Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 4 (2017), pp. 323-332
DOI: 10.4208/ata.2017.v33.n4.3

The Meromorphic Solutions of the Zakharov-Kuznetsov Modified Equal Width Equation

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Received 23 September 2016; Accepted (in revised version) 5 August 2017

Abstract. In this paper, we use the complex method to obtain all meromorphic solutions of the complex Zakharov-Kuznetsov modified equal width equation, then find the exact traveling wave solutions of the Zakharov-Kuznetsov modified equal width equation. At last, we give some computer simulations to illustrate our main results.

Key Words: Exact solution, meromorphic function, elliptic function.

AMS Subject Classifications: 30D35, 34A05

1 Introduction

One of the main topics of mathematical physics is to find exact solutions of nonlinear partial differential equations. These equations play an important role in fluid mechanics, plasma physics, optical fibres, solid-state physics and chemical physics and other fields. In 2009, A. Biswas [1, 2] obtained the solitary wave solutions, topological solitons and non-topological solitons of the generalized Zakharov-Kuznetsov modified equal width equation (1.1) by using the solitary wave ansatz method. In 2014, Y. Pandir [3] obtained the soliton solutions, rational solutions and elliptic integral function solutions of the generalized Zakharov-Kuznetsov modified equal width equation by using the extended trial equation method. In this paper, we use the complex method to obtain all meromorphic solutions of the Zakharov-Kuznetsov modified equal width (ZK-MEK) equation.

The Zakharov-kuznetsov modified equal width equation [1–3] is

$$u_t+a(u^3)_x+(bu_{xt}+cu_{yy})_x=0, \tag{1.1}$$

where a, b, c are constants.

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Using the traveling wave transformation

$$u=w(z),\quad z=x+ky+lt,$$

into the ZK-MEW equation, it gives a nonlinear ordinary differential equation

$$lw'+a(w^3)'+(blw''+ck^2w'')'=0,$$

and integrating it, we field the following ordinary differential equation

$$(bl+ck^2)w''+aw^3+lw+d=0, \tag{1.2}$$

where a, b, c, d, l, k are constants.

In order to clarify our main results, we need some basic concepts. $w(z)$ is a meromorphic function means that $w(z)$ is holomorphic in the complex plane $\mathbf{C}$ except for poles. We define a meromorphic function f belongs to the class W if f is an elliptic function, or a rational function of $e^{\alpha z}(\alpha\in\mathbf{C})$, or a rational function of z.

Our main results is the following Theorem 1.1.

Theorem 1.1. *Suppose that $a(bl+ck^2)\neq 0$, then all the general meromorphic solutions w of Eq.* (1.2) *are of the following forms:*
(I) The rational function solutions

$$w_{r1}(z)=\pm\sqrt{-\frac{2(bl+ck^2)}{a}}\frac{1}{z-z_0}, \tag{1.3}$$

and

$$w_{r2}(z)=\pm\sqrt{-\frac{2(bl+ck^2)}{az_1^2}}\Big(\frac{z_1}{z-z_0}-\frac{z_1}{z-z_0-z_1}-1\Big), \tag{1.4}$$

where $l=d=0$ in (1.3)*, or*

$$l(z_1^2-6b)=6ck^2,\quad d=\mp 2a\Big(\frac{-2(bl+ck^2)}{az_1^2}\Big)^{\frac{3}{2}},$$

in (1.4)*, $z_0,z_1\neq 0$ are arbitrary complex numbers.*
(II) The simply periodic solutions

$$w_{s1}(z)=\pm\alpha\sqrt{-\frac{bl+ck^2}{2a}}\tanh\frac{\alpha}{2}(z-z_0), \tag{1.5a}$$

$$w_{s2}(z)=\pm\alpha\sqrt{-\frac{bl+ck^2}{2a}}\coth\frac{\alpha}{2}(z-z_0), \tag{1.5b}$$

and

$$w_{s3}(z)=\alpha\sqrt{-\frac{bl+ck^2}{2a}}\Big(\coth\frac{\alpha}{2}(z-z_0)-\coth\frac{\alpha}{2}(z-z_0-z_1)-\coth\frac{\alpha}{2}z_1\Big), \tag{1.6}$$

