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Forward Scattering Series for 2-Parameter Acoustic Media: Analysis and Implications to the Inverse Scattering Task Specific Subseries

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Abstract. We study the 2-parameter acoustic Born series for an actual medium with constant velocity and a density distribution. Using a homogeneous background we define a perturbation, the difference between actual and reference medium (we use background and reference as synonyms), which exhibits an anisotropic behavior due to the density distribution. For an actual medium with a constant velocity, the reference velocity can be selected so that the waves in the actual medium travel with the same speed as the waves in the background medium. Scattering theory decomposes the actual wave field into an infinite series where each term contains the perturbation and the propagators in the background medium. Hence, in this formalism, all propagations occur in the background medium and the actual medium is included only through the perturbations which scatter the propagating waves. The density-only perturbation has an isotropic and an anisotropic component. The anisotropic component is dependent on the incident direction of the propagating waves and behaves as a *purposeful perturbation* in the sense that it annihilates the part of the Born series that acts to correct the time to build the actual wave field, an unnecessary activity when the reference velocity is equal to the one in the actual medium. This means that the forward series is not attempting to correct for an issue that does not exist. We define the purposeful perturbation concept as the intrinsic knowledge of precisely what a given term is designed to accomplish. This is a remarkable behavior for a formalism that predicts the scattered wave field with an infinite series. At each order of approximation the output of the series is consistent with the fact that the time is correct because the velocity is always constant. In the density-only perturbation, the forward series only seeks to predict the correct amplitudes. Finally, we extend the analysis to a wave propagating in a medium where both density and velocity change. By selecting a convenient set of parameters, we find a conceptual framework for the multiparameter Born series. This framework provides an insightful analysis that can be mapped and applied to the concepts and algorithms of the inverse scattering series.

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1 Introduction

The inverse scattering series has proven to be a good framework for solving the free surface and internal multiple problem [17, 18] without the need for a velocity model. Recently, the inverse scattering series has given results which indicate that it is also a good framework for doing imaging and inversion without a velocity model [5,6,13,14,19, 21]. In [13] it is shown that a 1D earth can be imaged without the velocity model. Later, [6] showed some early examples where the inverse scattering series is used to image a 2D earth without the velocity model.

In the above references, it is assumed that the inverse scattering series can be divided into different subseries, where each subseries is responsible for solving a single task of the inverse problem [19]. These single tasks are divided into the tasks which are common in a standard seismic processing workflow: 1) Free surface multiple removal, 2) Internal multiple removal, 3) Imaging, and 4) Inversion for earth parameters. In addition, the inverse scattering series will contain terms that contribute to solving more than a single task. These terms are omitted in the framework suggested in [19].

In this paper we study the forward scattering series (also known as Born or Neumann series) in order to identify or shed light on which terms in the inverse scattering series are important for performing imaging and inversion. In [7] it is shown analytically for the 1D one parameter acoustic wave equation the validity of the ideas and concepts introduced in [17–19] and used in the development of inverse scattering processing methods (see for example [19]). The mathematical analysis and study of the forward series and its relation with seismic events was revisited by [3,4,9–11]. In [3] absorption and velocity changes in the transmission analysis of the forward series were included. In [4] modeling of specific events (primaries, multiples, diffractions) with analytic, wave-theoretic expressions derived from the forward scattering series in complex 3D scalar media with only velocity changes were proposed. Padé approximants to improve the efficiency of the forward series when used to model acoustic wave field propagation in a vertically varying medium with constant density was introduced in [10, 11]. We will extend the analysis in [7, 8] by studying how the forward scattering series builds up the solution of the two parameter acoustic wave equation from a homogenous background. We will find that the terms containing a velocity perturbation are the terms that contribute to the construction of the actual travel-time for the wave propagating through the inhomogeneous medium. The density perturbation will only contribute to building up the correct amplitude response of the actual wave field.

In Section 2, we introduce the Lippmannn-Schwinger equation and the forward scattering series constructed from the two-parameter acoustic wave equation. It also gives an introduction to how we choose to interpret each term in the series.

In Section 3, we study the simple case of a 1D acoustic medium with only density variations. We show that, since the arrival time constructed by the first term in the series is correct, the forward series does not calculate higher order terms contributing to correcting the time of the actual wave field. Then, we extend the analysis to allow for an actual medium in which both parameters, velocity and density, change. In this case, we identify a special parametrization which helps us in interpreting the tasks of each term in the forward series. Hence, it is important to note that the interpretation of the tasks performed by each term in the forward series is dependent on the choice of parameters used.

