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Approximation by Nörlund Means of Hexagonal Fourier Series

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Abstract. Let f be an $H-$periodic Hölder continuous function of two real variables. The error $\|f-N_n(p;f)\|$ is estimated in the uniform norm and in the Hölder norm, where $p=(p_k)_{k=0}^{\infty}$ is a nonincreasing sequence of positive numbers and $N_n(p;f)$ is the nth Nörlund mean of hexagonal Fourier series of f with respect to $p=(p_k)_{k=0}^{\infty}$.

Key Words: Hexagonal Fourier series, Hölder class, Nörlund mean.
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1 Introduction

In general, approximation problems of functions of several real variables defined on cubes of the Euclidean space are studied by assuming that the functions are periodic in each of their variables (see, for example [10, Sections 5.3 and 6.3] and [12, Vol II, Chapter XVII]). But in the case of non tensor-product domains, for example in hexagonal domains of $\mathbb{R}^2$, another definition of periodicity is needed. For such domains most useful periodicity is the periodicity defined by lattices. We refer to [5] for general information about lattices.

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2},\frac{1}{2}\right)^2$, the simplest lattice is the hexagonal lattice and the simplest spectral set is the regular hexagon.

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H=\begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}$$

and

$$\Omega_H=\left\{(x_1,x_2)\in\mathbb{R}^2:\ -1\le x_2,\ \frac{\sqrt{3}}{2}x_1\pm\frac{1}{2}x_2<1\right\}.$$

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It is more convenient to use the homogeneous coordinates (t_1,t_2,t_3) that satisfy $t_1+t_2+t_3=0$. If we define

$$t_1:=-\frac{x_2}{2}+\frac{\sqrt{3}x_1}{2},\quad t_2:=x_2,\quad t_3:=-\frac{x_2}{2}-\frac{\sqrt{3}x_1}{2}, \tag{1.1}$$

the hexagon Ω_H becomes

$$\Omega=\big\{(t_1,t_2,t_3)\in\mathbb{R}^3:\ -1\le t_1,t_2,-t_3<1,\ t_1+t_2+t_3=0\big\},$$

which is the intersection of the plane $t_1+t_2+t_3=0$ with the cube $[-1,1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1+t_2+t_3=0$, that is

$$\mathbb{R}^3_H=\big\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3: t_1+t_2+t_3=0\big\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is $\mathbb{Z}^3_H=\mathbb{Z}^3\cap\mathbb{R}^3_H$.

A function $f:\mathbb{R}^2\to\mathbb{C}$ is called $H-$periodic if

$$f(x+Hk)=f(x)$$

for all $k\in\mathbb{Z}^2$ and $x\in\mathbb{R}^2$. If we define $\mathbf{t}\equiv\mathbf{s}\ (mod 3)$ as

$$t_1-s_1\equiv t_2-s_2\equiv t_3-s_3\ (mod 3)$$

for $\mathbf{t}=(t_1,t_2,t_3)$, $\mathbf{s}=(s_1,s_2,s_3)\in\mathbb{R}^3_H$, it follows that the function f is $H-$periodic if and only if $f(\mathbf{t})=f(\mathbf{t}+\mathbf{s})$ whenever $\mathbf{s}\equiv\mathbf{0}\ (mod 3)$. It is clear that

$$\int_\Omega f(\mathbf{t}+\mathbf{s})d\mathbf{t}=\int_\Omega f(\mathbf{t})d\mathbf{t},\quad \big(\mathbf{s}\in\mathbb{R}^3_H\big),$$

holds for $H-$periodic integrable function f (see [11]).

$L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f,g\rangle_H:=\frac{1}{|\Omega|}\int_\Omega f(\mathbf{t})\overline{g(\mathbf{t})}d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω. The functions

$$\phi_{\mathbf{j}}(\mathbf{t}):=e^{\frac{2\pi i}{3}\langle\mathbf{j},\mathbf{t}\rangle},\quad \big(\mathbf{t}\in\mathbb{R}^3_H\big),$$

are $H-$periodic and by a theorem of B. Fuglede (see [2]) the set

$$\big\{\phi_{\mathbf{j}}(\mathbf{t}):\mathbf{j}\in\mathbb{Z}^3_H\big\}$$

becomes an orthonormal basis of $L^2(\Omega)$ (see also [5]).

For every natural number n, we define a subset of $\mathbb{Z}_H^3$ by

$$\mathbb{H}_n := \left\{ \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \le j_1, j_2, j_3 \le n \right\}.$$

Note that, $\mathbb{H}_n$ consists of all integer points inside the hexagon $n\overline{\Omega}$. Members of the set

$$\mathcal{H}_n := \operatorname{span}\left\{ \phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n \right\}, \quad (n \in \mathbb{N}),$$

are called hexagonal trigonometric polynomials. It is clear that the dimension of $\mathcal{H}_n$ is $\#\mathbb{H}_n = 3n^2 + 3n + 1$.