where $l(2-b\alpha^2)=\alpha^2ck^2$, $d=0$ *in* (1.5a), (1.5b), *or*

$$l=(bl+ck^2)\alpha^2\Big(\frac{1}{2}+\frac{3}{2\sinh^2\frac{\alpha}{2}z_1}\Big),\quad d=\sqrt{-\frac{bl+ck^2}{2a}}\frac{\tanh\frac{\alpha}{2}z_1}{\sinh^2\frac{\alpha}{2}z_1},$$

in (1.6), $z_0,z_1\neq 0$ *are arbitrary complex numbers.*
(III) The elliptic general solutions

$$\begin{aligned}&w_d(z)\\=&\pm\frac{1}{2}\sqrt{-\frac{2(bl+ck^2)}{a}}\frac{(-\wp+C)(4\wp C^2+4\wp^2C+2\wp' D-\wp g_2-Cg_2)}{((12C^2-g_2)\wp+4C^3-3Cg_2)\wp'+(4\wp^3+12C\wp^2-3g_2\wp-Cg_2)D},\end{aligned} \tag{1.7}$$

where $D^2=4C^3-g_2C$, $g_3=0$, g_2, C *are arbitrary complex numbers.*

2 Pertinent lemmas and the complex method

In 2013, Yuan et al. [4] employed the complex method to obtain all meromorphic solutions of some nonlinear ordinary differential equations. In this section we will introduce the complex method and the lemmas.

Lemma 2.1 (see [5,6])**.** *Let* $k\in\mathbf{N}$, *then any meromorphic solution* $w\in W$ *of* k *order Briot-Bouquet equations*

$$F(w^{(k)},w)=\sum_{i=0}^{n}P_i(w)(w^{(k)})^i=0,$$

where $P_i(w)$ *are polynomials with constant coefficients and* w *has at least one pole.*

Set $m\in\mathbf{N}:=\{1,2,3,\cdots\}$, $r_j\in\mathbf{N}_0=\mathbf{N}\cup\{0\}$, $r=(r_0,r_1,\cdots,r_m)$, $j=0,1,\cdots,m$. Define differential monomial

$$M_r[w](z):=[w(z)]^{r_0}[w'(z)]^{r_1}[w''(z)]^{r_2}\cdots[w^{(m)}(z)]^{r_m}.$$

$p(r):=r_0+r_1+\cdots+r_m$ is called the degree of $M_r[w]$. Define differential polynomial

$$P(w,w',\cdots,w^{(m)}):=\sum_{r\in I}a_rM_r[w],$$

where a_r are constants, and I is a finite index set. The total degree of $P(w,w',\cdots,w^{(m)})$ is defined by

$$\deg P(w,w',\cdots,w^{(m)}):=\max_{r\in I}\{p(r)\}.$$

Now we observe the following complex ordinary differential equation

$$P(w,w',\cdots,w^{(m)})=bw^n+c, \tag{2.1}$$

where $n\in\mathbf{N}$, $b\neq 0$, c are constants.

Let p, $q\in\mathbf{N}$. Suppose that the Eq. (2.1) has a meromorphic solution w with at least one pole, we say that the Eq. (2.1) satisfies weak $\langle p,q\rangle$ condition if substituting Laurent series

$$w(z)=\sum_{k=-q}^{\infty}c_k z^k,\quad q>0,\quad c_{-q}\neq 0, \tag{2.2}$$

into the Eq. (2.1), we can determine p distinct Laurent singular parts below

$$\sum_{k=-q}^{-1}c_k z^k.$$

Lemma 2.2 (see [7–9]). *Let p, l, m, $n\in\mathbf{N}$, $\deg P(w,w^{(m)})<n$. Suppose that an m order Briot-Bouquet equation*

$$P(w^{(m)},w)=bw^n+c \tag{2.3}$$

satisfies weak $\langle p,q\rangle$ condition, then all meromorphic solutions w of Eq. (2.3) *belong to the class W. If for some values of parameters let solution w exists, then other meromorphic solutions are forming a one-parametric family $w(z-z_0),z_0\in\mathbf{C}$. Furthermore each elliptic solution with pole at $z=0$ can be written as*

$$\begin{aligned}w(z)=&\sum_{i=1}^{l-1}\sum_{j=2}^{q}\frac{(-1)^j c_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\Big(\frac{1}{4}\Big[\frac{\wp'(z)+B_i}{\wp(z)-A_i}\Big]^2-\wp(z)\Big)\\&+\sum_{i=1}^{l-1}\frac{c_{-i1}}{2}\frac{\wp'(z)+B_i}{\wp(z)-A_i}+\sum_{j=2}^{q}\frac{(-1)^j c_{-lj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z)+c_0,\end{aligned} \tag{2.4}$$

where c_{-ij} are given by (2.2), *$B_i^2=4A_i^3-g_2A_i-g_3$ and $\sum_{i=1}^{l}c_{-i1}=0$.*

