}\Big)\Big(\frac{k+r}{k+\rho}\Big)^{n-1}\right]M(P,r)\\&-\left[\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\left(\left(\Big(\frac{\rho+k}{r+k}\Big)^n-1\right)-n(\rho-r)\right)\right]m(P,k).\end{aligned}\tag{2.1}$$

The above lemma is due to Aziz and Zargar [2].

The above Lemma 2.1 was used to prove Theorem 1.1 and Theorem 1.2. One can find that in the above inequality (2.1), one of the term comes out to be $-n(\rho-r)$ instead of $-n(\frac{\rho-r}{r+k})$ due to miscalculation. The correct proof of Lemma 2.1 can be obtained by Lemma 2.3 for $\mu=1$.

Lemma 2.2. *If*

$$P(z)=a_0+\sum_{j=\mu}^n a_jz^j,\quad 1\le\mu\le n,$$

is a polynomial of degree n having no zeros in $|z|<k$, $k\ge1$, *then*

$$M(P',1)\le n\frac{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}{1+k^{\mu+1}+\frac{\mu}{n}|\frac{a_\mu}{a_0}|(k^{\mu+1}+k^{2\mu})}M(P,1).\tag{2.2}$$

The above lemma is due to Qazi [11].

Lemma 2.3. *If* $P(z)=a_0+\sum_{j=\mu}^n a_jz^j$, $1\le\mu\le n$, *is a polynomial of degree n having no zeros in* $|z|<k$, $k>0$, *then for* $0\le r\le\rho\le k$,

$$\begin{aligned}&M(P,\rho)\\&\le\Big(\frac{\rho^\mu+k^\mu}{k^\mu+r^\mu}\Big)^{\frac{n}{\mu}}\\&\times\left[1-\frac{k^\mu(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)n}{\mu\big(|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)\big)}\Big(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\Big)\Big(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\Big)^{\frac{n}{\mu}-1}\right]M(P,r)\\&-\left[\frac{(n|a_0|\rho+\mu k^{\mu+1}|a_\mu|)(r^\mu+k^\mu)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\left\{\left(\Big(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\Big)^{\frac{n}{\mu}}-1\right)-\frac{n}{\mu}\Big(\frac{\rho^\mu-r^\mu}{r^\mu+k^\mu}\Big)\right\}\right]m(P,k).\end{aligned}\tag{2.3}$$

The above lemma is due to Mir, Baba and Pukhta [10].

Now with the help of Lemma 2.3 (for $\mu=1$), Theorem 1.1 takes the form

Theorem 2.1. *If* $P(z)=\sum_{j=0}^n a_jz^j$ *is a polynomial of degree n having no zeros in* $|z|<k$, $k\ge1$, *then for* $0\le r\le\rho\le k$,

$$\begin{aligned}M(P',\rho)\le\frac{n}{\rho+k}&\left[\Big(\frac{\rho+k}{k+r}\Big)^n\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\frac{\rho-r}{k+\rho}\Big)\Big(\frac{k+r}{k+\rho}\Big)^{n-1}\right\}M(P,r)\right.\\&\left.-\left\{\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\left(\left(\Big(\frac{\rho+k}{r+k}\Big)^n-1\right)-n\Big(\frac{\rho-r}{r+k}\Big)\right)\right\}m(P,k)\right],\end{aligned}$$

and Theorem 1.2 takes the form

Theorem 2.2. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq\rho\leq k$,*

$$\begin{aligned}M(P',\rho)\leq&\frac{n(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\\&\times\Bigg[\Big(\frac{\rho+k}{k+r}\Big)^n\Big\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\frac{\rho-r}{k+\rho}\Big)\Big(\frac{k+r}{k+\rho}\Big)^{n-1}\Big\}M(P,r)\\&-\Big\{\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\Big(\Big(\Big(\frac{\rho+k}{r+k}\Big)^n-1\Big)-n\Big(\frac{\rho-r}{r+k}\Big)\Big)\Big\}m(P,k)\Bigg].\end{aligned}$$

Remark 2.1. If we take $\mu=1$ in Theorem 1.3, we get Theorem 2.1, for $k>0$, hence Theorem 1.3 is a generalization and an extension of Theorem 2.1.