In the last part of this work, we present an analysis of the framework introduced in the first two sections. The relations between the forward and inverse scattering series are studied. Even if these two series have completely different tasks to solve, symmetry relations between them can be found. Both the forward and the inverse series can be interpreted as a sum of Feynman diagrams [17,18]. By studying the series and its diagrams it is found that the time corrector diagrams, which resemble a transmission-like event, in the forward series correspond with the depth corrector diagrams in the inverse scattering series. In addition, it is found that diagrams that resemble a reflection like event in forward series correspond to a self-interaction like diagram in the inverse series. Both these diagrams are responsible for correcting the amplitudes of the actual wave field or the medium properties, respectively. It is also found that both forward and inverse series utilize the concept of purposeful perturbation [19], i.e. that the series know a priori that there is no series to sum if the task is not required. Each term in the task-specific subseries returns zero.

In the concluding section we sum up the results obtained throughout the paper.

2 The Lippmannn-Schwinger equation

The purpose of forward scattering series is to find the wave field produced by a localized source and propagated through a certain medium. The forward series constructs the solution by adding an infinite number of terms, each one corresponding to propagations in the reference medium separated by different orders of scattering interactions with a point scatterer earth model.

We present a brief background to scattering theory and forward scattering series, following the development provided in [3,7–9,16] wherein further detail, contributors and references can be found.

A simple, yet insightful, problem to consider is scattering from a half space where only the velocity is allowed to change, like the one shown in Fig. 1a. This model has been studied in [7–9], and led to a clear comprehension of the unique properties in the formulation of the forward series that can be used to understand and work with the inverse scattering series and its subseries. We want to build a similar understanding by

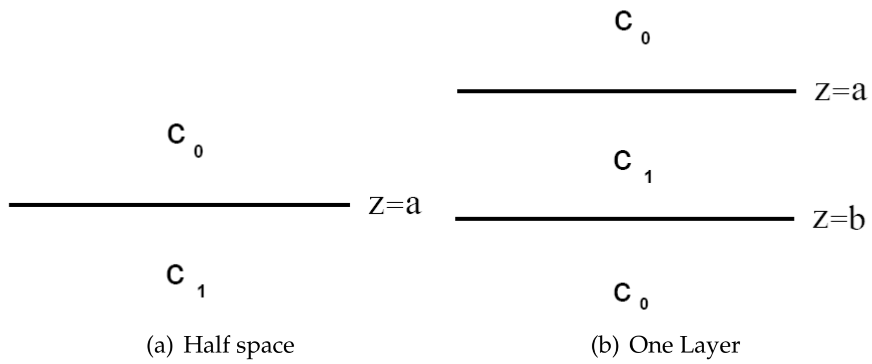


Figure 1: The figure displays the two 1D constant density acoustic models used by [7].

considering another simple case in scattering theory: scattering from a half space where only the density is allowed to change. We then extend this analysis to a change in both, density and velocity.

For a model with velocity and density distributions, Fig. 2, which are constant over intervals and discontinuous at the interval boundaries, the actual medium satisfies the acoustic wave equation,

$$\left(\nabla\cdot\frac{1}{\rho(\boldsymbol{x})}\nabla+\frac{\omega^2}{\rho(\boldsymbol{x})c^2(\boldsymbol{x})}\right)P(\boldsymbol{x}|\boldsymbol{x}';\omega)=\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{2.1}$$

where $P(\boldsymbol{x}|\boldsymbol{x}';\omega)$ is the actual pressure field at point $\boldsymbol{x}$ and frequency ω due to a source located at $\boldsymbol{x}'$ ignited at $t=0$; $\rho(\boldsymbol{x})$ is the density distribution and $c(\boldsymbol{x})$ is the velocity distribution.

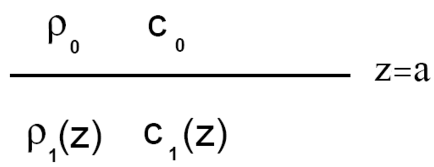


Figure 2: Model with a velocity and density distributions.

The reference medium will be chosen as a homogeneous whole space satisfying the acoustic wave equation,

$$\left(\nabla\cdot\frac{1}{\rho_0}\nabla+\frac{\omega^2}{\rho_0 c_0^2}\right)G_0(\boldsymbol{x}|\boldsymbol{x}';\omega)=\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{2.2}$$

where $G_0(\boldsymbol{x}|\boldsymbol{x}';\omega)$ is the causal free space Green's function.

