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Evaluation of Certain Integrals Involving the Product of Classical Hermite's Polynomials Using Laplace Transform Technique and Hypergeometric Approach

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Abstract. In this paper some novel integrals associated with the product of classical Hermite's polynomials

$$\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_r(x)\}^2dx,\quad \int_0^\infty \exp(-x^2)H_{2k}(x)H_{2s+1}(x)dx,$$
$$\int_0^\infty \exp(-x^2)H_{2k}(x)H_{2s}(x)dx \quad\text{and}\quad \int_0^\infty \exp(-x^2)H_{2k+1}(x)H_{2s+1}(x)dx,$$

are evaluated using hypergeometric approach and Laplace transform method, which is a different approach from the approaches given by the other authors in the field of special functions. Also the results may be of significant nature, and may yield numerous other interesting integrals involving the product of classical Hermite's polynomials by suitable simplifications of arbitrary parameters.

Key Words: Gauss's summation theorem, classical Hermite's polynomials, generalized hypergeometric function, generalized Laguerre's polynomials.

AMS Subject Classifications: 33C20, 33C45, 33C47

1 Introduction

In recent years, numerous integral formulae involving a variety of special functions have been established by many authors (see, [1–5,7]). Also many integral formulae associated with the general class of polynomials (Laguerre, Hermite, Legendre, Bessel, Tchebychev and as in Askey-scheme) and other special cases therein (see, [6,8–12]).

Many integral formulae involving the products of classical orthogonal polynomials have been developed and play an important role in several physical problems. In fact,

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Hermite's polynomials are associated with a wide range of problems in diverse areas of mathematics. These connections of Hermite's polynomials with various other research areas have led many researchers to the field of special functions. we aim at presenting four integral formulae involving the product of classical polynomials of Hermite using Laplace transform method and hypergeometric approach. Among many properties of Hermite's polynomials, they also have investigated some possible extensions of the Hermite's polynomials. Those integrals involving the general class of polynomials are not only of great interest in pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering.

The widely-used Pochhammer symbol $(\lambda)_\nu$, $(\lambda,\nu\in\mathbb{C})$ is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1, & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases} \tag{1.1}$$

it being understood *conventionally* that $(0)_0=1$ and assumed *tacitly* that the Γ quotient exists.

The *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1,\alpha_2,\cdots,\alpha_p$ and q denominator parameters $\beta_1,\beta_2,\cdots,\beta_q$, is defined by

$${}_pF_q\left[\begin{array}{l}\alpha_1,\alpha_2,\cdots,\alpha_p;\\ \beta_1,\beta_2,\cdots,\beta_q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1.2}$$

$(p,q\in\mathbb{N}_0;\ p\leqq q+1;\ p\leqq q$ and $|z|<\infty$, $p=q+1$ and $|z|<1$; $p=q+1$, $|z|=1$, and $\Re(\omega)>0$; $p=q+1$, $|z|=1$, $z\neq 1$ and $0\geq\Re(\omega)>-1)$, where

$$\omega:=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j,\quad \big(\alpha_j\in\mathbb{C},\quad (j=1,2,\cdots,p);\quad \beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\,(j=1,2,\cdots,q)\big).$$

Laplace transform of $t^{\alpha-1}$:

$$\int_0^\infty e^{-st}t^{\alpha-1}dt=\frac{\Gamma(\alpha)}{s^\alpha}, \tag{1.3}$$

where $\Re(s)>0$, $0<\Re(\alpha)<\infty$ or $\Re(s)=0$, $0<\Re(\alpha)<1$.

Guass's summation theorem: The Gauss's summation theorem plays a vital role in the proof of many interesting results and some physical problems [13, pp. 49(Theorem 18)]

$${}_2F_1\left[\begin{array}{c}a,b;\\ c;\end{array} 1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.4}$$

where $c\neq 0,-1,-2,-3,\cdots$, and $\Re(c-a-b)>0$.