The hexagonal Fourier series of an $H-$periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{1.2}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}) \overline{\phi_{\mathbf{j}}(\mathbf{t})} d\mathbf{t}, \quad \left(\mathbf{j} \in \mathbb{Z}_H^3 \right).$$

The nth partial sum of the series (1.2) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \quad (n \in \mathbb{N}).$$

It is clear that

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t} - \mathbf{s}) D_n(\mathbf{s}) d\mathbf{s}, \tag{1.3}$$

where D_n is the Dirichlet kernel, defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \phi_{\mathbf{j}}(\mathbf{t}).$$

It is known that (see [5,9]) the Dirichlet kernel can be expressed as

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}), \quad (n \in \mathbb{N}), \tag{1.4}$$

where

$$\Theta_n(\mathbf{t}) := \frac{\sin \frac{(n+1)(t_1 - t_2)\pi}{3} \sin \frac{(n+1)(t_2 - t_3)\pi}{3} \sin \frac{(n+1)(t_3 - t_1)\pi}{3}}{\sin \frac{(t_1 - t_2)\pi}{3} \sin \frac{(t_2 - t_3)\pi}{3} \sin \frac{(t_3 - t_1)\pi}{3}} \tag{1.5}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3$.

More detailed information on hexagonal Fourier series can be found in [5] and [11].

2 Main results

Let $C_H(\overline{\Omega})$ be the Banach space of complex valued $H-$periodic continuous functions defined on $\mathbb{R}^3_H$, whose norm is the uniform norm:

$$\|f\|_{C_H(\overline{\Omega})}=\sup\{|f(\mathbf{t})|:\mathbf{t}\in\overline{\Omega}\}.$$

A function $f\in C_H(\overline{\Omega})$ is said to belong to the Hölder space $H^\alpha(\overline{\Omega})$ $(0<\alpha\leq 1)$ if

$$A^\alpha(f):=\sup_{\mathbf{t}\neq\mathbf{s}}\frac{|f(\mathbf{t})-f(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\alpha}<\infty,$$

where

$$\|\mathbf{t}\|:=\max\{|t_1|,|t_2|,|t_3|\},\quad \mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3_H.$$

The Hölder norm on $H^\alpha(\overline{\Omega})$ $(0<\alpha\leq 1)$ is defined by

$$\|f\|_{H^\alpha(\overline{\Omega})}:=\|f\|_{C_H(\overline{\Omega})}+A^\alpha(f),\quad \left(f\in H^\alpha(\overline{\Omega})\right).$$

Fejér and Abel summability of hexagonal Fourier series of functions belong to $C_H(\overline{\Omega})$ was studied by Y. Xu in [11]. In this paper, the author proved that Fejér and Abel-Poisson means of hexagonal Fourier series of a function $f\in C_H(\overline{\Omega})$ converges uniformly to this function on $\overline{\Omega}$. Later, in [4] and [3], the order of convergence of Fejér and Abel-Poisson means of hexagonal Fourier series of functions belong to $H^\alpha(\overline{\Omega})$ was estimated in uniform and Hölder norms, respectively. In this work we give estimates for the order of approximation of Nörlund means of hexagonal Fourier series in uniform and Hölder norms, and by this way we obtain analogues of theorems given in [8] and [1].

Let $p=(p_k)_{k=0}^\infty$ be a nonincreasing sequence of positive numbers. The nth Nörlund mean of the series (1.2) with respect to the sequence $p=(p_k)_{k=0}^\infty$ is defined by

$$N_n(p;f)(\mathbf{t}):=\frac{1}{P_n}\sum_{k=0}^n p_{n-k}S_k(f)(\mathbf{t}),\quad (n\in\mathbb{N}),$$

where

$$P_n:=\sum_{k=0}^n p_k.$$

By considering (1.3), we get

$$N_n(p;f)(\mathbf{t})=\frac{1}{|\Omega|}\int_\Omega f(\mathbf{t}-\mathbf{s})F_n(p;\mathbf{s})d\mathbf{s},$$

where

$$F_n(p;\mathbf{t}):=\frac{1}{P_n}\sum_{k=0}^n p_{n-k}D_k(\mathbf{t}).$$

In the case $p_k=1$, $(k=0,1,\cdots)$, $N_n(p;f)(\mathbf{t})$ coincide with the Fejér means

$$S_n^{(1)}(f)(\mathbf{t}):=\frac{1}{n+1}\sum_{k=0}^{n}S_k(f)(\mathbf{t}),\quad (n\in\mathbb{N}).$$

We shall write $A\lesssim B$ for the quantities A and B, if there exists a constant $K>0$ (K is an absolute constant, or a constant depending only on parameters which are not important for the questions involve in the paper) such that $A\le KB$ holds.

We estimate the order of approximation of Nörlund means of hexagonal Fourier series as follows:

Theorem 2.1. *Let $p=(p_k)_{k=0}^{\infty}$be a nonincreasing sequence of positive numbers and let $f\in H^{\alpha}\left(\overline{\Omega}\right)$, $(0<\alpha\le 1)$. Then,*

$$\|f-N_n(p;f)\|_{C_H(\overline{\Omega})}\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}. \tag{2.1}$$

Proof. Since $f\in H^{\alpha}\left(\overline{\Omega}\right)$, we have

$$\begin{aligned}|f(\mathbf{t})-N_n(p;f)(\mathbf{t})|&\le\frac{1}{|\Omega|}\int_{\Omega}|f(\mathbf{t})-f(\mathbf{t}-\mathbf{s})||F_n(p;\mathbf{s})|d\mathbf{s}\\&\lesssim\int_{\Omega}\|\mathbf{s}\|^{\alpha}|F_n(p;\mathbf{s})|d\mathbf{s}=\frac{1}{P_n}\int_{\Omega}\|\mathbf{s}\|^{\alpha}\left|\sum_{k=0}^{n}p_{n-k}D_k(\mathbf{s})\right|d\mathbf{s}\\&=\frac{1}{P_n}\int_{\Omega}\|\mathbf{s}\|^{\alpha}\left|p_n+\sum_{k=1}^{n}p_{n-k}(\Theta_k(\mathbf{s})-\Theta_{k-1}(\mathbf{s}))\right|d\mathbf{s}.\end{aligned}$$