The velocity and density distributions in Eq. (2.1) can be written in a convenient form, described by a constant reference velocity c_0 and density ρ_0, and their corresponding perturbations, $\alpha(\boldsymbol{x})$ and $\beta(\boldsymbol{x})$, so that

$$\frac{1}{c(\boldsymbol{x})^2}=\frac{1}{c_0^2}(1-\alpha(\boldsymbol{x})), \tag{2.3}$$

$$\frac{1}{\rho(\boldsymbol{x})}=\frac{1}{\rho_0}(1-\beta(\boldsymbol{x})). \tag{2.4}$$

The perturbation, V, is the difference between the reference and actual medium properties defined by the wave equation operators in Eqs. (2.1) and (2.2),

$$\begin{aligned} V(\boldsymbol{x};\omega)&=\nabla\frac{1}{\rho_0}\nabla+\frac{\omega^2}{\rho_0 c_0^2}-\left(\nabla\cdot\frac{1}{\rho(\boldsymbol{x})}\nabla+\frac{\omega^2}{\rho(\boldsymbol{x})c^2(\boldsymbol{x})}\right)\\ &=\nabla\cdot\left(\frac{1}{\rho_0}-\frac{1}{\rho(\boldsymbol{x})}\right)\nabla+\omega^2\left(\frac{1}{\rho_0 c_0^2}-\frac{1}{\rho(\boldsymbol{x})c^2(\boldsymbol{x})}\right)\\ &=\frac{1}{\rho_0}\nabla\cdot\beta(\boldsymbol{x})\nabla+\frac{\omega^2}{\rho_0 c_0^2}(1-(1-\beta(\boldsymbol{x}))(1-\alpha(\boldsymbol{x}))). \end{aligned} \tag{2.5}$$

Using G_0 as the reference wave field we can write the Lippmannn-Schwinger equation,

$$P(\boldsymbol{x}|\boldsymbol{x}';\omega)=G_0(\boldsymbol{x}|\boldsymbol{x}';\omega)+\int_{-\infty}^{\infty}G_0(\boldsymbol{x}|\boldsymbol{x}';\omega)V(\boldsymbol{x}';\omega)P(\boldsymbol{x}|\boldsymbol{x}';\omega), \tag{2.6}$$

or in operator form,

$$P=G_0+G_0VP, \tag{2.7}$$

which is an integral equation corresponding to Eq. (2.1) and its physical boundary conditions (G_0 is a causal Green's function and V contains the properties of the actual medium). The Lippmann-Schwinger equation is a mathematical identity that describes the relationship between two wave fields, G_0 and P. The two wave fields satisfy two different wave equations. The fields are connected through the difference in the wave equation operators, V, and sources. A formal solution to the Lippmann-Schwinger equation can be found through a series expansion,

$$\begin{aligned} &P=G_0+G_0VP\\ &\quad\Updownarrow\\ &P=(I-G_0V)^{-1}G_0=(I+G_0V+G_0VG_0V+G_0VG_0VG_0V+\cdots)G_0\\ &\quad\equiv P_0+P_1+P_2+\cdots, \end{aligned} \tag{2.8}$$

where $P_0=G_0$. When convergent, the forward scattering series, Eq. (2.8), gives a solution for the actual wave field, P, in terms of the reference wave field, G_0, and the perturbation operator, V. In other words, the forward series is able to predict the correct amplitude and phase of a wave field by summing an infinite amount of terms involving interactions between the reference wave field, with its own amplitude and phase, and the perturbation.

Since the reference wave field, G_0, travels with a velocity given by the chosen background medium, it is obvious that the perturbation operator, V, is responsible for obtaining the correct time and amplitude of the actual wave field by interacting with G_0. The perturbation operator is the only entity in the Born series that contain information about the actual medium. How this process is taking place is not obvious, and one of the main objectives of this paper is to show how the Born series obtain the correct phase and amplitude of the actual wave field by summing an infinite amount of terms in an acoustic model, i.e. with density and velocity contrasts.

