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Energy Measures on a Homogeneous Hierarchical Gasket

Donglei Tang and Rui Hu*

School of Science, Nanjing Audit University, Nanjing 211815, Jiangsu, China

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Abstract. On a homogeneous hierarchical gasket we give a linear extension method to compute the energy measures of harmonic functions with respect to the standard energy.

Key Words: Homogeneous hierarchical gasket, energy measures, harmonic function.
AMS Subject Classifications: 28A80

1 Introduction

Dirichlet forms (energies) play an important role in the area of analysis on fractals. There is a measure associated to the energies, called energy measure, which is essentially the Carrë du Champ from the Beurling-Deny-LeJan theory. Energy measures have attracted a lot of attention in recent years (see [1–3, 7–18, 20] and the references therein). However even for the self-similar fractals, the energy measure may be not self-similar. Hence it is very interesting to understand the explicit structure of energy measures.

In [1] Azzam etc. obtained a linear extension method to compute the energy measures of harmonic functions at different levels of the Sierpinski gasket. This was unexpected because the energy is a quadratic quantity. Later the authors in [18] derived a linear extension method for energy measures on connected p.c.f. self-similar sets. It would be nice to show that the linear extension method holds for energy measures on random hierarchical gaskets (see [4–6, 16, 19]), but up to now it is still not clear how to do this. The ultimate goal is to obtain the linear extension method on wider classes of fractals. Therefor it is worthwhile to have a basic example worked out in detail.

A homogeneous hierarchical gasket K (see [4–6, 16, 19]) is a fractal constructed level by level. Here we take a special construction (see Fig. 1). If the level of the gasket K is odd, we take the same construction as the Sierpinski gasket. If the level of the gasket K is even, we take the same construction as the level 3 Sierpinski gasket.

In this paper we will provide details of how to establish a linear extension method for energy measures of harmonic functions on a homogeneous hierarchical gasket K.

*Corresponding author. *Email addresses:* `tdonglei@nau.edu.cn` (D. L. Tang), `hurui@nau.edu.cn` (R. Hu)

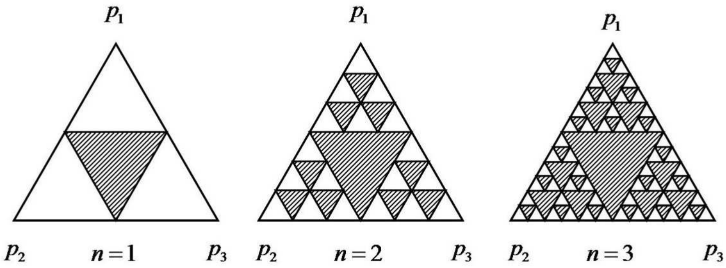

Figure 1: The construction of the homogeneous hierarchical gasket.

2 Some basic results on K

In this section we first summarize some basic facts about a homogeneous hierarchical gasket K from [4–6,16,19].

Let p_1, p_2, p_3 are the vertices of an equilateral triangle, and $V_0=\{p_1,p_2,p_3\}$. If k is odd, then $V_{k+1}=\bigcup_{j=1}^{3}F_jV_k$. If k is even, then $V_{k+1}=\bigcup_{j=1}^{6}F_jV_k$. We approximate the fractal K by a sequence of graphs $G_0,G_1,\cdots$ with vertices $V_0\subseteq V_1\subseteq V_2\cdots$. For $x,y\in V_m$ and $x\neq y$, the edge relation is defined if there exists a word with length $|w|=m$ such that $x,y\in F_wV_0$, denoted by $x\sim_m y$. Let u and v denote continuous functions on K.

The Dirichlet form $(\varepsilon,\mathcal{F})$ on K are defined as follows. The energy ε and the domain $\mathcal{F}$ are given by

$$\varepsilon(u,u)=\lim_{m\to\infty}\sum_{|w|=m}\sum_{x\sim_m y}\frac{1}{r_w}(u(x)-u(y))^2,\quad \mathcal{F}=\{u\in C(K):\varepsilon(u,u)<\infty\},\tag{2.1}$$

where $r_w=r_{w_1}\cdots r_{w_m}$. Note that $r_{w_i}=3/5$ if i is odd, and $r_{w_i}=7/15$ if i is even.

A function h on V_m (for $m\geq 1$) is said to be graph harmonic if it satisfies

$$h(x)=\begin{cases}\dfrac{1}{4}\sum\limits_{y\sim_m x}h(y), & \text{if } \sharp\{y:y\sim_m x\}=4,\\ \dfrac{1}{6}\sum\limits_{y\sim_m x}h(y), & \text{if } \sharp\{y:y\sim_m x\}=6,\end{cases}\tag{2.2}$$

for all junction point x, and $\sharp A$ denotes the number of the elements of a set A.

A function h is said to be harmonic on K if it satisfies $\varepsilon(h,h)=\min_u\{\varepsilon(u,u):u|_{V_0}=h|_{V_0}\}$. Then h is harmonic if and only if the restriction of h on each V_m is graph harmonic for any $m\geq 1$. For any harmonic function h, $\varepsilon_m(h,h)=\varepsilon(h,h)$ is a constant independent of m.

