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On a Difference Matrix and Its Properties

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Abstract. In the present paper, a new difference matrix via difference operator D is introduced. Let $x=(x_k)$ be a sequence of real numbers, then the difference operator D is defined by $D(x)_n=\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}x_k$, where $n=0,1,2,3,\cdots$. Several interesting properties of the new operator D are discussed.

Key Words: Difference operators Δ^α, $B(r,s)$, $B(r,s,t,u)$, D, Cesàro operator $C(1,1)$.
AMS Subject Classifications: 46A45, 46A35, 46B45

1 Introduction, preliminaries and definitions

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of all real numbers and nonnegative integers, respectively. Let w be the space of all real valued sequences and X and Y be two subspaces of w. Then we define a matrix mapping $A:X\to Y$, as

$$(Ax)_n:=\sum_k a_{nk}x_k, \quad n\in\mathbb{N}. \tag{1.1}$$

In fact, for $x=(x_k)\in X$, Ax is called as the $A-$transform of x provided the series in (1.1) converges for each $n\in\mathbb{N}$. Moreover, the matrix $A=(a_{nk})$, $(n,k\in\mathbb{N})$ is also regarded as a linear operator. By ℓ_∞, c and c_0, we denote the spaces of all bounded, convergent and null sequences, respectively, normed by $\|x\|_\infty=\sup_k|x_k|$. Initially, Kızmaz [14] introduced the idea of difference sequence spaces associated with the spaces ℓ_∞,c and c_0 by defining the forward difference operator Δ of order one, where

$$(\Delta x)_k=x_k-x_{k+1}, \quad k\in\mathbb{N}.$$

Later on, these sequence spaces have been generalized to the case of integral order m by Et and Çolak [12] using the operator Δ^m and

$$(\Delta^m x)_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i}, \quad k\in\mathbb{N}.$$

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Recently, Baliarsingh [3] (see also [4–6, 8]) generalized the above spaces by introducing fractional difference operator Δ^{α}, where

$$(\Delta^{\alpha}x)_k=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}x_{k+i},\quad k\in\mathbb{N}.$$

In fact, for most of the cases the new operators generated on various sequence spaces can be derived from respective limiting conditions of the triangular matrix A. The summation operator S (see [11]) derived from n-th partial sum of the sequence x is defined by $S=(s_{nk})$, where

$$s_{nk}=\begin{cases}1, & k\le n,\\ 0, & \text{otherwise.}\end{cases}$$

The well known Cesàro operator $C(1,1)$ of order one is defined by $C(1,1)=(c_{nk})$ (see [15]), where

$$c_{nk}=\begin{cases}\dfrac{1}{n+1}, & k\le n,\\ 0, & \text{otherwise.}\end{cases}$$

The backward difference operator $\Delta^{(r)}$ of order r is defined by $\Delta^{(r)}=(\delta^{(r)}_{nk})$ (see [1]), where

$$\delta^{(r)}_{nk}=\begin{cases}(-1)^{n-k}\binom{r}{n-k}, & k\le n,\\ 0, & \text{otherwise.}\end{cases}$$

Similarly, difference operator associated with four tuple band matrix $B(r,s,t,u)=(b_{nk})$ (see [7]) is defined by

$$b_{nk}=\begin{cases}r, & k=n,\\ s, & k=n-1,\\ t, & k=n-2,\\ u, & k=n-3,\\ 0, & \text{otherwise,}\end{cases}$$

where $r,s,t,u\in\mathbb{R}$ with the condition that $r\neq 0$. In particular, for $t=0$ and $u=0$, $B(r,s,t,u)$ reduces to the difference operator $B(r,s)$, studied by Altay and Başar [2] (see also [9]) whereas for $u=0$, it reduces to the difference operator $B(r,s,t)$, studied by Furkan et al. [13].

Let $x=(x_k)$ be a sequence in w. Now, we define the generalized difference operator D as

$$(Dx)_n=\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}x_k,\quad n\in\mathbb{N}. \tag{1.2}$$

Clearly, the difference operator D is represented by the lower triangular matrix $D=(d_{nk})$, where

$$d_{nk}=\begin{cases}(-1)^k\binom{n}{n-k}, & k\le n,\\ 0, & \text{otherwise.}\end{cases} \tag{1.3}$$

Equivalently, in componentwise, one may also write

$$D=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots\\ 1 & -1 & 0 & 0 & 0 & \cdots\\ 1 & -2 & 1 & 0 & 0 & \cdots\\ 1 & -3 & 3 & -1 & 0 & \cdots\\ 1 & -4 & 6 & -4 & 1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, we illustrate the following numerical examples concerning the sequence Dx:

- Let $x=e=(1,1,1,\cdots)$, then $(Dx)_n=0$ for all $n\ge 1$ and 1 for $n=0$.
- Let $x=(x_n)$ with $x_n=(-1)^n$, then $(Dx)_n=2^n$, for all $n\in\mathbb{N}$.
- Let $x=(x_n)$ with $x_n=\frac{1+(-1)^n}{2}$, then $(Dx)_n=2^{n-1}$, for all $n\in\mathbb{N}$.
- Let $x=(x_n)$ with $x_n=n$, then

$$(Dx)_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}k=-n\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}=0 \quad \text{for all } n\in\mathbb{N}.$$

Now, we state some known identities (see [10, pp. 14-17]) involving binomial coefficients which are being directly used in deriving certain new results on the difference matrix D.

Lemma 1.1. **(i)** *Let $m,n\ge 0$ and $0\le k\le \min\{m,n\}$, then*

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}=\binom{m+n}{k}.$$

(ii) *Let $0\le k\le n$, then*

$$\sum_{i=0}^{k}\binom{n+i}{i}\binom{n-i}{k-i}=\binom{2n+1}{k}.$$

(iii) *Let $0\le k\le n$, then*

$$\sum_{i=0}^{k}(-1)^i\binom{n}{i}=\begin{cases}1, & n=0,\\ (-1)^k\binom{n-1}{k}, & n\ge 1.\end{cases}$$

2 Main results

In this section, we provide certain new results involving binomial coefficients and apply them while taking the transform of the operator D.

Theorem 2.1. **(i)** *Let* $0\le k\le n$ *and* $r\in\mathbb{N}$*, then*

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{n+r}{i}\binom{n-i}{k-i}=(-1)^k\frac{\Gamma(-r+1)}{k!\Gamma(-r-k+1)}.$$

(ii) *Let* $0\le k\le n$*, then*

$$\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{k+i}}{n+1-i}\binom{n}{i}\binom{n-j}{k-j}=\begin{cases}1, & n=0,\\ 0, & n\ge 1.\end{cases}$$

(iii) *Let* $n\in\mathbb{N}$*, then*

$$\sum_{i=0}^{n}(-1)^i i\binom{n}{n-i}=0.$$

Proof. **(i)** Suppose $0\le k\le n$ and $r\in\mathbb{N}$, then

$$\begin{aligned}(-1)^{k-i}\binom{n+r}{i}&=(-1)^{k-i}\frac{(n+r)(n+r-1)\cdots(n+r-i+1)(n+r-i)!}{i!((n+r-i)!}\\&=(-1)^k\frac{(-n-r)(-n-r+1)\cdots(-n-r+i-1)}{i!}\\&=(-1)^k\frac{(-n-r-1)!(-n-r)(-n-r+1)\cdots(-n-r+i-1)}{i!(-n-r-1)!}\\&=(-1)^k\binom{-n-r+i-1}{i}.\end{aligned}$$

Now, the left hand side of (i) becomes

$$\begin{aligned}\sum_{i=0}^{k}(-1)^{k-i}\binom{n+r}{i}\binom{n-i}{k-i}&=\sum_{i=0}^{k}(-1)^k\binom{-n-r+i-1}{i}\binom{n-i}{k-i}\\&=(-1)^k\binom{-n-r+i-1+n-i+1}{k}\\&=(-1)^k\binom{-r}{k}=(-1)^k\frac{\Gamma(-r+1)}{k!\Gamma(-r-k+1)}.\end{aligned}$$

(ii) Suppose $n=0$, then the only possible value of k is 0, thus

$$\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{k+i}}{n+1-i}\binom{n}{i}\binom{n-j}{k-j}=1.$$

Again for $n\geq 1$, from (i) of Lemma 1.1, we have

$$\begin{aligned}\sum_{j=0}^{k}\sum_{i=0}^{j}\frac{(-1)^{k+i}}{n+1-i}\binom{n}{i}\binom{n-j}{k-j}&=\frac{1}{n+1}\sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{k+i}\binom{n+1}{i}\binom{n-j}{k-j}\\&=\frac{(-1)^k}{n+1}\sum_{j=0}^{k}\binom{n-j}{k-j}\sum_{i=0}^{j}(-1)^i\binom{n+1}{i}\\&=\frac{(-1)^k}{n+1}\sum_{j=0}^{k}\binom{n-j}{k-j}(-1)^j\binom{n}{j}\\&=\frac{(-1)^k}{n+1}\binom{n}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}=0.\end{aligned}$$