Now, using the wave fields and perturbation defined in Eqs. (2.2)-(2.5) we have the following representation of the Born series in Eq. (2.8),

$$\begin{aligned}P(\boldsymbol{x}_g|\boldsymbol{x}_s)=&G_0(\boldsymbol{x}_g|\boldsymbol{x}_s)+\int_{-\infty}^{\infty}d\boldsymbol{x}G_0(\boldsymbol{x}_g|\boldsymbol{x})V(\boldsymbol{x})G_0(\boldsymbol{x}|\boldsymbol{x}_s)\\&+\int_{-\infty}^{\infty}d\boldsymbol{x}d\boldsymbol{x}'G_0(\boldsymbol{x}_g|\boldsymbol{x})V(\boldsymbol{x})G_0(\boldsymbol{x}|\boldsymbol{x}')V(\boldsymbol{x}')G_0(\boldsymbol{x}'|\boldsymbol{x}_s)\\&+\int_{-\infty}^{\infty}d\boldsymbol{x}d\boldsymbol{x}'d\boldsymbol{x}''G_0(\boldsymbol{x}_g|\boldsymbol{x})V(\boldsymbol{x})G_0(\boldsymbol{x}|\boldsymbol{x}')V(\boldsymbol{x}')G_0(\boldsymbol{x}'|\boldsymbol{x}'')V(\boldsymbol{x}'')G_0(\boldsymbol{x}''|\boldsymbol{x}_s)\\&+\cdots,\end{aligned}\tag{2.9}$$

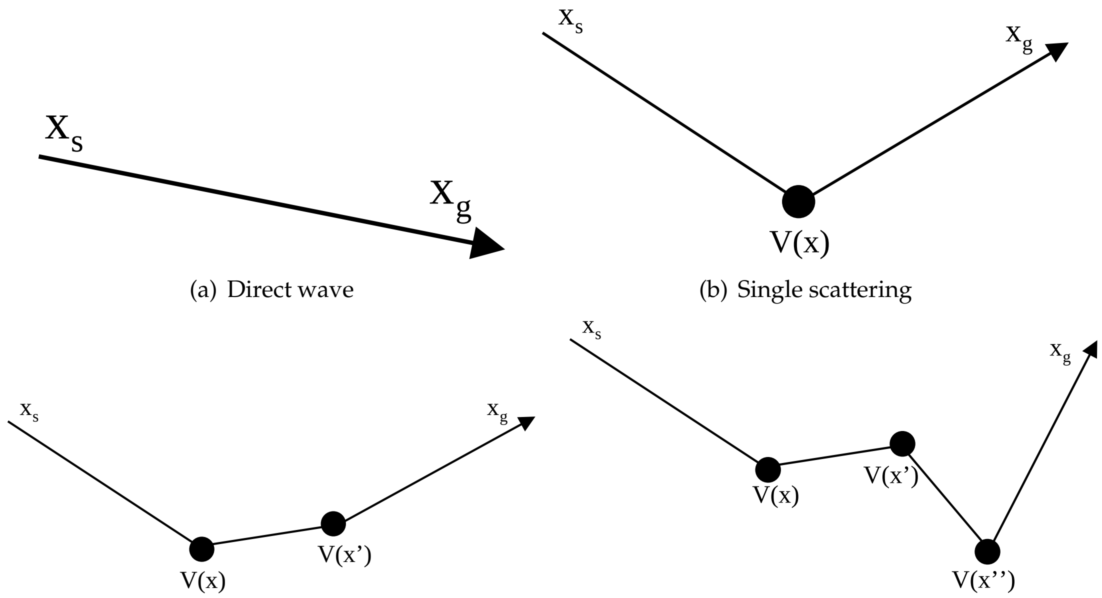
where we have omitted the frequency dependency for convenience.

The Born series can be interpreted as a sequence of infinitely many scattering processes, where the first term is the reference Green's function (a wave propagating in the reference medium from the source at $\boldsymbol{x}_s$ directly to the measurement point, $\boldsymbol{x}_g$, as shown in Fig. 3a).

The second term in the Born series contains $V(\boldsymbol{x})$ sitting between two wave fields propagating in the reference medium. The wave field on the right represents a wave propagating in the reference medium from the source, $\boldsymbol{x}_s$, down to a scattering point at $\boldsymbol{x}$. The wave field is now scattered by the perturbation $V(\boldsymbol{x})$ before it propagates to the measurement $\boldsymbol{x}_g$, as described by the wave field on the left. The integration means a sum over all possible scattering configurations for a given perturbation. This process is displayed in Fig. 3b with a single scattering diagram. Diagrams are tools of interpretation and analysis for the inverse and forward scattering series that are equivalent to Feynman diagrams in quantum mechanics. They were first introduced and applied in exploration seismology in [17–19].