Each rational function solution $w:=R(z)$ is of the form

$$R(z)=\sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(z-z_i)^j}+c_0, \tag{2.5}$$

with $l(\leq p)$ distinct poles of multiplicity q.

Each simply periodic solution is a rational function $R(\xi)$ of $\xi=e^{\alpha z}$ $(\alpha\in\mathbf{C})$. $R(\xi)$ has $l(\leq p)$ distinct poles of multiplicity q, and is of the form

$$R(\xi)=\sum_{i=1}^{l}\sum_{j=1}^{q}\frac{c_{ij}}{(\xi-\xi_i)^j}+c_0. \tag{2.6}$$

In order to give the representations of elliptic solutions, we need some notations [9].

Let ω_1, ω_2 be two fixed complex numbers such that $\mathrm{Im}\frac{\omega_1}{\omega_2}>0$, $L=L[2\omega_1,2\omega_2]$ be discrete subset $L[2\omega_1,2\omega_2]=\{\omega|\omega=2n\omega_1+2m\omega_2,\ n,m\in\mathbf{Z}\}$, which is isomorphic to $\mathbf{Z}\times\mathbf{Z}$. The discriminant $\Delta=\Delta(c_1,c_2):=c_1^3-27c_2^2$ and

$$s_n=s_n(L):=\sum_{\omega\in L\setminus\{0\}}\frac{1}{\omega^n},\quad n\geq 3,\quad n\in\mathbb{N}.$$

Weierstrass elliptic function $\wp(z):=\wp(z,g_2,g_3)$ is a meromorphic function with two periods $2\omega_1$, $2\omega_2$ and satisfying

$$(\wp'(z))^2=4\wp(z)^3-g_2\wp(z)-g_3, \tag{2.7}$$

where $g_2=60s_4$, $g_3=140s_6$ and $\Delta(g_2,g_3)\neq 0$.

If modify (2.7) to the following equation

$$(\wp'(z))^2=4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3), \tag{2.8}$$

we have $e_1=\wp(\omega_1)$, $e_2=\wp(\omega_2)$, $e_3=\wp(\omega_1+\omega_2)$.

Reversely, given two complex numbers g_2 and g_3 such that $\Delta(g_2,g_3)\neq 0$, then there exists a double periodic $2\omega_1$, $2\omega_2$ Weierstrass elliptic function $\wp(z)$ such that above hold.

Lemma 2.3 (see [9,10]). *Two successive degeneracies and addition formula of Weierstrass elliptic functions $\wp(z):=\wp(z,g_2,g_3)$ are*
(I) Degeneracy to simply periodic functions (i.e., rational functions of one exponential e^{kz}) according to

$$\wp(z,3d^2,-d^3)=2d-\frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}z, \tag{2.9}$$

if one root e_j is double ($\Delta(g_2,g_3)=0$).
(II) Degeneracy to rational functions of z according to

$$\wp(z,0,0)=\frac{1}{z^2}$$

if one root e_j is triple ($g_2=g_3=0$).
(III) Addition formula

$$\wp(z-z_0)=-\wp(z)-\wp(z_0)+\frac{1}{4}\Big[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\Big]^2. \tag{2.10}$$

By these lemmas, Yuan et al. [4] give the complex method to find exact solutions of some PDEs:

Step 1 Substituting the transform $T:u(x,y,t)\to w(z)$, $(x,y,t)\to z$ into a given PDE yields a non-linear ordinary differential equations (2.1) or (2.3).

Step 2 Substitute (2.2) into Eqs. (2.1) or (2.3) to determine that weak $\langle p,q\rangle$ condition holds.

Step 3 By indeterminant relation (2.4)-(2.6) we find the elliptic, rational and simply periodic solutions $w(z)$ of Eqs. (2.1) or (2.3) with pole at $z=0$, respectively.