The function

$$\|\mathbf{t}\|^{\alpha}\left|p_n+\sum_{k=1}^{n}p_{n-k}(\Theta_k(\mathbf{t})-\Theta_{k-1}(\mathbf{t}))\right|$$

is symmetric with respect to variables t_1,t_2 and t_3, where $\mathbf{t}=(t_1,t_2,t_3)\in\Omega$, hence it is sufficient to estimate the integral

$$I_n:=\int_{\Delta}\|\mathbf{t}\|^{\alpha}\left|p_n+\sum_{k=1}^{n}p_{n-k}(\Theta_k(\mathbf{t})-\Theta_{k-1}(\mathbf{t}))\right|d\mathbf{t},$$

where

$$\begin{aligned}\Delta:&=\left\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3_H:0\le t_1,t_2,-t_3\le 1\right\}\\&=\left\{(t_1,t_2):t_1\ge 0,\ t_2\ge 0,\ t_1+t_2\le 1\right\},\end{aligned}$$

which is one of the six equilateral triangles in $\overline{\Omega}$. By considering the formula (1.5), we obtain

$$\begin{aligned} I_n &= \int_{\Delta} \|\mathbf{t}\|^{\alpha} \left| p_n + \sum_{k=1}^{n} p_{n-k} \left(\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t}) \right) \right| d\mathbf{t} \\ &= \int_{\Delta} (t_1+t_2)^{\alpha} \left| p_n + \sum_{k=1}^{n} p_{n-k} \left(\begin{array}{c} \frac{\sin\frac{(k+1)(t_1-t_2)\pi}{3} \sin\frac{(k+1)(t_2-t_3)\pi}{3} \sin\frac{(k+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3} \sin\frac{(t_2-t_3)\pi}{3} \sin\frac{(t_3-t_1)\pi}{3}} \\ -\frac{\sin\frac{k(t_1-t_2)\pi}{3} \sin\frac{k(t_2-t_3)\pi}{3} \sin\frac{k(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3} \sin\frac{(t_2-t_3)\pi}{3} \sin\frac{(t_3-t_1)\pi}{3}} \end{array} \right) \right| d\mathbf{t}. \end{aligned}$$

If we use the change of variables

$$s_1 := \frac{t_1-t_3}{3} = \frac{2t_1+t_2}{3}, \quad s_2 := \frac{t_2-t_3}{3} = \frac{t_1+2t_2}{3}, \tag{2.2}$$

as in [11], we get

$$I_n = 3\int_{\widetilde{\Delta}} (s_1+s_2)^{\alpha} \left| p_n + \sum_{k=1}^{n} p_{n-k} \left(\begin{array}{c} \frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \\ -\frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \end{array} \right) \right| ds_1 ds_2,$$

where $\widetilde{\Delta}$ is the image of Δ in the plane, that is

$$\widetilde{\Delta} := \{(s_1,s_2) : 0 \le s_1 \le 2s_2,\ 0 \le s_2 \le 2s_1,\ s_1+s_2 \le 1\}.$$

Since the integrated function is symmetric with respect to s_1 and s_2, we have

$$I_n = 6\int_{\Delta^*} (s_1+s_2)^{\alpha} \left| p_n + \sum_{k=1}^{n} p_{n-k} \left(\begin{array}{c} \frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \\ -\frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)} \end{array} \right) \right| ds_1 ds_2,$$

where Δ^* is the half of $\widetilde{\Delta}$:

$$\Delta^* := \left\{ (s_1,s_2) \in \widetilde{\Delta} : s_1 \le s_2 \right\} = \{(s_1,s_2) : s_1 \le s_2 \le 2s_1,\ s_1+s_2 \le 1\}.$$

The change of variables

$$s_1 := \frac{u_1-u_2}{2}, \quad s_2 := \frac{u_1+u_2}{2}, \tag{2.3}$$

transforms the triangle Δ^* to the triangle

$$\Gamma := \left\{ (u_1,u_2) : 0 \le u_2 \le \frac{u_1}{3},\ 0 \le u_1 \le 1 \right\},$$

hence we have

$$I_n = 3\int_{\Gamma} u_1^{\alpha} \left| p_n + \sum_{k=1}^{n} p_{n-k} D_k^*(u_1,u_2) \right| du_1 du_2,$$

where

$$D_k^*(u_1,u_2)=\frac{\sin((k+1)(u_2)\pi)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin((u_2)\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$
$$-\frac{\sin(k(u_2)\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(k\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin((u_2)\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

By using elementary trigonometric identities, we obtain

$$D_k^*(u_1,u_2)=D_{k,1}^*(u_1,u_2)+D_{k,2}^*(u_1,u_2)+D_{k,3}^*(u_1,u_2), \tag{2.4}$$

where

$$D_{k,1}^*(u_1,u_2)=2\cos\left(\left(k+\frac{1}{2}\right)u_2\pi\right)\frac{\sin\left(\frac{1}{2}u_2\pi\right)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$
$$D_{k,2}^*(u_1,u_2)=2\cos\left(\left(k+\frac{1}{2}\right)\frac{u_1+u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin\left(\frac{1}{2}\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$
$$D_{k,3}^*(u_1,u_2)=2\cos\left(\left(k+\frac{1}{2}\right)\frac{u_1-u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{1}{2}\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