The proof of (iii) is obvious, hence omitted. □

Theorem 2.2. *The difference operator D is a linear operator and has its own inverse i.e., $D=D^{-1}$ or $D^2=I$, where I represents the identity operator.*

Proof. Linearity of D is clear. It is found from Eq. (1.2) that

$$\begin{aligned}(Dx)_n&=\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}x_k,\quad n\in\mathbb{N},\\&=\binom{n}{n}x_0-\binom{n}{n-1}x_1+\binom{n}{n-2}x_2-\binom{n}{n-3}x_3+\cdots+(-1)^n\binom{n}{0}x_n.\end{aligned}$$

Now, if we apply the operator D^2, the composition of the operator D with itself, to the sequence (x_n), then we have

$$\begin{aligned}(D^2x)_n=&D\left(\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}x_k\right),\quad n\in\mathbb{N},\\=&\binom{n}{n}D(x)_0-\binom{n}{n-1}D(x)_1+\binom{n}{n-2}D(x)_2-\binom{n}{n-3}D(x)_3+\cdots+(-1)^n\binom{n}{0}(Dx)_n\\=&x_0-n(x_0-x_1)+\frac{n(n-1)}{2}(x_0-2x_1+x_2)-\frac{n(n-1)(n-2)}{6}(x_0-3x_1+3x_2-x_3)+\cdots\\&+(-1)^n(x_0-nx_1+\frac{n(n-1)}{2}x_2-\frac{n(n-1)(n-2)}{6}x_3+\cdots+(-1)^nx_n)\\=&\left[\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^n\binom{n}{n}\right]x_0\\&+\binom{n}{1}\left[\binom{n-1}{0}-\binom{n-1}{1}+\binom{n-1}{2}-\binom{n-1}{3}+\cdots+(-1)^{n-1}\binom{n-1}{n-1}\right]x_1\\&+\binom{n}{2}\left[\binom{n-2}{0}-\binom{n-2}{1}+\binom{n-2}{2}-\binom{n-2}{3}+\cdots+(-1)^{n-2}\binom{n-2}{n-2}\right]x_2\\&+\cdots\binom{n}{n-1}\left[\binom{1}{0}-\binom{1}{1}\right]x_{n-1}+\binom{n}{n}x_n=x_n.\end{aligned}$$

This completes the proof. □

Theorem 2.3. **(i)** *Let $\Delta^{(r)}$ be the difference operator of order r, then $D\Delta^{(r)}D=\Delta^{(-r)}$, where $\Delta^{(-r)}$ represents the inverse operator of $\Delta^{(r)}$.*

(ii) *Let $C(1,1)$ be the Cesàro operator of order one, then $DC(1,1)D=C(1,1)_d$, where $C(1,1)_d$ represents a diagonal matrix with the diagonal elements $1/(n+1)$ for all $n\in\mathbb{N}$.*

(iii) *Let S be the summation operator, then $DSD=\Delta$, where Δ is the difference operator of order one.*

(iv) *Let $B(r,s)$ be the double band matrix, then $DB(r,s)D=B_1$, where $B_1=(b^1_{nk})$ with*

$$b^1_{nk}=\begin{cases} r, & k=n,\\ -s, & 0\le k<n,\\ 0, & k>n.\end{cases}$$

(v) *Let $B(r,s,t)$ be the triple band matrix, then $DB(r,s,t)D=B_2$, where $B_2=(b^2_{nk})$ with*

$$b^2_{nk}=\begin{cases} r, & k=n,\\ -s+(n-k-1)t, & 0\le k<n,\\ 0, & k>n.\end{cases}$$

(vi) *Let $B(r,s,t,u)$ be the fourth order band matrix, then $DB(r,s,t,u)D=B_3$, where $B_3=(b^3_{nk})$ and*

$$b^3_{nk}=\begin{cases} r, & k=n,\\ -s+(n-k-1)t+\dfrac{(n-k-1)(n-k-2)}{2}u, & 0\le k<n,\\ 0, & k>n.\end{cases}$$

Proof. **(i)** Let r be a nonnegative integer, then the rth backward difference of the sequence $x=(x_n)$ is given by

$$(\Delta^{(r)}x)_n=\sum_{k=0}^{n}(-1)^k\binom{r}{k}x_{n-k},\quad n\in\mathbb{N},$$

and subsequently, the inverse transform of the rth difference of the sequence x is defined by

$$(\Delta^{(-r)}x)_n=\sum_{k=0}^{n}(-1)^k\frac{\Gamma(-r+1)}{k!\Gamma(-r-k+1)}x_{n-k},\quad n\in\mathbb{N},$$

where Γ denotes well known Euler gamma function.