In this paper, as we have already corrected Theorem 1.1 and Theorem 1.2. Next we obtain the following result which generalizes Theorem 2.2 and also refines the inequality (1.10). More precisely, we prove

Theorem 2.3. *If $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu\leq n$, is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, then for $0\leq r\leq\rho\leq k$,*

$$\begin{aligned}&M(P',\rho)\\\leq& n\rho^{\mu-1}\Big\{\frac{n|a_0|\rho+\mu k^{\mu+1}|a_\mu|}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\Big\}\\&\times\Bigg[\Big(\frac{\rho^\mu+k^\mu}{k^\mu+r^\mu}\Big)^{\frac{n}{\mu}}\Big\{1-\frac{k^\mu(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)n}{\mu\Big(|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)\Big)}\Big(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\Big)\Big(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\Big)^{\frac{n}{\mu}-1}\Big\}M(P,r)\\&-\Big\{\frac{(n|a_0|\rho+\mu k^{\mu+1}|a_\mu|)(r+k)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\Big(\Big(\Big(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\Big)^{\frac{n}{\mu}}-1\Big)-\frac{n}{\mu}\Big(\frac{\rho^\mu-r^\mu}{r^\mu+k^\mu}\Big)\Big)\Big\}m(P,k)\Bigg].\end{aligned}$$

Remark 2.2. For $\mu=1$, Theorem 2.3 reduces to Theorem 2.2, for $k>0$. Also it is well known (for example see [4, Proof of Lemma 2.4]), that if $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu\leq n$, is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, then

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m(P,k)}k^\mu\leq 1,$$

which implies

$$\frac{\mu}{n}\Big|\frac{a_\mu}{a_0}\Big|k^\mu\leq 1,$$

which is equivalent to

$$\frac{n|a_0|\rho+\mu k^{\mu+1}|a_\mu|}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\leq\frac{1}{k^\mu+\rho^\mu}.$$

From this it is easy to see that Theorem 2.3 is an improvement of inequality (1.10).

If we put $n|a_0|=\mu|a_\mu|k^\mu$ in Theorem 2.3, we get the following result.

Corollary 2.1. If $P(z)=a_0+\sum_{j=\mu}^n a_j z^j$, $1\le\mu\le n$, is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, then for $0\le r\le\rho\le k$,

$$M(P',\rho)\le n\rho^{\mu-1}\left(\frac{\rho+k}{k^{\mu+1}+\rho^{\mu+1}+k^\mu\rho+k\rho^\mu}\right)\left(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}$$
$$\times\left[M(P,r)-\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)\left\{1-\left(\frac{r^\mu+k^\mu}{\rho^\mu+k^\mu}\right)^{\frac{n}{\mu}}-\frac{n}{\mu}\frac{(\rho^\mu-r^\mu)(r^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+\rho^\mu)^{\frac{n}{\mu}}}\right\}m(P,k)\right].$$

The result is best possible and equality holds for the polynomial $P(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

3 Proof of theorem

Proof of Theorem 2.3. Since $P(z)$ has no zeros in $|z|<k$, $k>0$, then for $0<\rho\le k$, it follows that $T(z)=P(\rho z)$ has no zero in $|z|<\frac{k}{\rho}$, where $\frac{k}{\rho}\ge 1$. Applying Lemma 2.2 to the polynomial $T(z)$, we get

$$M(T',1)\le n\frac{1+\frac{\mu}{n}\left|\frac{\rho^\mu a_\mu}{a_0}\right|\left(\frac{k}{\rho}\right)^{\mu+1}}{1+\left(\frac{k}{\rho}\right)^{\mu+1}+\frac{\mu}{n}\left|\frac{\rho^\mu a_\mu}{a_0}\right|\left(\left(\frac{k}{\rho}\right)^{\mu+1}+\left(\frac{k}{\rho}\right)^{2\mu}\right)}M(T,1),$$

which implies,

$$M(P',\rho)\le n\rho^{\mu-1}\left[\frac{n|a_0|\rho+\mu k^{\mu+1}|a_\mu|}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right]M(P,\rho). \tag{3.1}$$

Now if $0\le r\le\rho\le k$, then from (3.1) it follows with the help of Lemma 2.3 that

$$\begin{aligned}&M(P',\rho)\\ \le& n\rho^{\mu-1}\left\{\frac{n|a_0|\rho+\mu k^{\mu+1}|a_\mu|}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\right\}\\ &\times\left[\left(\frac{\rho^\mu+k^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}}\left\{1-\frac{k^\mu(k-\rho)(n|a_0|-k^\mu\mu|a_\mu|)n}{\mu\left(|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)\right)}\left(\frac{\rho^\mu-r^\mu}{k^\mu+\rho^\mu}\right)\left(\frac{k^\mu+r^\mu}{k^\mu+\rho^\mu}\right)^{\frac{n}{\mu}-1}\right\}M(P,r)\right.\\ &\left.-\left\{\frac{(n|a_0|\rho+\mu k^{\mu+1}|a_\mu|)(r+k)}{n|a_0|(k^{\mu+1}+\rho^{\mu+1})+\mu|a_\mu|(k^{2\mu}\rho+k^{\mu+1}\rho^\mu)}\left(\left(\left(\frac{\rho^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right)-\frac{n}{\mu}\left(\frac{\rho^\mu-r^\mu}{r^\mu+k^\mu}\right)\right)\right\}m(P,k)\right],\end{aligned}$$

this completes the proof of Theorem 2.3. □

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