The third term represents a sum over waves which propagate in the reference medium and undergo two scattering interactions. Following this interpretation of the scattering process, each term in the Born series involves a series of propagations and interactions with points within the scattering region. See Fig. 3c.

A cartoon of the fourth term in the forward series which involves three scatterers is shown in Fig. 3d.



(a) Direct wave (b) Single scattering (c) Double scattering (d) Triple scattering

Figure 3: The figure displays the first four terms in the Born series, where (a) shows the direct wave propagating from source to receiver; (b) a propagation from source down to the scattering point, x, and propagation to receiver; (c) a propagation from source down to the scattering point, x, propagation from x to x', and propagation to receiver; (d) a propagation from source down to the scattering point, x, propagation from x to x', propagation from x' to x'', and propagation to receiver.

The perturbation operator in the Born series is responsible for scattering the incoming wave, i.e. giving it a new direction and amplitude. And since the perturbation operator is dependent on the model used, the scattering pattern is dependent on the medium. As already mentioned, it is the scatterer acting on the field in the reference medium which is responsible for generating the correct amplitude and exact phase of the actual wave field.

The perturbation operator contains the differences in medium properties which the two wave fields have experienced, e.g. density and velocity in the acoustic case. In the forward series the reference field interacts with the whole perturbation operator to generate the correct field. Intuitively, in a 1D acoustic world with constant velocity and a density distribution, one would expect this velocity to be responsible for giving the correct time of the actual wave field. And the task for the perturbation to create the correct amplitudes. This is not obvious and it is one of the main objectives of this paper to use a simple analytical model to study the behavior of the Born series and how it generates the output field in the acoustic case with changes in density only.

The forward series is an approach to wave field modeling for a known model using a convenient reference medium. The modeling or creation of data is performed order by order in terms of the perturbation and reference medium operators. With specific mathematical manipulations which are being effectively pursued in [10,11] the forward series can be an efficient wave field modeling tool. In this paper, however, we are not attempting to find a more efficient way to model data but to understand the inner workings of the forward series and to use this understanding as a guide to solve or improve the inverse problem in which the data is processed, not created. The analysis of the forward series and its scattering approach to wave field modeling in which seismic events can be identified will be our guide to shed light on the more complicated process of using seismic data to invert and image reflectors at depth, i.e. if we want to do imaging with the inverse scattering series, how should we select the model to study and test? Can we have a model type independent imaging algorithm? Is velocity information sufficient to perform the imaging step? Will the densities be necessary to consider? What about using the Lamé parameter, $\lambda = \rho c^2$? What is the best choice of parameters to do imaging? What are the implications to non-linear AVO based on inverse scattering? The exercise we do here will give us a hint about how to choose the best model parameters. Using symmetry relations between the forward and inverse scattering series we will show how the best parameters for modeling data give also an advantage when selected to perform imaging and inversion.

3 Analytical examples

The perturbation operator in the Born series contains the differences in medium properties which the two wave fields have experienced, e.g. density and velocity in the acoustic case. In the forward series the reference field interacts with the whole perturbation operator to generate the actual field. Intuitively, in a 1D acoustic world with constant velocity and changing densities, one would expect this velocity to be responsible for the prediction of the correct time of the actual wave field. Hence, the task for the perturbation and all non-linear forward series activity would be to correct the amplitudes. In this section we will go through some analytical examples for a 1D model with a single interface to demonstrate how the forward series generates the actual wave field.

3.1 A 1D earth with constant velocity and changing densities

There are some questions we want to answer: What happens if the perturbation operator only contains a difference in the densities? In that case we know that the reference wave field has the same time behavior as the actual wave field and there is no reason for the forward series to correct the time. How does the series accomplish this? Will it add and subtract non-zero terms an infinite number of times gradually converging to zero, or will it know from the first term that the time is correct? In order to answer these questions, we will study a simple acoustic 1D model with a single interface where the velocity is constant over the interface and the density changes. The model is displayed in Fig. 4.