Since

$$\sin 2x+\sin 2y+\sin 2z=-4\sin x\sin y\sin z$$

for $x+y+z=0$, we also get the expression

$$D_k^*(u_1,u_2)=H_{k,1}(u_1,u_2)+H_{k,2}(u_1,u_2)+H_{k,3}(u_1,u_2), \tag{2.5}$$

where

$$H_{k,1}(u_1,u_2)=\frac{1}{2}\frac{\cos((2k+1)u_2\pi)}{\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$
$$H_{k,2}(u_1,u_2)=-\frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1+u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$
$$H_{k,3}(u_1,u_2)=\frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)}.$$

Since (p_n) and $\left(P_n/n^{1+\alpha}\right)$ are nonincreasing,

$$\int_\Gamma u_1^\alpha p_n du_1 du_2\le p_n\le\frac{P_n}{n+1}\le\frac{P_n}{n^\alpha}=\frac{nP_n}{n^{1+\alpha}}=\sum_{k=1}^n\frac{P_n}{n^{1+\alpha}}\le\sum_{k=1}^n\frac{P_k}{k^{1+\alpha}}.$$

Hence,

$$I_n\lesssim I_n^*+\sum_{k=1}^n\frac{P_k}{k^{1+\alpha}},$$

where

$$I_n^* := \int_\Gamma u_1^\alpha \left| \sum_{k=1}^n p_{n-k} D_k^*(u_1,u_2) \right| du_1 du_2.$$

If we partition the triangle Γ as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned} \Gamma_1 &= \left\{ (u_1,u_2) \in \Gamma : u_1 \le \frac{1}{n+1} \right\}, \\ \Gamma_2 &= \left\{ (u_1,u_2) \in \Gamma : u_1 \ge \frac{1}{n+1},\ u_2 \le \frac{1}{3(n+1)} \right\}, \\ \Gamma_3 &= \left\{ (u_1,u_2) \in \Gamma : u_1 \ge \frac{1}{n+1},\ u_2 \ge \frac{1}{3(n+1)} \right\}, \end{aligned}$$

we have

$$I_n^* = I_{n,1} + I_{n,2} + I_{n,3},$$

where

$$I_{n,j} := \int_{\Gamma_j} u_1^\alpha \left| p_n + \sum_{k=1}^n p_{n-k} D_k^*(u_1,u_2) \right| du_1 du_2, \quad j = 1,2,3.$$

We shall need the inequalities

$$\left| \frac{\sin nt}{\sin t} \right| \le n, \quad (n \in \mathbb{N}), \tag{2.6}$$

and

$$\sin t \ge \frac{2}{\pi} t, \quad \left(0 \le t \le \frac{\pi}{2} \right), \tag{2.7}$$

to estimate integrals $I_{n,1}$, $I_{n,2}$ and $I_{n,3}$. By using (2.4), (2.6) and considering that $\left(P_n / n^{1+\alpha} \right)$ is nonincreasing we get

$$\begin{aligned} I_{n,1} &= \int_{\Gamma_l} u_1^\alpha \left| \sum_{k=1}^n p_{n-k} D_k^*(u_1,u_2) \right| du_1 du_2 \le \int_{\Gamma_l} u_1^\alpha \left(\sum_{k=1}^n p_{n-k} |D_k^*(u_1,u_2)| \right) du_1 du_2 \\ &\le \int_{\Gamma_l} u_1^\alpha \left(\sum_{k=1}^n p_{n-k} (k+1)^2 \right) du_1 du_2 \le (n+1)^2 \int_{\Gamma_l} u_1^\alpha \left(\sum_{k=1}^n p_{n-k} \right) du_1 du_2 \\ &\le (n+1)^2 P_n \int_{\Gamma_l} u_1^\alpha du_1 du_2 \le \frac{P_n}{(n+1)^\alpha} \le \sum_{k=1}^n \frac{P_k}{k^{1+\alpha}}. \end{aligned}$$

To estimate the integral $I_{n,2}$ we write the rectangle Γ_2 as $\Gamma_2 = \Gamma_2' \cup \Gamma_2''$, where

$$\Gamma_2' := \left\{ (u_1,u_2) \in \Gamma_2 : u_2 \le \frac{p_n}{3(n+1) P_n} \right\}$$

and

$$\Gamma_2'':=\left\{(u_1,u_2)\in\Gamma_2: u_2\geq\frac{p_n}{3(n+1)P_n}\right\}.$$

By (2.7),

$$\begin{aligned}
&\int_{\Gamma_2'} u_1^{\alpha}\left|\sum_{k=1}^{n}p_{n-k}D_{k,1}^{*}(u_1,u_2)\right|du_1du_2\\
\leq&\int_{\Gamma_2'} u_1^{\alpha}\left(\sum_{k=1}^{n}p_{n-k}\left|D_{k,1}^{*}(u_1,u_2)\right|\right)du_1du_2\\
\lesssim&\int_{\Gamma_2'} u_1^{\alpha-2}\left(\sum_{k=1}^{n}p_{n-k}\right)du_1du_2\leq P_n\int_{\Gamma_2'}u_1^{\alpha-2}du_1du_2\\
\lesssim& p_n\begin{cases}\dfrac{1}{(n+1)^{\alpha}}, & \alpha<1,\\ \dfrac{\log(n+1)}{n+1}, & \alpha=1.\end{cases}
\end{aligned}$$