Special case of Gauss's summation theorem: By using the Gauss's summation theorem, it is easy to prove (see, [13, pp. 69(Q.N.4)])

$${}_2F_1\left[\begin{array}{c}-n,b;\\ c;\end{array} 1\right]=\frac{(c-b)_n}{(c)_n}, \tag{1.5}$$

where $c \neq 0,-1,-2,-3,\cdots$, and $n=0,1,2,3,\cdots$.

Classical Hermite's polynomials: Here, we are interested to use hypergeometric form of classical Hermite's polynomials (see, [13, pp. 191])

$$H_n(x)=(2x)^n {}_2F_0\left[\begin{array}{ccc} \frac{-n}{2},\frac{-n+1}{2}; & & \frac{-1}{x^2} \\ \overline{\qquad\quad}; & & \end{array}\right], \tag{1.6}$$

where $n=0,1,2,3,\cdots$.

Generalized Laguerre's polynomials: Here, we are interested to use hypergeometric form of generalized Laguerre's polynomials (see, [13, pp. 200])

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}{}_1F_1\left[\begin{array}{cc} -n; & x \\ 1+\alpha; & \end{array}\right]. \tag{1.7}$$

Special cases of classical Hermite's polynomials: By using the definition of generalized Laguerre's polynomials, it is easy to prove (see, [13, pp. 216(Q.N.1)])

$$H_{2n}(x)=(-1)^n 2^{2n} n! L_n^{(-\frac{1}{2})}(x^2)=(-1)^n 2^{2n}\left(\frac{1}{2}\right)_n {}_1F_1\left[\begin{array}{cc} -n; & x^2 \\ \frac{1}{2}; & \end{array}\right], \tag{1.8a}$$

$$H_{2n+1}(x)=(-1)^n 2^{2n+1} n! x L_n^{(\frac{1}{2})}(x^2)=(-1)^n 2^{2n+1} x\left(\frac{3}{2}\right)_n {}_1F_1\left[\begin{array}{cc} -n; & x^2 \\ \frac{3}{2}; & \end{array}\right]. \tag{1.8b}$$

Combinatorial Identity:

$$\sum_{K=0}^{N}(-1)^K\binom{N}{K}=\sum_{K=0}^{N}\frac{(-N)_K}{K!}=\begin{cases} 0, & \text{if } N=1,2,3,\cdots, \\ 1, & \text{if } N=0. \end{cases} \tag{1.9}$$

Some summation identities: Suppose $\{\Phi(K)\}$ is a well defined sequence of complex numbers and M, N are positive integers.

(i) If $M>N$, then

$$\sum_{K=0}^{N}(-N)_K(-K)_M\Phi(K)=0. \tag{1.10}$$

(ii) If $M=N$, then

$$\sum_{K=0}^{N}(-N)_K\,(-K)_N\Phi(K)=(N!)^2\Phi(N). \tag{1.11}$$

(iii) If $M<N$, then

$$\sum_{K=0}^{N}(-N)_K(-K)_M\,\Phi(K)=\sum_{K=0}^{N-M}(-N)_{K+M}(-K-M)_M\,\Phi(K+M). \tag{1.12}$$

Above identities (1.9)-(1.12) can be proved and verified easily.

2 First integral

Consider the first integral:

$$I_1=\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_r(x)\}^2dx, \tag{2.1}$$

where $m,r\in\mathbb{N}_0=\{0,1,2,3,\cdots\}$.