In a medium with constant velocity, the actual medium satisfies the acoustic wave equation in Eq. (2.1), with $c(x) = c_0$, and the reference medium satisfies Eq. (2.2). The

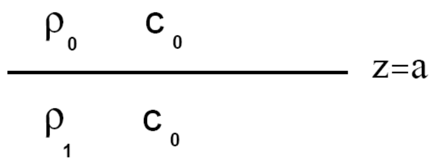


Figure 4: Model with constant velocity and density perturbation.

perturbation is in this case given by

$$V=\frac{\omega^2\beta(\boldsymbol{x})}{\rho_0 c_0^2}+\frac{\partial}{\partial \boldsymbol{x}}\frac{\beta(\boldsymbol{x})}{\rho_0}\frac{\partial}{\partial \boldsymbol{x}}, \tag{3.1}$$

which is Eq. (2.5), with $\alpha(\boldsymbol{x})$ set to zero. In 1D, the perturbation will depend on depth only, and the perturbation for the model in Fig. 4 is given by

$$\begin{aligned} V&=\frac{\omega^2\beta(z)}{\rho_0 c_0^2}+\frac{\partial}{\partial z}\frac{\beta(z)}{\rho_0}\frac{\partial}{\partial z} \\ &=\frac{\omega^2\beta H(z-a)}{\rho_0 c_0^2}+\frac{\beta}{\rho_0}\frac{\partial}{\partial z}H(z-a)\frac{\partial}{\partial z}, \end{aligned} \tag{3.2}$$

where $H(z-a)$ is the unit step or Heaviside function.

We are going to define the first term on the right hand side of the perturbation in Eq. (3.2) as isotropic and the second term, containing the gradients, as anisotropic. The isotropic part depends on the background velocity and treats all directions with equal weight. The anisotropic part depends only on the density, and it has two gradients. These gradients, will generate factors that depend on the direction of the incident reference wave field.

The isotropic part in Eq. (3.2) is analogous in form and behavior to the velocity perturbation that Matson [7] used in his work, which is the perturbation in Eq. (2.5) with $\beta(\boldsymbol{x})=0$ and $\alpha(\boldsymbol{x})=\alpha H(z-a)$.

In a 1D homogenous earth, the solution to the wave equation in Eq. (2.2) is

$$G_0(z|z_s;\omega)=\rho_0\frac{e^{ik|z-z_s|}}{2ik}, \tag{3.3}$$

where $k=\omega/c_0$, z is the receiver depth and z_s is the source depth. The argument of the exponential, $ik|z-z_s|$, can be written as

$$i\omega\Big|\frac{z}{c_0}-\frac{z_s}{c_0}\Big|=i\omega|t-t_s|,$$

where t_s is the time when the source went off and t is the arrival time at z of a wave propagating with velocity c_0. Note that G_0 has the correct time since the velocity in the reference medium, c_0, agrees with the velocity in the actual medium. Hence, the travel time prediction is correct at the first term, $P_0 = G_0$, in the forward series for this model. We would expect it to remain correct at every order of approximation of the wave field, P_i, since each term propagates with the actual velocity.

The forward series has terms, which we represent with transmission-like diagrams (Fig. 6), that add to generate the correct arrival time of the scattered wave field. The reference Green's function, G_0, travels with the correct velocity in a forward series where the perturbation is due to a change in density properties only. Hence, the task of the forward series must only be to construct the correct amplitude of the wave field.

In the following examples, the model in Fig. 4, the perturbation in Eq. (3.2) and the reference wave field in Eq. (3.3) will be used.

3.1.1 Transmission case

We start off by considering the transmitted wave field, i.e. the actual wave field calculated below the interface at $z = a$. The source-receiver configuration is shown in Fig. 5. The zeroth order term is G_0.

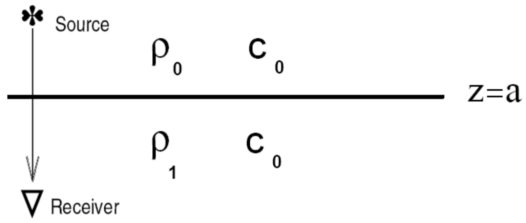


Figure 5: Transmission case with constant velocity.

For the first order term in the Born series, Eq. (2.9), we have

$$\begin{aligned} G_0VG_0 &= \int_{-\infty}^{\infty} \rho_0 \frac{e^{ik|z_g - z|}}{2ik} \left(\frac{\omega^2 \beta H(z-a)}{\rho_0 c_0^2} \right) \rho_0 \frac{e^{ik|z-z_s|}}{2ik}\, dz \\ &\quad + \int_{-\infty}^{\infty} \rho_0 \frac{e^{ik|z_g - z|}}{2ik} \left(\frac{\beta}{\rho_0} \frac{\partial}{\partial z} H(z-a) \frac{\partial}{\partial z} \right) \rho_0 \frac{e^{ik|z-z_s|}}{2ik}\, dz \\ &= I_1 + I_2, \end{aligned} \tag{3.4}$$