The space of harmonic functions is 3-dimensional. Each harmonic function is determined uniquely by its boundary values. If the values of h on V_0 are known, then the values $h(x)$ for $x\in V_{m+1}$ are obtained as follows.

$$\begin{pmatrix} h(F_w p_1) \\ h(F_w p_2) \\ h(F_w p_3) \end{pmatrix} = D_w \begin{pmatrix} h(p_1) \\ h(p_2) \\ h(p_3) \end{pmatrix}, \tag{2.3}$$

here $D_w = D_{w_m}\circ\cdots\circ D_{w_1}$ for a word $w=w_1\cdots w_m$. $D_{w_i}=D'_{w_i}$If i is odd, otherwise $D_{w_i}=D''_{w_i}$. Where stochastic matrices D'_i for $i=1,2,3$ and D''_j for $j=1,2,\cdots,6$ are defined as

$$D'_i = T^{-(i-1)} \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix} T^{i-1}, \quad i=1,2,3, \quad T=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$D''_j = T^{-(j-1)} \begin{pmatrix} 1 & 0 & 0 \\ \frac{8}{15} & \frac{4}{15} & \frac{3}{15} \\ \frac{8}{15} & \frac{3}{15} & \frac{4}{15} \end{pmatrix} T^{j-1}, \quad j=1,2,3,$$

$$D''_j = T^{-(j-1)} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{15} & \frac{8}{15} & \frac{4}{15} \\ \frac{3}{15} & \frac{4}{15} & \frac{8}{15} \end{pmatrix} T^{j-1}, \quad j=4,5,6.$$

Note that for the matrix T, $T^3=I$.

The energy measure ν_u [1,3,8–11,15,18,20] is defined for an open set O by

$$\nu_u(O) = \lim_{m\to\infty} r^{-m} \sum_{y\sim_m x,\ x,\ y\in V_m\cap O} (u(x)-u(y))^2.$$

Indeed, $\varepsilon(u,u)=\nu_u(K)$ for $u\in\mathcal{F}$.

3 Energy distribution

In this section we will show the details of linear extension method for energy measures of harmonic functions on the homogeneous hierarchical gasket K.

Theorem 3.1. Assume h is harmonic on K. If $|w|=m$ is even, and $i=1,2,3$, then we have

$$\begin{cases} \begin{pmatrix} \nu_h(F_{wi1}K) \\ \nu_h(F_{wi2}K) \\ \nu_h(F_{wi3}K) \end{pmatrix} = T^{-i}M_1T^i \begin{pmatrix} \nu_h(F_{w1}K) \\ \nu_h(F_{w2}K) \\ \nu_h(F_{w3}K) \end{pmatrix}, \\ \begin{pmatrix} \nu_h(F_{wi4}K) \\ \nu_h(F_{wi5}K) \\ \nu_h(F_{wi6}K) \end{pmatrix} = T^{-i}M_2T^i \begin{pmatrix} \nu_h(F_{w1}K) \\ \nu_h(F_{w2}K) \\ \nu_h(F_{w3}K) \end{pmatrix}, \end{cases} \tag{3.1}$$

for all words $w=w_1w_2\cdots w_m$. And

$$\begin{cases} \begin{pmatrix} \nu_h(F_{wij1}K) \\ \nu_h(F_{wij2}K) \\ \nu_h(F_{wij3}K) \end{pmatrix} = T^{-j}M_3T^j \begin{pmatrix} \nu_h(F_{wi1}K) \\ \nu_h(F_{wi2}K) \\ \nu_h(F_{wi3}K) \end{pmatrix}, \quad j=1,2,3, \\ \begin{pmatrix} \nu_h(F_{wij1}K) \\ \nu_h(F_{wij2}K) \\ \nu_h(F_{wij3}K) \end{pmatrix} = T^{-j}M_4T^j \begin{pmatrix} \nu_h(F_{wi1}K) \\ \nu_h(F_{wi2}K) \\ \nu_h(F_{wi3}K) \end{pmatrix}, \quad j=4,5,6, \end{cases} \tag{3.2}$$

where

$$M_1 = \begin{pmatrix} \frac{54}{525} & -\frac{36}{525} & \frac{59}{525} \\ -\frac{36}{525} & \frac{54}{525} & \frac{59}{525} \\ -\frac{18}{525} & -\frac{18}{525} & \frac{257}{525} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} -\frac{27}{1050} & \frac{18}{1050} & \frac{133}{1050} \\ \frac{18}{1050} & -\frac{27}{1050} & \frac{133}{1050} \\ \frac{9}{1050} & \frac{9}{1050} & \frac{34}{1050} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} \frac{287}{6750} & -\frac{238}{6750} & \frac{1337}{6750} \\ -\frac{238}{6750} & \frac{287}{6750} & \frac{1337}{6750} \\ -\frac{49}{6750} & -\frac{49}{6750} & \frac{4076}{6750} \end{pmatrix}, \qquad M_4 = \begin{pmatrix} \frac{1969}{13500} & \frac{769}{13500} & -\frac{506}{13500} \\ \frac{769}{13500} & \frac{1969}{13500} & -\frac{506}{13500} \\ \frac{187}{13500} & \frac{187}{13500} & \frac{562}{13500} \end{pmatrix},$$

and T is the same matrix as before.