Case I: When $r=2n$ in the Eq. (2.1), we get

$$I_2=\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n}(x)\}^2dx. \tag{2.2}$$

Applying the definition of classical Hermite's polynomials (1.8a) in the integral of Eq. (2.2), we get

$$\begin{aligned}I_2=&2^{4n}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2){}_1F_1\left[\begin{array}{c}-n;\\ \frac{1}{2};\end{array}x^2\right]{}_1F_1\left[\begin{array}{c}-n;\\ \frac{1}{2};\end{array}x^2\right]dx\\ =&2^{4n+1}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\sum_{p=0}^{n}\frac{(-n)_p}{(\frac{1}{2})_p p!}\sum_{q=0}^{n}\frac{(-n)_q}{(\frac{1}{2})_q q!}\int_0^{\infty}\exp(-x^2)x^{2m+2p+2q}dx.\end{aligned} \tag{2.3}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the integral of Eq. (2.3), we get

$$\begin{aligned}I_2=&2^{4n}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\sum_{p=0}^{n}\frac{(-n)_p}{(\frac{1}{2})_p p!}\sum_{q=0}^{n}\frac{(-n)_q}{(\frac{1}{2})_q q!}\Gamma(m+p+q+(1/2))\\ =&2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{1}{2}+m)_p}{(\frac{1}{2})_p p!}\sum_{q=0}^{n}\frac{(-n)_q(\frac{1}{2}+m+p)_q}{(\frac{1}{2})_q q!}\\ =&2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{1}{2}+m)_p}{(\frac{1}{2})_p p!}{}_2F_1\left[\begin{array}{c}-n,\frac{1}{2}+m+p;\\ \frac{1}{2};\end{array}1\right],\end{aligned} \tag{2.4a}$$

$$\begin{aligned}&\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n}(x)\}^2dx\\ =&2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{1}{2}+m)_p(-m-p)_n}{(\frac{1}{2})_p p!}.\end{aligned} \tag{2.4b}$$

Case I(a): When $m\geq n$ in the Eq. (2.4b), then

$$I_2=2^{4n}\sqrt{\pi}(-1)^n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{1}{2}+m)_p(m+p)!}{(\frac{1}{2})_p(m+p-n)!p!}$$

$$=2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m(-m)_n\sum_{p=0}^{n}\frac{(-n)_p(\frac{1}{2}+m)_p(1+m)_p}{(\frac{1}{2})_p(1+m-n)_p p!}, \tag{2.5a}$$

$$\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n}(x)\}^2dx$$
$$=2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m(-m)_n\,{}_3F_2\left[\begin{array}{c}-n,\frac{1}{2}+m,1+m;\\ \frac{1}{2},1+m-n;\end{array}1\right], \tag{2.5b}$$

where $m\geq n$.

Case I(b): When $m<n$ in the Eq. (2.4b), then

$$I_2=2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m\sum_{p=n-m}^{n}\frac{(-n)_p(\frac{1}{2}+m)_p(-m-p)_n}{(\frac{1}{2})_p p!}. \tag{2.6}$$

Replacing p by $p+n-m$ in the Eq. (2.6), we get

$$I_2=2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_m(-1)^n\sum_{p=0}^{m}\frac{(-n)_{p+n-m}(\frac{1}{2}+m)_{p+n-m}(p+n)!}{(\frac{1}{2})_{p+n-m}(p+n-m)!p!}, \tag{2.7a}$$

$$\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n}(x)\}^2dx=2^{4n}\sqrt{\pi}(-1)^n\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-n\right)_m(-n)_m(-n)_{n-m}$$
$$\times{}_3F_2\left[\begin{array}{c}-m,\frac{1}{2}+n,1+n;\\ \frac{1}{2}+n-m,1+n-m;\end{array}1\right], \tag{2.7b}$$

where $m<n$.

Case I(c): Put $m=0$ and $n\geq 1$ in the Eq. (2.7b), we get

$$\int_{-\infty}^{+\infty}\exp(-x^2)\{H_{2n}(x)\}^2dx=2^{4n}\sqrt{\pi}\left(\frac{1}{2}\right)_n n!. \tag{2.8}$$

Case I(d): Put $m=1$ and $n\geq 2$ in the Eq. (2.7b), we get

$$\int_{-\infty}^{+\infty}(x^2)\exp(-x^2)\{H_{2n}(x)\}^2dx=2^{4n-1}\sqrt{\pi}\left(\frac{1}{2}\right)_n n!(4n+1). \tag{2.9}$$