where

$$I_1 = \frac{-1}{4} \int_a^{z_g} \beta \rho_0 e^{ik(z_g - z_s)}\, dz - \frac{1}{4} \int_{z_g}^{\infty} \beta \rho_0 e^{2ikz} e^{ik(-z_g - z_s)}\, dz, \tag{3.5}$$

$$\begin{aligned} I_2 &= -\frac{1}{4k^2} \int_a^{\infty} \beta\, sgn(z_g - z) ik \rho_0 e^{ik|z_g - z|} ik e^{ik(z-z_s)}\, dz \\ &= -\frac{i^2}{4} \int_a^{z_g} \beta \rho_0 e^{ik(z_g - z_s)}\, dz - \frac{i^2}{4} \int_{z_g}^{\infty} \beta (-1) \rho_0 e^{2ikz} e^{ik(-z_g - z_s)}\, dz. \end{aligned} \tag{3.6}$$

We can represent the terms in Eqs. (3.5) and (3.6) with transmission-like and reflection-like diagrams as shown in Fig. 6, and note that the transmission-like terms cancel out. Thus,

$$G_0VG_0=I_1+I_2=\frac{-\beta}{2}\int_{z_g}^{\infty}\rho_0e^{2ikz}e^{ik(-z_g-z_s)}\,dz=\frac{\beta}{2}\frac{\rho_0e^{ik(z_g-z_s)}}{2ik}. \tag{3.7}$$

This result shows that the first order approximation to the Born series in this simple model generates a wave traveling with the correct velocity c_0 from the source at z_s down to the receiver at z_g below the interface. However, the amplitude of this wave is not correct.

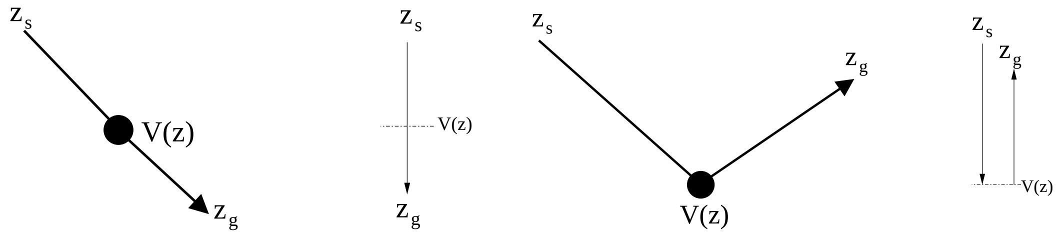
We have seen earlier that the isotropic part of the density perturbation has the exact same form as the velocity perturbation studied in [7]. This means that considering only the isotropic part of the density perturbation, the first order approximation to the Born series will try to correct the time of the actual wave field as well as its amplitude. This is not correct in the constant velocity case. The gradients in I_2, have a directionality feature. This feature plays a major and important role for how this particular forward series behaves. It is responsible for selecting the terms that contribute to the amplitude prediction of the scattered wave field and cancelling the terms whose task are to correct the arrival time.

The anisotropic part of the density perturbation (the one involving $\frac{\partial}{\partial \boldsymbol{x}}\frac{\beta(\boldsymbol{x})}{\rho_0}\frac{\partial}{\partial \boldsymbol{x}}$) acts as the exact negative of the time correction part of the isotropic density perturbation, i.e. the equations eliminate the integrals dependent on the depth difference (z_g-a) at every order of approximation. This is due to the *signum* function in Eq. (3.6) that derives from the gradient operation on the reference wave field. When the reference wave field leaves the perturbation going downwards, the *signum* function gives a $(+1)$ factor; and when it leaves the perturbation going upwards, the *signum* function gives a (-1) factor. These factors help eliminating the integrals where $a<z<z_g$, and adding the integrals where $z>z_g$.

The depth difference factor, (z_g-a), contained in the forward series for a model with changes in velocity was identified in [7] to be a part of a Taylor series of an exponential function which corrects the arrival time of the predicted wave field to generate the actual scattered wave field. This exponential was identified as a time corrector whose first order diagrammatic representation is shown in Fig. 6a. For the model we are considering, the action of the time correctors are not needed, hence the anisotropic part of the density perturbation annihilates the time correction contributions.