Step 4 By Lemmas 2.1 and 2.2, we obtain all meromorphic solutions $w(z-z_0)$.

Step 5 Substituting the inverse transform T^{-1} into these meromorphic solutions $w(z-z_0)$, then we get all exact solutions $u(x,y,t)$ of the original given PDE.

3 Proof of Theorem 1.1

Proof. Substituting 2.2 into Eq. (1.2), we have $q=1$, $p=2$, $c_{-1}=\pm\sqrt{\frac{-2(bl+ck^2)}{a}}$, $c_0=0$, $c_1=\pm\frac{l}{3ac_{-1}}$, $c_2=\frac{d}{4(bl+ck^2)}$. Therefore, Eq. (1.2) satisfies the weak $\langle 2,1\rangle$ condition and is a second-order Briot-Bouquet differential equation. Obviously, Eq. (1.2) satisfies the dominant condition. Hence by Lemma 2.2, we get that all meromorphic solutions of Eq. (1.2) belong to the class W. Now we will derive the forms of all meromorphic solutions of Eq. (1.2).

By (2.6), we infer that the indeterminant rational solutions of Eq. (1.2) with pole are

$$R_1(z)=\frac{c_1}{z}+\frac{c_2}{z-z_1}+c_0.$$

Substituting $R_1(z)$ into Eq. (1.2), we get that

$$R_{11}(z)=\pm\sqrt{-\frac{2bl+2ck^2}{a}}\frac{1}{z},$$

where $l=d=0$, and

$$R_{12}=\pm\sqrt{-\frac{2bl+2ck^2}{az_1^2}}\Big(\frac{z_1}{z}-\frac{z_1}{z-z_1}-1\Big),$$

where $l(z_1^2-6b)=6ck^2$, $d=\mp 2a(\frac{-2bl-2ck^2}{az_1^2})^{\frac{3}{2}}$.

Thus all rational solutions of Eq. (1.2) are

$$w_{r1}(z)=\pm\sqrt{-\frac{2(bl+ck^2)}{a}}\frac{1}{z-z_0},$$

$$w_{r2}(z)=\pm\sqrt{-\frac{2(bl+ck^2)}{az_1^2}}\Big(\frac{z_1}{z-z_0}-\frac{z_1}{z-z_0-z_1}-1\Big),$$

where $l=d=0$ in $w_{r1}(z)$, or

$$l(z_1^2-6b)=6ck^2,\quad d=\mp 2a\Big(\frac{-2(bl+ck^2)}{az_1^2}\Big)^{3/2},$$

in w_{r2}, and $z_0, z_1 \neq 0$ are arbitrary complex numbers.

In order to derive simply periodic solutions, set $\xi = e^{\alpha z}$, put $w = R(\xi)$ into Eq. (1.2), then

$$(bl+ck^2)\alpha^2[\xi R' + \xi^2 R''] + lR + aR^3 + d = 0. \tag{3.1}$$

Substituting

$$R_2(\xi) = \frac{c_{21}}{\xi - 1} + \frac{c_{22}}{\xi - \xi_1} + c_{20}$$

into Eq. (3.1), we get that

$$\begin{aligned}
R_{21}(z) &= \pm\alpha\sqrt{-\frac{bl+ck^2}{2}}\Big(-\frac{2}{\xi+1}+1\Big),\\
R_{22}(z) &= \pm\alpha\sqrt{-\frac{bl+ck^2}{2}}\Big(\frac{2}{\xi-1}+1\Big),\\
R_{23}(z) &= \frac{\alpha}{1-\xi_1}\sqrt{-\frac{bl+ck^2}{2a}}\Big(-2(\xi_1-1)\Big(\frac{1}{\xi-1}-\frac{\xi_1}{\xi-\xi_1}\Big)+(\xi_1+1)\Big),
\end{aligned}$$

where $l(2-b\alpha^2) = \alpha^2 ck^2$, $d = 0$ in $R_{21}(z)$, $R_{22}(z)$, or

$$l = \Big(\frac{1}{2} + \frac{6\xi_1}{(1-\xi_1)^2}\Big)(bl+ck^2)\alpha^2, \quad d = \frac{8a\xi_1(\xi_1+1)\alpha^3}{(1-\xi_1)^3}\Big(-\frac{bl+ck^2}{2a}\Big)^{\frac{3}{2}},$$

in $R_{23}(z)$.