Case II: When $r=2n+1$ in the Eq. (2.1), we get

$$I_3=\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n+1}(x)\}^2dx. \tag{2.10}$$

Applying the definition of classical Hermite's polynomials (1.8b) in the integral of E-q. (2.10), we get

$$\begin{aligned} I_3 =& 2^{4n+2}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\int_{-\infty}^{+\infty} x^{2m+2}\exp(-x^2)\,{}_1F_1\left[\begin{matrix}-n; & x^2\\ \frac{3}{2}; & \end{matrix}\right]{}_1F_1\left[\begin{matrix}-n; & x^2\\ \frac{3}{2}; & \end{matrix}\right]dx\\ =& 2^{4n+3}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\sum_{p=0}^{n}\frac{(-n)_p}{(\frac{3}{2})_p p!}\sum_{q=0}^{n}\frac{(-n)_q}{(\frac{3}{2})_q q!}\int_0^{\infty}\exp(-x^2)x^{2m+2p+2q+2}dx. \end{aligned} \tag{2.11}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the integral of Eq. (2.11), we get

$$\begin{aligned} I_3 =& 2^{4n+2}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\sum_{p=0}^{n}\frac{(-n)_p}{(\frac{3}{2})_p p!}\sum_{q=0}^{n}\frac{(-n)_q}{(\frac{3}{2})_q q!}\Gamma(m+p+q+(3/2))\\ =& 2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{3}{2}+m)_p}{(\frac{3}{2})_p p!}\sum_{q=0}^{n}\frac{(-n)_q(\frac{3}{2}+m+p)_q}{(\frac{3}{2})_q q!}\\ =& 2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{3}{2}+m)_p}{(\frac{3}{2})_p p!}\,{}_2F_1\left[\begin{matrix}-n,\frac{3}{2}+m+p; & 1\\ \frac{3}{2}; & \end{matrix}\right], \end{aligned} \tag{2.12a}$$

$$\begin{aligned} &\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n+1}(x)\}^2dx\\ =& 2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{3}{2}+m)_p(-m-p)_n}{(\frac{3}{2})_p p!}. \end{aligned} \tag{2.12b}$$

Case II(a): When $m\geq n$ in the Eq. (2.12a), then

$$\begin{aligned} I_3 =& 2^{4n+1}\sqrt{\pi}(-1)^n\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m\sum_{p=0}^{n}\frac{(-n)_p(\frac{3}{2}+m)_p(m+p)!}{(\frac{3}{2})_p(m+p-n)!p!}\\ =& 2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m(-m)_n\sum_{p=0}^{n}\frac{(-n)_p(\frac{3}{2}+m)_p(1+m)_p}{(\frac{3}{2})_p(1+m-n)_p p!} \end{aligned}$$

$$\begin{aligned} &\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n+1}(x)\}^2dx\\ =& 2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m(-m)_n\,{}_3F_2\left[\begin{matrix}-n,\frac{3}{2}+m,1+m; & 1\\ \frac{3}{2},1+m-n; & \end{matrix}\right], \end{aligned} \tag{2.13}$$

where $m\geq n$.

Case II(b): When $m<n$ in the Eq. (2.12a), then

$$I_3 = 2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m\sum_{p=n-m}^{n}\frac{(-n)_p(\frac{3}{2}+m)_p(-m-p)_n}{(\frac{3}{2})_p p!}. \tag{2.14}$$

Replacing p by $p+n-m$ in the Eq. (2.14), we get

$$I_3=2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_m(-1)^n\sum_{p=0}^{m}\frac{(-n)_{p+n-m}(\frac{3}{2}+m)_{p+n-m}(p+n)!}{(\frac{3}{2})_{p+n-m}(p+n-m)!p!}, \tag{2.15a}$$

$$\int_{-\infty}^{+\infty}(x^2)^m\exp(-x^2)\{H_{2n+1}(x)\}^2dx=2^{4n+1}\sqrt{\pi}(-1)^n\left(\frac{3}{2}\right)_n\left(-\frac{1}{2}-n\right)_m(-n)_m\,(-n)_{n-m}$$
$$\times{}_3F_2\left[\begin{array}{c}-m,\frac{3}{2}+n,1+n;\\ \frac{3}{2}+n-m,1+n-m;\end{array}1\right], \tag{2.15b}$$

where $m<n$.