For $j=2,3$, by considering (2.6) and (2.7),

$$\begin{aligned}
&\int_{\Gamma_2'} u_1^{\alpha}\left|\sum_{k=1}^{n}p_{n-k}D_{k,j}^{*}(u_1,u_2)\right|du_1du_2\\
\leq&\int_{\Gamma_2'} u_1^{\alpha}\left(\sum_{k=1}^{n}p_{n-k}\left|D_{k,j}^{*}(u_1,u_2)\right|\right)du_1du_2\\
\lesssim&\int_{\Gamma_2'} u_1^{\alpha-1}\left(\sum_{k=1}^{n}p_{n-k}k\right)du_1du_2\\
\leq& nP_n\int_{\Gamma_2'}u_1^{\alpha-1}du_1du_2\lesssim p_n.
\end{aligned}$$

To estimate the integral

$$\int_{\Gamma_2''} u_1^{\alpha}\left|\sum_{k=1}^{n}p_{n-k}D_{k}^{*}(u_1,u_2)\right|du_1du_2,$$

we shall use the expression (2.5) of $D_k^*(u_1,u_2)$.

By (2.7),

$$\begin{aligned}
&\int_{\Gamma_2''} u_1^{\alpha}\left|\sum_{k=1}^{n}p_{n-k}H_{k,1}(u_1,u_2)\right|du_1du_2\\
\leq&\int_{\Gamma_2''} u_1^{\alpha}\left(\sum_{k=1}^{n}p_{n-k}\left|H_{k,1}(u_1,u_2)\right|\right)du_1du_2
\end{aligned}$$

$$\lesssim \int_{\Gamma_2''} u_1^{\alpha-2}\left(\sum_{k=1}^{n} p_{n-k}\right) du_1 du_2 \le P_n \int_{\Gamma_2''} u_1^{\alpha-2} du_1 du_2$$
$$\lesssim P_n \begin{cases} \dfrac{1}{(n+1)^{\alpha}}, & \alpha<1, \\ \dfrac{\log(n+1)}{n+1}, & \alpha=1. \end{cases}$$

Lemma 5.11 of [6] yields

$$\left|\sum_{k=1}^{n} p_{n-k}\cos((2k+1)u_2\pi)\right| \lesssim P\left(\frac{1}{2\pi u_2}\right)$$

and

$$\left|\sum_{k=1}^{n} p_{n-k}\cos\left((2k+1)\frac{u_1-u_2}{2}\pi\right)\right| \lesssim P\left(\frac{1}{(u_1-u_2)\pi}\right)$$

for $(u_1,u_2)\in\Gamma_2''\cup\Gamma_3$, where $P(t):=P_{[t]}$. By Lemmas 5.11 and 5.10 of [6], the fact

$$\sin\frac{u_1\pi}{2} \le \frac{2}{\sqrt{3}}\sin\left(\frac{u_1+u_2}{2}\pi\right),$$

and (2.7), we get

$$\left|\sum_{k=1}^{n} p_{n-k}\cos\left((2k+1)\frac{u_1+u_2}{2}\pi\right)\right| \lesssim P\left(\frac{1}{u_1\pi}\right)$$

for $(u_1,u_2)\in\Gamma_2''\cup\Gamma_3$. Hence by considering these inequalities and (2.7) we obtain

$$\left|\sum_{k=1}^{n} p_{n-k}H_{k,1}(u_1,u_2)\right| \lesssim \frac{1}{u_1^2}P\left(\frac{1}{2\pi u_2}\right) \tag{2.8}$$

and

$$\left|\sum_{k=1}^{n} p_{n-k}H_{k,j}(u_1,u_2)\right| \lesssim \frac{1}{u_1u_2}P\left(\frac{3}{2\pi u_1}\right), \quad (j=2,3), \tag{2.9}$$

for $(u_1,u_2)\in\Gamma_2''\cup\Gamma_3$.

By (2.9), for $j=2,3$,

$$
\begin{aligned}
&\int_{\Gamma_2''} u_1^{\alpha}\left|\sum_{k=1}^{n} p_{n-k}H_{k,j}(u_1,u_2)\right| du_1du_2\\
\lesssim&\int_{\frac{p_n}{3(n+1)P_n}}^{\frac{1}{3(n+1)}}\int_1^{\frac{1}{n+1}} u_1^{\alpha-1}\frac{1}{u_2}P\left(\frac{3}{2\pi u_1}\right)du_1du_2\\
=&\log\left(\frac{P_n}{p_n}\right)\int_{\frac{1}{n+1}}^{1} u_1^{\alpha-1}P\left(\frac{3}{2\pi u_1}\right)du_1\\
\lesssim&\log\left(\frac{P_n}{p_n}\right)\int_{\frac{3}{2\pi}}^{\frac{3}{2\pi}(n+1)}\frac{P(t)}{t^{1+\alpha}}dt\\
=&\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\left(\int_{\frac{3}{2\pi}k}^{\frac{3}{2\pi}}(k+1)\frac{P(t)}{t^{1+\alpha}}dt\right)\\
\lesssim&\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{1}{k^{1+\alpha}}\left(\int_{\frac{3}{2\pi}k}^{\frac{3}{2\pi}(k+1)}P(t)dt\right)\\
\le&\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P\left(\frac{3}{2\pi}(k+1)\right)}{k^{1+\alpha}}\le\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}.
\end{aligned}
$$

If we combine above estimates we get

$$I_{n,2}\lesssim\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}.$$

By (2.8),

$$\int_{\Gamma_3} u_1^{\alpha}\left|\sum_{k=1}^{n} p_{n-k}H_{k,1}(u_1,u_2)\right| du_1du_2\lesssim\int_{\Gamma_3} u_1^{\alpha-2}P\left(\frac{1}{2\pi u_2}\right)du_1du_2.$$