Now, consider the rth difference of the D transformation of the sequence $x=(x_n)$, then

$$
\begin{aligned}
&\Delta^r(Dx)_n\\
=&\Delta^r\left[(-1)^n\binom{n}{0}x_n+(-1)^{n-1}\binom{n}{1}x_{n-1}+\cdots+\binom{n}{n-2}x_2-\binom{n}{n-1}x_1+\binom{n}{n}x_0\right]\\
=&(-1)^n\binom{n}{0}\left[\binom{r}{0}x_n-\binom{r}{1}x_{n-1}+\cdots+(-1)^{r-1}\binom{r}{r-1}x_{n-r-1}+(-1)^r\binom{r}{r}x_{n-r}\right]\\
&+(-1)^{n-1}\binom{n}{1}\left[\binom{r}{0}x_{n-1}-\binom{r}{1}x_{n-2}+\cdots+(-1)^{r-2}\binom{r}{r-1}x_{n-r-2}+(-1)^r\binom{r}{r}x_{n-r-1}\right]+\cdots\\
&+\binom{n}{n-2}\left[\binom{r}{0}x_2-\binom{r}{1}x_1+\binom{r}{2}x_0\right]-\binom{n}{n-1}\left[\binom{r}{0}x_1-\binom{r}{1}x_0\right]+\binom{n}{n}\binom{r}{0}x_0\\
=&(-1)^nx_n+(-1)^{n-1}\left[\binom{n}{0}\binom{r}{1}+\binom{n}{1}\binom{r}{0}\right]x_{n-1}\\
&+(-1)^{n-2}\left[\binom{n}{0}\binom{r}{2}+\binom{n}{1}\binom{r}{1}+\binom{n}{2}\binom{r}{0}\right]x_{n-2}+(-1)^{n-3}\left[\sum_{i=0}^{3}\binom{n}{i}\binom{r}{3-i}\right]x_{n-2}\\
&+(-1)^{n-4}\left[\sum_{i=0}^{4}\binom{n}{i}\binom{r}{4-i}\right]x_{n-2}+\cdots+\left[\sum_{i=0}^{n}\binom{n}{i}\binom{r}{n-i}\right]x_0\\
=&(-1)^nx_n+(-1)^{n-1}\binom{n+r}{1}x_{n-1}+(-1)^{n-2}\binom{n+r}{2}x_{n-2}+\cdots+\binom{n+r}{n}x_0.
\end{aligned}
$$

Now, applying the operator D to the sequence $\Delta^r(Dx)$, it is observed that

$$
\begin{aligned}
&D(\Delta^r(Dx))_n\\
=&D\left[(-1)^nx_n+(-1)^{n-1}\binom{n+r}{1}x_{n-1}+(-1)^{n-2}\binom{n+r}{2}x_{n-2}+\cdots+\binom{n+r}{n}x_0\right]\\
=&(-1)^n\left[(-1)^n\binom{n}{0}x_n+(-1)^{n-1}\binom{n}{1}x_{n-1}+\cdots+\binom{n}{n-2}x_2-\binom{n}{n-1}x_1+\binom{n}{n}x_0\right]\\
&+(-1)^{n-1}\binom{n+r}{1}\left[(-1)^{n-1}\binom{n-1}{0}x_{n-1}+(-1)^{n-2}\binom{n-1}{1}x_{n-2}+\cdots\right.\\
&\left.-\binom{n-1}{n-2}x_1+\binom{n-1}{n-1}x_0\right]+\cdots+\binom{n+r}{n}x_0\\
=&x_n+\left[-\binom{n+r}{0}\binom{n}{1}+\binom{n+r}{1}\binom{n-1}{0}\right]x_{n-1}+\left[\sum_{i=0}^{2}(-1)^{2-i}\binom{n+r}{i}\binom{n-i}{2-i}\right]x_{n-2}+\cdots\\
&+\left[\sum_{i=0}^{n}(-1)^{n-i}\binom{n+r}{i}\right]x_0\\
=&x_n+(r)x_{n-1}+\left(\frac{r(r+1)}{2}\right)x_{n-2}+\cdots+\left(\frac{r(r+1)\cdots(r+n-1)}{n!}\right)x_0\\
=&(\Delta^{-r}x)_n.
\end{aligned}
$$