Case II(c): Put $m=0$ and $n\geq 1$ in the Eq. (2.15a), we get

$$\int_{-\infty}^{+\infty}\exp(-x^2)\{H_{2n+1}(x)\}^2dx=2^{4n+1}\sqrt{\pi}\left(\frac{3}{2}\right)_n n!. \tag{2.16}$$

Case II(d): Put $m=1$ and $n\geq 2$ in the Eq. (2.15a), we get

$$\int_{-\infty}^{+\infty}(x^2)\exp(-x^2)\{H_{2n+1}(x)\}^2dx=2^{4n}\sqrt{\pi}\left(\frac{3}{2}\right)_n n!(4n+3). \tag{2.17}$$

Above orthogonal properties (2.8), (2.16) and the resulting integrals (2.9), (2.17) were proved by the researchers with the help of Rodrigue's formula, successive integration by parts, some recurrence relations associated with classical Hermite's polynomials and Hermite's differential equation.

3 Second integral

Consider the second integral:

$$I_4=\int_0^{\infty}\exp(-x^2)H_{2k}(x)H_{2s+1}(x)dx. \tag{3.1}$$

Applying the definition of classical Hermite's polynomials (1.8a) and (1.8b) in the E-q. (3.1), we get

$$\begin{aligned}I_4&=(-1)^{k+s}2^{2k+2s+1}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_s\int_0^{\infty}x\exp(-x^2)\,{}_1F_1\left[\begin{array}{c}-k;\\ \frac{1}{2};\end{array}x^2\right]{}_1F_1\left[\begin{array}{c}-s;\\ \frac{3}{2};\end{array}x^2\right]dx\\ &=(-1)^{k+s}2^{2k+2s+1}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_s\sum_{p=0}^{k}\frac{(-k)_p}{(\frac{1}{2})_p p!}\sum_{q=0}^{s}\frac{(-s)_q}{(\frac{3}{2})_q q!}\int_0^{\infty}\exp(-x^2)x^{2p+2q+1}dx.\end{aligned} \tag{3.2}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the Eq. (3.2), we get

$$\begin{aligned} I_4 &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_s\sum_{p=0}^{k}\frac{(-k)_p(1)_p}{(\frac{1}{2})_p p!}\sum_{q=0}^{s}\frac{(-s)_q(1+p)_q}{(\frac{3}{2})_q q!} \\ &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_s\sum_{p=0}^{k}\frac{(-k)_p(1)_p}{(\frac{1}{2})_p p!}\,{}_2F_1\left[\begin{matrix} -s,1+p; \\ \frac{3}{2}; \end{matrix}\quad 1\right]. \end{aligned} \tag{3.3}$$

Applying the Gauss's summation theorem (1.5) in the Eq. (3.3), we get

$$\begin{aligned} I_4 &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\sum_{p=0}^{k}\frac{(-k)_p(1)_p}{(\frac{1}{2})_p p!}(\frac{1}{2}-p)_s \\ &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_s\sum_{p=0}^{k}\frac{(-k)_p(1)_p}{(\frac{1}{2}-s)_p p!} \\ &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_s\,{}_2F_1\left[\begin{matrix} -k,1; \\ \frac{1}{2}-s; \end{matrix}\quad 1\right]. \end{aligned} \tag{3.4}$$