For $\alpha<1$,

$$
\begin{aligned}
&\int_{\Gamma_3} u_1^{\alpha-2}P\left(\frac{1}{2\pi u_2}\right)du_1du_2\\
=&\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\int_{3u_2}^{1} u_1^{\alpha-2}P\left(\frac{1}{2\pi u_2}\right)du_1du_2\\
=&\frac{1}{1-\alpha}\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\left((3u_2)^{\alpha-1}-1\right)P\left(\frac{1}{2\pi u_2}\right)du_2\\
\lesssim&\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} u_2^{\alpha-1}P\left(\frac{1}{2\pi u_2}\right)du_2=\int_{\frac{3}{2\pi}}^{\frac{3}{2\pi}(n+1)}\frac{P(t)}{t^{1+\alpha}}dt\\
\lesssim&\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}.
\end{aligned}
$$

For $\alpha=1$,

$$\begin{aligned}&\int_{\Gamma_3} u_1^{-1} P\left(\frac{1}{2\pi u_2}\right) du_1 du_2\\ =&\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \int_{3u_2}^{1} u_1^{-1} P\left(\frac{1}{2\pi u_2}\right) du_1 du_2\\ =&\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \log\left(\frac{1}{3u_2}\right) P\left(\frac{1}{2\pi u_2}\right) du_2\\ \le&\log(n+1)\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} P\left(\frac{1}{2\pi u_2}\right) du_2\\ =&\frac{1}{2\pi}\log(n+1)\int_{\frac{3}{2\pi}}^{\frac{3}{2\pi}(n+1)} \frac{P(t)}{t^2} dt \lesssim \log(n+1)\sum_{k=1}^{n}\frac{P_k}{k^2}.\end{aligned}$$

For $j=2,3$, if we use (2.9),

$$\begin{aligned}&\int_{\Gamma_3} u_1^{\alpha}\left|\sum_{k=1}^{n} p_{n-k} H_{k,j}(u_1,u_2)\right| du_1 du_2\\ \lesssim&\int_{\Gamma_3} u_1^{\alpha-1}\frac{1}{u_2} P\left(\frac{3}{2\pi u_1}\right) du_1 du_2\\ =&\int_{\frac{1}{n+1}}^{1}\int_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}} u_1^{\alpha-1}\frac{1}{u_2} P\left(\frac{3}{2\pi u_1}\right) du_1 du_2\\ =&\int_{\frac{1}{n+1}}^{1} \log((n+1)u_1) u_1^{\alpha-1} P\left(\frac{3}{2\pi u_1}\right) du_1\\ \le&\log(n+1)\int_{\frac{1}{n+1}}^{1} u_1^{\alpha-1} P\left(\frac{3}{2\pi u_1}\right) du_1\\ \lesssim&\log(n+1)\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}.\end{aligned}$$

Hence we obtain

$$I_{n,3} \lesssim \log(n+1)\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}$$

for $0<\alpha\le 1$. Thus, if we consider these above estimates and the fact $(n+1)p_n \le P_n$, we obtain

$$I_n \lesssim \log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}},$$

and this completes the proof. □

Remark 2.1. Analogue of Theorem 2.1 was proved by B. N. Sahney and D. S. Goel in [8] for classical Fourier series.

Remark 2.2. If we take $p_k=1$, $(k=0,1,\cdots)$, (2.1) gives the estimate

$$\left\|f-S_n^{(1)}(f)\right\|_{C_H(\overline{\Omega})}\lesssim\begin{cases}\dfrac{\log n}{n^\alpha}, & \alpha<1,\\ \dfrac{(\log n)^2}{n}, & \alpha=1.\end{cases}$$

for Fejér means.

In the following theorem, we give an estimate for the order of approximation of $N_n(p;f)$ in the Hölder norm. This is an analogue of a Theorem proved by P. Chandra for Nörlund means of classical Fourier series (see [1]).

Theorem 2.2. *Let $p=(p_k)_{k=0}^\infty$ be a nonincreasing sequence of positive numbers, $0\le\beta<\alpha\le1$ and $f\in H^\alpha\left(\overline{\Omega}\right)$. Then,*

$$\|f-N_n(p;f)\|_{H^\beta(\overline{\Omega})}\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_n}\right)\left(\sum_{k=1}^n\frac{P_k}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\left(\sum_{k=1}^n\frac{P_k}{k}\right)^{\frac{\beta}{\alpha}}. \tag{2.10}$$

Proof. Set $e_n(\mathbf{t}):=f(\mathbf{t})-N_n(p;f)(\mathbf{t})$. Hence,

$$\begin{aligned}\|f-N_n(p;f)\|_{H^\beta(\overline{\Omega})}&=\|e_n\|_{C_H(\overline{\Omega})}+A^\beta(e_n)\\&\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^n\frac{P_k}{k^{1+\alpha}}+A^\beta(e_n).\end{aligned}$$

Since,

$$e_n(\mathbf{t})-e_n(\mathbf{s})=\frac{1}{|\Omega|}\int_\Omega(f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})-f(\mathbf{s})+f(\mathbf{s}-\mathbf{u}))F_n(p;\mathbf{u})d\mathbf{u},$$

we have

$$\begin{aligned}|e_n(\mathbf{t})-e_n(\mathbf{s})|&\lesssim\int_\Omega|f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})-f(\mathbf{s})+f(\mathbf{s}-\mathbf{u})||F_n(p;\mathbf{u})|d\mathbf{u}\\&=\frac{1}{P_n}\int_\Omega|f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})-f(\mathbf{s})+f(\mathbf{s}-\mathbf{u})|\left|\sum_{k=0}^np_{n-k}D_k(\mathbf{u})\right|d\mathbf{u}\\&=:\frac{1}{P_n}J_n.\end{aligned}$$