(ii) Applying Cesàro mean $C(1,1)$ of order one to the $D-$transform of the sequence

$x=(x_n)$, it is seen that

$$\begin{aligned}
&C(1,1)(Dx)_n\\
=&C(1,1)\left[(-1)^n\binom{n}{0}x_n+(-1)^{n-1}\binom{n}{1}x_{n-1}+\cdots+\binom{n}{n-2}x_2-\binom{n}{n-1}x_1+\binom{n}{n}x_0\right]\\
=&\frac{(-1)^n}{n+1}\binom{n}{0}[x_n+x_{n-1}+x_{n-2}+\cdots+x_0]+\frac{(-1)^{n-1}}{n}\binom{n}{1}[x_{n-1}+x_{n-2}+x_{n-3}+\cdots+x_0]\\
&+\frac{(-1)^{n-2}}{n-1}\binom{n}{2}[x_{n-2}+x_{n-3}+x_{n-4}+\cdots+x_0]+\cdots-\frac{1}{2}\binom{n}{n-1}[x_1+x_0]+\binom{n}{n}x_0\\
=&\frac{(-1)^n}{n+1}x_n+\left[\frac{(-1)^n}{n+1}\binom{n}{0}+\frac{(-1)^{n-1}}{n}\binom{n}{1}\right]x_{n-1}+\left[\sum_{i=0}^{2}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\right]x_{n-2}+\cdots\\
&+\left[\sum_{i=0}^{n}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\right]x_0.
\end{aligned}$$

Now, performing the difference operator D in the above sequence, we obtain

$$\begin{aligned}
&D(C(1,1)(Dx))_n\\
=&D\left[\frac{(-1)^n}{n+1}x_n+\left[\sum_{i=0}^{1}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\right]x_{n-1}+\cdots+\left[\sum_{i=0}^{n}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\right]x_0\right]\\
=&\frac{(-1)^n}{n+1}\left[(-1)^n\binom{n}{0}x_n+(-1)^{n-1}\binom{n}{1}x_{n-1}+\cdots+\binom{n}{n-2}x_2-\binom{n}{n-1}x_1+\binom{n}{n}x_0\right]\\
&+\sum_{i=0}^{1}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\left[(-1)^{n-1}\binom{n-1}{0}x_{n-1}+(-1)^{n-2}\binom{n-1}{1}x_{n-2}+\cdots-\binom{n-1}{n-2}x_1+\binom{n-1}{n-1}x_0\right]\\
&+\cdots+\left[\sum_{i=0}^{n}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\right]x_0\\
=&\frac{x_n}{n+1}+\left[\frac{(-1)^{2n-1}}{n+1}\binom{n}{1}+\sum_{i=0}^{1}\frac{(-1)^{2n-1-i}}{n+1-i}\binom{n}{i}\binom{n-1}{0}\right]x_{n-1}\\
&+\left[\frac{(-1)^{2n-2}}{n+1}\binom{n}{2}+\sum_{i=0}^{1}\frac{(-1)^{2n-2-i}}{n+1-i}\binom{n}{i}\binom{n-1}{1}+\sum_{i=0}^{2}\frac{(-1)^{2n-2-i}}{n+1-i}\binom{n}{i}\binom{n-2}{0}\right]x_{n-2}+\cdots\\
&+\left[\frac{(-1)^n}{n+1}+\sum_{i=0}^{1}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}+\cdots+\sum_{i=0}^{n}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\right]x_0\\
=&\frac{x_n}{n+1}+\left[\sum_{j=0}^{1}\sum_{i=0}^{j}\frac{(-1)^{2n-1-i}}{n+1-i}\binom{n}{i}\binom{n-j}{1-j}\right]x_{n-1}+\left[\sum_{j=0}^{2}\sum_{i=0}^{j}\frac{(-1)^{2n-2-i}}{n+1-i}\binom{n}{i}\binom{n-j}{2-j}\right]x_{n-2}\\
&+\cdots+\left[\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{(-1)^{n-i}}{n+1-i}\binom{n}{i}\right]x_0\\
=&\frac{x_n}{n+1}.
\end{aligned}$$

(iii) It is noted that the inverse of the summation operator S is equal to the the well known 1st order backward difference operator Δ. From **(i)**, it is concluded that

$$(DSD)^{-1}=DS^{-1}D=D\Delta D=\Delta^{-1}.$$

Taking inverse on both sides we get the result as desired.

Proofs of **(iv)**, **(v)** and **(vi)** are similar, hence omitted. □

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