Again applying the Gauss's summation theorem (1.5) in the Eq. (3.4), we get

$$\begin{aligned} I_4 &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_s\frac{(-\frac{1}{2}-s)_k}{(\frac{1}{2}-s)_k} \\ &= (-1)^{k+s}2^{2k+2s}\left(-\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_s\frac{(\frac{1}{2}-k)_s}{(\frac{3}{2}-k)_s}, \end{aligned} \tag{3.5a}$$

$$\int_0^\infty \exp(-x^2)H_{2k}(x)H_{2s+1}(x)dx = \frac{(-1)^{k+s}2^{2k+2s}(\frac{1}{2})_k(\frac{3}{2})_s}{(1-2k+2s)}. \tag{3.5b}$$

4 Third integral

Consider the third integral:

$$I_5 = \int_0^\infty \exp(-x^2)H_{2k}(x)H_{2s}(x)dx. \tag{4.1}$$

Applying the definition of classical Hermite's polynomials (1.8a) in the Eq. (4.1), we get

$$\begin{aligned} I_5 &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_s\int_0^\infty \exp(-x^2)\,{}_1F_1\left[\begin{matrix} -k; \\ \frac{1}{2}; \end{matrix}\quad x^2\right]{}_1F_1\left[\begin{matrix} -s; \\ \frac{1}{2}; \end{matrix}\quad x^2\right]dx \\ &= (-1)^{k+s}2^{2k+2s}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_s\sum_{p=0}^{k}\frac{(-k)_p}{(\frac{1}{2})_p p!}\sum_{q=0}^{s}\frac{(-s)_q}{(\frac{1}{2})_q q!}\int_0^\infty \exp(-x^2)x^{2p+2q}dx. \end{aligned} \tag{4.2}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the Eq. (4.2), we get

$$\begin{aligned} I_5 &= (-1)^{k+s} 2^{2k+2s-1} \sqrt{\pi} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_s \sum_{p=0}^{k} \frac{(-k)_p}{p!} \sum_{q=0}^{s} \frac{(-s)_q (\frac{1}{2}+p)_q}{(\frac{1}{2})_q q!} \\ &= (-1)^{k+s} 2^{2k+2s-1} \sqrt{\pi} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_s \sum_{p=0}^{k} \frac{(-k)_p}{p!} {}_2F_1 \left[\begin{matrix} -s, \frac{1}{2}+p; \\ \frac{1}{2}; \end{matrix} \; 1 \right]. \end{aligned} \tag{4.3}$$

Applying the Gauss's summation theorem (1.5) in the Eq. (4.3), we get

$$\int_0^\infty \exp(-x^2) H_{2k}(x) H_{2s}(x) dx = (-1)^{k+s} 2^{2k+2s-1} \sqrt{\pi} \left(\frac{1}{2}\right)_k \sum_{p=0}^{k} \frac{(-k)_p (-p)_s}{p!}. \tag{4.4}$$

Case I: When $s > k$ in the Eq. (4.4) and using the identity (1.10), we get

$$\int_0^\infty \exp(-x^2) H_{2k}(x) H_{2s}(x) dx = 0. \tag{4.5}$$

Case II: When $s < k$ in the Eq. (4.4) and using the identity (1.12), we get

$$\begin{aligned} \int_0^\infty \exp(-x^2) H_{2k}(x) H_{2s}(x) dx &= (-1)^{k+s} 2^{2k+2s-1} \sqrt{\pi} \left(\frac{1}{2}\right)_k k! \sum_{p=0}^{k-s} \frac{(-1)^p}{(k-p-s)! p!} \\ &= (-1)^{k+s} 2^{2k+2s-1} \sqrt{\pi} \left(\frac{1}{2}\right)_k k! \frac{1}{(k-s)!} \sum_{p=0}^{k-s} (-1)^p \binom{k-s}{p}. \end{aligned}$$

Now using combinatorial identity (1.9), we get

$$\int_0^\infty \exp(-x^2) H_{2k}(x) H_{2s}(x) dx = 0. \tag{4.6}$$

Case III: When $s = k$ in the Eq. (4.4) and using the identity (1.11), we get

$$\int_0^\infty \exp(-x^2) \{H_{2k}(x)\}^2 dx = 2^{4k-1} \left(\frac{1}{2}\right)_k \sqrt{\pi} k!. \tag{4.7}$$