Since

$$|f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})-f(\mathbf{s})+f(\mathbf{s}-\mathbf{u})|\lesssim\|\mathbf{u}\|^\alpha,$$

by considering Theorem 2.1, we get

$$
\begin{aligned}
(J_n)^{1-\frac{\beta}{\alpha}} &= \left(\int_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left| \sum_{k=0}^{n} p_{n-k} D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{1-\frac{\beta}{\alpha}} \\
&\lesssim \left(\int_{\Omega} \|\mathbf{u}\|^{\alpha} \left| \sum_{k=0}^{n} p_{n-k} D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{1-\frac{\beta}{\alpha}} \\
&\lesssim \left[\log\left(\frac{P_n}{p_n}\right) \sum_{k=1}^{n} \frac{P_k}{k^{1+\alpha}} \right]^{1-\frac{\beta}{\alpha}}.
\end{aligned}
$$

Since $f \in H^{\alpha}\left(\overline{\Omega}\right)$ we also have

$$
|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \lesssim \|\mathbf{t}-\mathbf{s}\|^{\alpha},
$$

and hence

$$
\begin{aligned}
(J_n)^{\frac{\beta}{\alpha}} &= \left(\int_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left| \sum_{k=0}^{n} p_{n-k} D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{\frac{\beta}{\alpha}} \\
&\lesssim \left(\int_{\Omega} \|\mathbf{t}-\mathbf{s}\|^{\alpha} \left| \sum_{k=0}^{n} p_{n-k} D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{\frac{\beta}{\alpha}} \\
&= \|\mathbf{t}-\mathbf{s}\|^{\beta} \left(\int_{\Omega} \left| p_n + \sum_{k=1}^{n} p_{n-k} (\Theta_k(\mathbf{u}) - \Theta_{k-1}(\mathbf{u})) \right| d\mathbf{u} \right)^{\frac{\beta}{\alpha}}.
\end{aligned}
$$

As in proof of Theorem 2.1, it is sufficient to estimate the integral

$$
\int_{\Delta} \left| p_n + \sum_{k=1}^{n} p_{n-k} (\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t})) \right| d\mathbf{t}.
$$

If we use the transformations (2.2) and (2.3), we get

$$
\int_{\Delta} \left| p_n + \sum_{k=1}^{n} p_{n-k} (\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t})) \right| d\mathbf{t} = 3 \int_{\Gamma} \left| p_n + \sum_{k=1}^{n} p_{n-k} D_k^{*}(u_1, u_2) \right| du_1 du_2.
$$

It is clear that

$$
\int_{\Gamma} p_n du_1 du_2 = p_n \leq p_n \leq \frac{P_n}{n} \leq \sum_{k=1}^{n} \frac{P_k}{k}.
$$

If we consider (2.6), we obtain

$$
\int_{\Gamma_l} \left| \sum_{k=1}^{n} p_{n-k} D_k^{*}(u_1, u_2) \right| du_1 du_2 \leq (n+1)^2 P_n \int_{\Gamma_l} du_1 du_2 \leq P_n,
$$

and by (2.7),

$$\int_{\Gamma_2'}\left|\sum_{k=1}^{n}p_{n-k}D_{k,1}^{*}(u_1,u_2)\right|du_1du_2\lesssim P_n\int_0^{\frac{p_n}{3(n+1)P_n}}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^2}du_1du_2=\frac{p_n}{3(n+1)}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^2}du_1du\le p_n.$$

By using (2.6) and (2.7),

$$\int_{\Gamma_2'}\left|\sum_{k=1}^{n}p_{n-k}D_{k,j}^{*}(u_1,u_2)\right|du_1du_2\lesssim (n+1)P_n\int_0^{\frac{p_n}{3(n+1)P_n}}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1}du_1du_2\le p_n\log(n+1)$$

for $j=2,3$. By considering (2.7) again we get

$$\int_{\Gamma_2''}\left|\sum_{k=1}^{n}p_{n-k}H_{k,1}(u_1,u_2)\right|du_1du_2\lesssim P_n\int_{\frac{p_n}{3(n+1)P_n}}^{\frac{1}{3(n+1)}}\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1^2}du_1du_2=nP_n\int_{\frac{p_n}{3(n+1)P_n}}^{\frac{1}{3(n+1)}}du_2\le P_n.$$

For $j=2,3$, by (2.9),

$$\begin{aligned}&\int_{\Gamma_2''}\left|\sum_{k=1}^{n}p_{n-k}H_{k,j}(u_1,u_2)\right|du_1du_2\\ \lesssim&\int_{\frac{1}{n+1}}^{1}\int_{\frac{p_n}{3(n+1)P_n}}^{\frac{1}{3(n+1)}}\frac{1}{u_1u_2}P\left(\frac{3}{2\pi u_1}\right)du_2du_1\\ =&\log\left(\frac{P_n}{p_n}\right)\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1}P\left(\frac{3}{2\pi u_1}\right)du_1\\ =&\log\left(\frac{P_n}{p_n}\right)\int_{\frac{3}{2\pi}}^{\frac{3}{2\pi}(n+1)}\frac{P(t)}{t}dt\le\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k}{k}.\end{aligned}$$