Proof. We only prove for $w=\emptyset$. Without loss of generality, we may assume that $h(p_1)=0$, $h(p_2)=a$, $h(p_3)=b$. A direct computation shows that

$$\begin{aligned} \nu_h(F_1K) &= \frac{2}{5}a^2 + \frac{2}{5}b^2 + \frac{2}{5}ab, \\ \nu_h(F_2K) &= \frac{6}{5}a^2 + \frac{2}{5}b^2 - \frac{6}{5}ab, \\ \nu_h(F_3K) &= \frac{2}{5}a^2 + \frac{6}{5}b^2 - \frac{6}{5}ab. \end{aligned}$$

Calculating the energy measures on each cell $F_{i1}K, F_{i2}K, \cdots, F_{i6}K$, for $i=1,2,3$, we see that

$$T^{-1}M_1T\begin{pmatrix}\nu_h(F_1K)\\ \nu_h(F_2K)\\ \nu_h(F_3K)\end{pmatrix}=\begin{pmatrix}\frac{257}{525} & -\frac{18}{525} & -\frac{18}{525}\\ \frac{59}{525} & \frac{54}{525} & -\frac{36}{525}\\ \frac{59}{525} & -\frac{36}{525} & \frac{54}{525}\end{pmatrix}\begin{pmatrix}\nu_h(F_1K)\\ \nu_h(F_2K)\\ \nu_h(F_3K)\end{pmatrix}$$

$$=\frac{1}{525}\begin{pmatrix}74 & 74 & 146\\ 74 & 2 & 2\\ 2 & 74 & 2\end{pmatrix}\begin{pmatrix}a^2\\ b^2\\ ab\end{pmatrix}=\begin{pmatrix}\nu_h(F_{11}K)\\ \nu_h(F_{12}K)\\ \nu_h(F_{13}K)\end{pmatrix},$$

$$T^{-1}M_2T\begin{pmatrix}\nu_h(F_1K)\\ \nu_h(F_2K)\\ \nu_h(F_3K)\end{pmatrix}=\begin{pmatrix}\frac{34}{1050} & \frac{9}{1050} & \frac{9}{1050}\\ \frac{133}{1050} & -\frac{27}{1050} & \frac{18}{1050}\\ \frac{133}{1050} & \frac{18}{1050} & -\frac{27}{1050}\end{pmatrix}\begin{pmatrix}\nu_h(F_1K)\\ \nu_h(F_2K)\\ \nu_h(F_3K)\end{pmatrix}$$

$$=\frac{1}{525}\begin{pmatrix}14 & 14 & -4\\ 14 & 32 & 32\\ 32 & 14 & 32\end{pmatrix}\begin{pmatrix}a^2\\ b^2\\ ab\end{pmatrix}=\begin{pmatrix}\nu_h(F_{14}K)\\ \nu_h(F_{15}K)\\ \nu_h(F_{16}K)\end{pmatrix}.$$

For $i=2,3$, the proof is similar. Thus we finish the proof of (3.1).

For $i=1$, $j=1$,

$$T^{-1}M_3T\begin{pmatrix}\nu_h(F_{11}K)\\ \nu_h(F_{12}K)\\ \nu_h(F_{13}K)\end{pmatrix}=\frac{1}{7875}\begin{pmatrix}662 & 662 & 1322\\ 266 & 182 & 434\\ 182 & 266 & 434\end{pmatrix}\begin{pmatrix}a^2\\ b^2\\ ab\end{pmatrix}=\begin{pmatrix}\nu_h(F_{111}K)\\ \nu_h(F_{112}K)\\ \nu_h(F_{113}K)\end{pmatrix}.$$

For $i=1$, $j=4$,

$$T^{-1}M_4T\begin{pmatrix}\nu_h(F_{11}K)\\ \nu_h(F_{12}K)\\ \nu_h(F_{13}K)\end{pmatrix}=\frac{1}{7875}\begin{pmatrix}62 & 62 & 92\\ 122 & 26 & -76\\ 26 & 122 & -76\end{pmatrix}\begin{pmatrix}a^2\\ b^2\\ ab\end{pmatrix}=\begin{pmatrix}\nu_h(F_{141}K)\\ \nu_h(F_{142}K)\\ \nu_h(F_{143}K)\end{pmatrix}.$$

By symmetry, the rest of the proof is similar, we omit the details. Thus we finish the proof of (3.2). □

Remark 3.1. The reason why we suppose that a harmonic function h defined on the boundary of K always has values of $0, a, b$ is that we subtract off a constant function. Note that even if we subtract a constant function, the energy of this function on the subcells will still be the same, and the energy measures also keep the same.

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