Substituting $\xi = e^{\alpha z}$ into above relations, and then we get two simply periodic solutions of Eq. (1.2) with pole at $z = 0$

$$\begin{aligned}
w_{s0,1}(z) &= \pm\alpha\sqrt{-\frac{bl+ck^2}{2a}}\tanh\frac{\alpha}{2}z,\\
w_{s0,2}(z) &= \pm\alpha\sqrt{-\frac{bl+ck^2}{2a}}\coth\frac{\alpha}{2}z,\\
w_{s0,3}(z) &= \alpha\sqrt{-\frac{bl+ck^2}{2a}}\Big(\coth\frac{\alpha}{2}z - \coth\frac{\alpha}{2}(z-z_1) - \coth\frac{\alpha}{2}z_1\Big).
\end{aligned}$$

So all simply periodic solutions of Eq. (1.2) are obtained by

$$\begin{aligned}
w_{s1}(z) &= \pm\alpha\sqrt{-\frac{bl+ck^2}{2a}}\tanh\frac{\alpha}{2}(z-z_0),\\
w_{s2}(z) &= \pm\alpha\sqrt{-\frac{bl+ck^2}{2a}}\coth\frac{\alpha}{2}(z-z_0),\\
w_{s3}(z) &= \alpha\sqrt{-\frac{bl+ck^2}{2a}}\Big(\coth\frac{\alpha}{2}(z-z_0) - \coth\frac{\alpha}{2}(z-z_0-z_1) - \coth\frac{\alpha}{2}z_1\Big),
\end{aligned}$$

where $l(2-b\alpha^2)=\alpha^2ck^2$, $d=0$ in $w_{s1}(z)$, $w_{s2}(z)$, or

$$l=(bl+ck^2)\alpha^2\Big(\frac{1}{2}+\frac{3}{2\sinh^2\frac{\alpha}{2}z_1}\Big),\quad d=\sqrt{-\frac{bl+ck^2}{2a}}\frac{\tanh\frac{\alpha}{2}z_1}{\sinh^2\frac{\alpha}{2}z_1},$$

in $w_{s3}(z)$, and $z_0,z_1\neq 0$ are arbitrary complex numbers.

By (2.4), we get the indeterminant relations of elliptic solutions of Eq. (1.2) with pole at $z=0$ are

$$w_{d0}(z)=c_{-1}\frac{\wp'(z)+F}{\wp(z)-E}+c_{30},$$

where $F^2=4E^3-g_2E-g_3$. According to the conclusion of Lemma 2.2, and noting that the results of rational solutions above, we derive that $c_{30}=E=F=g_3=0$. Then we get that

$$w_{d0}(z)=\pm\frac{1}{2}\sqrt{-\frac{2bl+2ck^2}{a}}\frac{\wp'(z)}{\wp(z)}.$$

Therefore, all elliptic solutions of Eq. (1.2) are

$$w_{d1}(z)=\pm\frac{1}{2}\sqrt{-\frac{2bl+2ck^2}{a}}\frac{\wp'(z-z_0)}{\wp(z-z_0)},$$

where $z_0\in\mathbf{C}$, $g_3=0$. By using the addition formula of Lemma 2.3, we rewrite it to the form

$$w_d(z)=\pm\frac{1}{2}\sqrt{-\frac{2(bl+ck^2)}{a}}\frac{(-\wp+C)(4\wp C^2+4\wp^2C+2\wp'D-\wp g_2-Cg_2)}{((12C^2-g_2)\wp+4C^3-3Cg_2)\wp'+(4\wp^3+12C\wp^2-3g_2\wp-Cg_2)D'},$$

where $g_3=0$, $D^2=4C^3-g_2C$, g_2 and C are arbitrary complex numbers. □

4 Computer simulations

In this section, we give some computer simulations to illustrate our main results. Here we take $\alpha=2$, $k=2$, $a=2$, $b=1$, $c=1$, $l=-8$, $z_0=0$, $x\in[-50,50]$, $y\in[-50,50]$ in the simply periodic solutions (1.5a) and (1.5b), and have two kink wave type solutions of the ZK-MEW equation (see (1.1) and Figs. 1-4)

$$u_1(x,y,t)=2\tanh(x+2y-8t),\quad u_2(x,y,t)=2\coth(x+2y-8t).$$

5 Conclusions

In this article, the complex method has been implemented to find the exact traveling wave solutions of the ZK-MEW eqution, containing rational solutions, simply periodic solutions and elliptic solutions. It is important to observe that, comparing to other methods, the complex method is much powerful. Also it is quite capable for finding exact solutions of other PDEs in mathematical physics.