If we use (2.8), we obtain

$$\begin{aligned}&\int_{\Gamma_3}\left|\sum_{k=1}^{n}p_{n-k}H_{k,1}(u_1,u_2)\right|du_1du_2\\ \lesssim&\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\int_{3u_2}^{1}\frac{1}{u_1^2}P\left(\frac{1}{2\pi u_2}\right)du_1du_2\\ =&\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\left(\frac{1}{3u_2}-1\right)P\left(\frac{1}{2\pi u_2}\right)du_2\\ \le&\int_{\frac{1}{3(n+1)}}^{\frac{1}{3}}\frac{1}{u_2}P\left(\frac{1}{2\pi u_2}\right)du_2=\int_{\frac{3}{2\pi}}^{\frac{3}{2\pi}(n+1)}\frac{P(t)}{t}dt\le\sum_{k=1}^{n}\frac{P_k}{k},\end{aligned}$$

and by (2.9),

$$
\begin{aligned}
&\int_{\Gamma_3}\left|\sum_{k=1}^{n}p_{n-k}H_{k,j}^{*}(u_1,u_2)\right|du_1du_2\\
\lesssim&\int_{\frac{1}{n+1}}^{1}\int_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}}\frac{1}{u_1u_2}P\left(\frac{3}{2\pi u_1}\right)du_2du_1\\
=&\int_{\frac{1}{n+1}}^{1}\log((n+1)u_1)\frac{1}{u_1}P\left(\frac{3}{2\pi u_1}\right)du_1\\
\le&\log(n+1)\int_{\frac{1}{n+1}}^{1}\frac{1}{u_1}P\left(\frac{3}{2\pi u_1}\right)du_1\\
\le&\log(n+1)\sum_{k=1}^{n}\frac{P_k}{k}.
\end{aligned}
$$

Combining these estimates, we get

$$\int_{\Delta}\left|p_n+\sum_{k=1}^{n}p_{n-k}(\Theta_k(\mathbf{t})-\Theta_{k-1}(\mathbf{t}))\right|d\mathbf{t}\lesssim\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k}{k},$$

and this implies that

$$(J_n)^{\frac{\beta}{\alpha}}\lesssim\|\mathbf{t}-\mathbf{s}\|^{\beta}\left(\log\left(\frac{P_n}{p_n}\right)\sum_{k=1}^{n}\frac{P_k}{k}\right)^{\frac{\beta}{\alpha}}.$$

Thus, we obtain

$$J_n\lesssim\|\mathbf{t}-\mathbf{s}\|^{\beta}\log\left(\frac{P_n}{p_n}\right)\left(\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\left(\sum_{k=1}^{n}\frac{P_k}{k}\right)^{\frac{\beta}{\alpha}}.$$

Hence,

$$\frac{|e_n(\mathbf{t})-e_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^{\beta}}\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_n}\right)\left(\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\left(\sum_{k=1}^{n}\frac{P_k}{k}\right)^{\frac{\beta}{\alpha}},$$

which means that

$$A^{\beta}(e_n)\lesssim\frac{1}{P_n}\log\left(\frac{P_n}{p_n}\right)\left(\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\left(\sum_{k=1}^{n}\frac{P_k}{k}\right)^{\frac{\beta}{\alpha}}.$$

This estimate and Theorem 2.1 completes proof of Theorem 2.2. □

Remark 2.3. In the case $p_k=1$, $(k=0,1,\cdots)$, (2.10) reduces to

$$\left\|f-S_n^{(1)}(f)\right\|_{H^\beta(\overline{\Omega})}\lesssim\begin{cases}\dfrac{\log n}{n^{\alpha-\beta}}, & 0<\alpha<1,\\ \dfrac{(\log n)^2}{n^{1-\beta}}, & \alpha=1.\end{cases}\tag{2.11}$$

Analogue of (2.11) was obtained by S. Prössdorf in [7] for classical Fourier series.

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References

[1] P. Chandra, On the generalized Fejér means in the metric of Hölder space, Math. Nachr.,109 (1982), 39–45.
[2] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal., 16 (1974), 101–121.
[3] A. Guven, Approximation by means of hexagonal Fourier series in Hölder norms, J. Classical Anal., 1 (2012), 43–52.
[4] A. Guven, Approximation by $(C,1)$ and Abel-Poisson means of Fourier series on hexagonal domains, Math. Inequal. Appl., 16 (2013), 175–191.
[5] H. Li, J. Sun and Y. Xu, Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle, SIAM J. Numer. Anal., 46 (2008), 1653–1681.
[6] L. Mcfadden, Absolute Nörlund summability, Duke Math. J., 9 (1942), 168–207.
[7] S. Prössdorf, Zur konvergenz der Fourierreihen hölderstetiger funktionen, Math. Nachr., 69 (1975), 7–14.
[8] B. N. Sahney and D. S. Goel, On the degree of approximation of continuous functions, Ranchii Univ. Math. J., 4 (1973), 50–53.
[9] J. Sun, Multivariate Fourier series over a class of non tensor-product partition domains, J. Comput. Math., 21(2003), 53–62.
[10] A. F. Timan, Theory of Approximation of Functions of a Real Variable, Pergamon Press, 1963.
[11] Y. Xu, Fourier series and approximation on hexagonal and triangular domains, Constr. Approx., 31 (2010), 115–138.
[12] A. Zygmund, Trigonometric Series, Vols. I-II, Cambridge Univ. Press, 2nd edition, 1959.