5 Fourth integral

Consider the fourth integral:

$$I_6 = \int_0^\infty \exp(-x^2) H_{2k+1}(x) H_{2s+1}(x) dx. \tag{5.1}$$

Applying the definition of classical Hermite's polynomials (1.8b) in the Eq. (5.1), we get

$$\begin{aligned} I_6 &= (-1)^{k+s} 2^{2k+2s+2} \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_s \int_0^\infty \exp(-x^2) x^2 {}_1F_1\left[\begin{array}{c} -k; \\ \frac{3}{2}; \end{array} x^2\right] {}_1F_1\left[\begin{array}{c} -s; \\ \frac{3}{2}; \end{array} x^2\right] dx \\ &= (-1)^{k+s} 2^{2k+2s+2} \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_s \sum_{p=0}^{k} \frac{(-k)_p}{(\frac{3}{2})_p p!} \sum_{q=0}^{s} \frac{(-s)_q}{(\frac{3}{2})_q q!} \int_0^\infty \exp(-x^2) x^{2p+2q+2} dx. \end{aligned} \tag{5.2}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the Eq. (5.2), we get

$$\begin{aligned} I_6 &= (-1)^{k+s} 2^{2k+2s} \sqrt{\pi} \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_s \sum_{p=0}^{k} \frac{(-k)_p}{p!} \sum_{q=0}^{s} \frac{(-s)_q (\frac{3}{2}+p)_q}{(\frac{3}{2})_q q!} \\ &= (-1)^{k+s} 2^{2k+2s} \sqrt{\pi} \left(\frac{3}{2}\right)_k \left(\frac{3}{2}\right)_s \sum_{p=0}^{k} \frac{(-k)_p}{p!} {}_2F_1\left[\begin{array}{c} -s, \frac{3}{2}+p; \\ \frac{3}{2}; \end{array} 1\right]. \end{aligned} \tag{5.3}$$

Applying the Gauss's summation theorem (1.5) in the Eq. (5.3), we get

$$\int_0^\infty \exp(-x^2) H_{2k+1}(x) H_{2s+1}(x) dx = (-1)^{k+s} 2^{2k+2s} \sqrt{\pi} \left(\frac{3}{2}\right)_k \sum_{p=0}^{k} \frac{(-k)_p (-p)_s}{p!}. \tag{5.4}$$

Case I: When $s>k$ in the Eq. (5.4) and using the identity (1.10), we get

$$\int_0^\infty \exp(-x^2) H_{2k+1}(x) H_{2s+1}(x) dx = 0. \tag{5.5}$$

Case II: When $s<k$ in the Eq. (5.4) and using the identity (1.12), we get

$$\begin{aligned} \int_0^\infty \exp(-x^2) H_{2k+1}(x) H_{2s+1}(x) dx &= (-1)^{k+s} 2^{2k+2s} \sqrt{\pi} \left(\frac{3}{2}\right)_k k! \sum_{p=0}^{k-s} \frac{(-1)^p}{(k-p-s)! p!} \\ &= (-1)^{k+s} 2^{2k+2s} \sqrt{\pi} \left(\frac{3}{2}\right)_k k! \frac{1}{(k-s)!} \sum_{p=0}^{k-s} (-1)^p \binom{k-s}{p}. \end{aligned}$$

Now using combinatorial identity (1.9), we get

$$\int_0^\infty \exp(-x^2) H_{2k+1}(x) H_{2s+1}(x) dx = 0. \tag{5.6}$$

Case III: When $s=k$ in the Eq. (5.4) and using the identity (1.11), we get

$$\int_0^\infty \exp(-x^2) \{H_{2k+1}(x)\}^2 dx = 2^{4k} \sqrt{\pi} \left(\frac{3}{2}\right)_k k!. \tag{5.7}$$

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