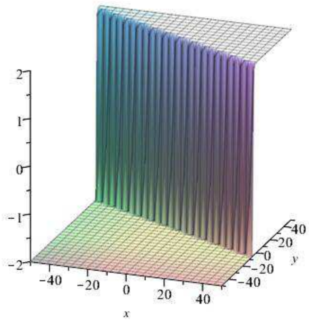

Figure 1: Graph of the solution $u_1(x,y,t)$, $t=5$.

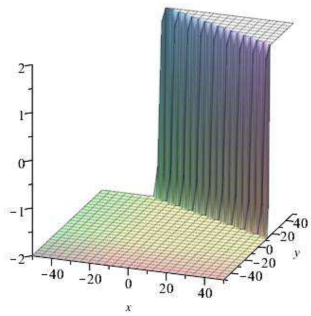

Figure 2: Graph of the solution $u_1(x,y,t)$, $t=10$.

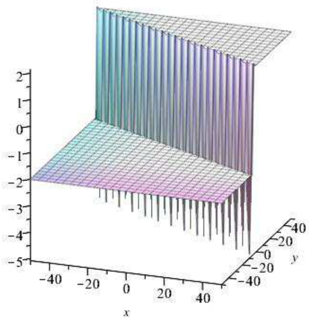

Figure 3: Graph of the solution $u_2(x,y,t)$, $t=5$.

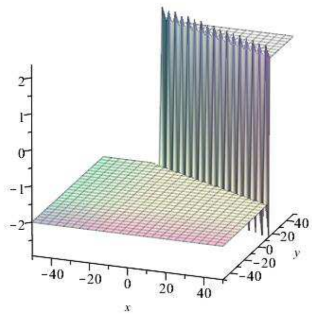

Figure 4: Graph of the solution $u_2(x,y,t)$, $t=10$.

Acknowledgements

The authors thanks the editors and referees for their useful suggestions and helpful comments. This work was supported by NSF of China (11271090) and NSF of Guangdong (2016A030310257 and 2015A030313346).

References

[1] A. Biswas, Topological and non-topological solitons for the generalized Zakharov-Kuznetsov modified equal width equation, Int. J. Theor. Phys., 48 (2009), 2698–2703.
[2] A. Biswas, 1-soliton solution of the generalized Zakharov-Kuznetsov modified equal width equation, Appl. Math. Lett., 22 (2009), 1775–1777.
[3] Y. Pandir, New exact solutions of the generalized Zakharov-Kuznetsov modified equal width equation, Pramana J. Phys., 82 (2014), 949–964.
[4] W. J. Yuan, Q. H. Chen, J. M. Qi and Y. Z. Li, The general traveling wave solutions of the Fisher equation with degree three, Adv. Math. Phys., (2013).

[5] A. Eremenko, Meromorphic solutions of equations of Briot-Bouquet type, Teor. Funktsii, Funk. Anal. i Prilozh, 38 (1982), 48–56. English translation, Amer. Math. Soc. Transl., 133 (1986), 15–23.

[6] A. Eremenko, L. W. Liao and T. W. Ng, Meromorphic solutions of higher order Briot-Bouquet differential equations, Math. Proc. Camb. Phil. Soc., 146 (2009), 197–206.

[7] N. A. Kudryashov, Meromorphic solutions of nonlinear ordinary differential equations, Commun. Nonlinear Sci. Numer. Simul., 15 (2010), 2778–2790.

[8] W. J. Yuan, Y. D. Shang, Y. Huang and H. Wang, The representation of meromorphic solutions of certain ordinary differential equations and its applications, Scientia Sinica Math., 43 (2013), 563–575.

[9] S. Lang, Elliptic Functions, Springer Verlag, New York, 2nd Ed, 1987.

[10] R. Conte and M. Musette, Elliptic general analytic solutions, Stud. Appl. Math., 123 (2009), 63–81.