The density-only perturbation exhibits the feature of a purposeful perturbation. Its only task is to correct the amplitude of the predicted transmitted wave field order by order in the perturbation. It does not allow the creation of time corrector diagrams because their action is not needed. So we end up with the addition of amplitude correctors diagrams as shown in Fig. 7.

Calculating the higher order terms of the Born series in Eq. (2.9) using the same ap-



(a) Transmission-like diagram (b) Reflection-like diagram

Figure 6: The figure displays the diagrams representing the time and the amplitude correction parts of the forward series in $1D$: (a) Shows a time-corrector diagram. Downward propagation from source to the scattering point z and to the receiver. (b) Shows an amplitude corrector diagram. Downward propagation from source to the scattering point, z and upward propagation to the receiver.

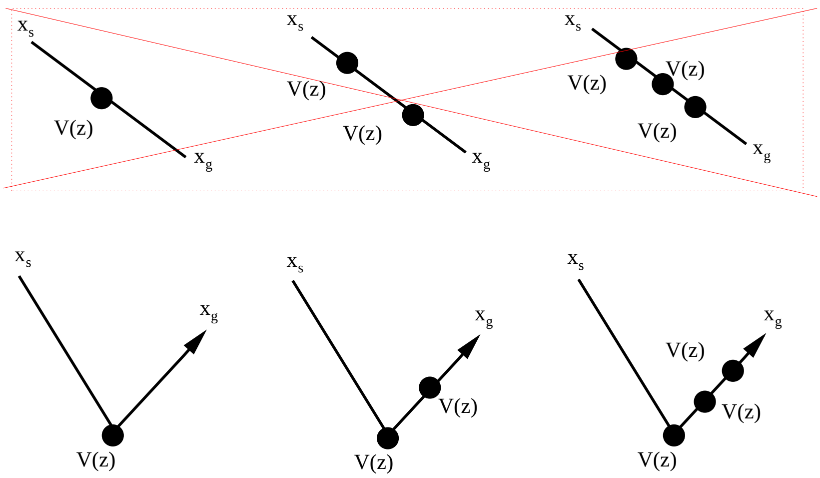


Figure 7: The density-only perturbation is a purposeful perturbation. The diagrams on the top are time correctors; since their action is not needed in this example, the forward series cancels them out.

proach as in Eq. (3.7), we obtain

$$\begin{aligned} &G_0 = \frac{\rho_0 e^{ik(z_g - z_s)}}{2ik}, \quad G_0 V G_0 = \frac{\beta}{2}\frac{\rho_0 e^{ik(z_g - z_s)}}{2ik}, \\ &G_0 V G_0 V G_0 = \frac{\beta^2}{4}\frac{\rho_0 e^{ik(z_g - z_s)}}{2ik}, \\ &G_0 V G_0 V G_0 V G_0 = \frac{\beta^3}{8}\frac{\rho_0 e^{ik(z_g - z_s)}}{2ik}, \\ &G_0 V G_0 V G_0 V G_0 V G_0 = \frac{\beta^4}{16}\frac{\rho_0 e^{ik(z_g - z_s)}}{2ik}, \\ &\qquad\vdots \end{aligned} \tag{3.8}$$

Observe that each term in the forward series has a common G_0 factor traveling with the actual velocity directly from the source to the receiver location. The terms in the forward series differ only in the amplitudes. Summing all these terms gives us the Born series representation of the actual transmitted wave field,

$$\begin{aligned} &P=G_0+G_0VG_0+G_0VG_0VG_0+G_0VG_0VG_0VG_0+\cdots, \\ &P(z_g>a|z_s;k)=\left(1+\frac{\beta}{2}+\frac{\beta^2}{4}+\frac{\beta^3}{8}+\frac{\beta^4}{16}+\cdots\right)\frac{\rho_0 e^{ik(z_g-z_s)}}{2ik}. \end{aligned} \tag{3.9}$$

The forward series does its job given the tools at its disposal. The perturbation has the correct information of the change in parameters, including the exact depth where the density changed and the knowledge of the constant velocity throughout the whole medium. Each order of approximation in the forward series provides the correct wave type, *i.e.* a transmitted wave with the correct arrival time, G_0. However, the amplitude of the transmitted wave is incorrect at each order, and it requires an infinite number of terms to be corrected.

We have yet to establish the connection between the Born series representation of the actual wave field and the analytical solution to the wave equation for the model in Fig. 4 given by

$$P(z_g>a|z_s;k)=T_{01}\frac{\rho_0 e^{ik(z_g-z_s)}}{2ik}, \tag{3.10}$$

where the transmission coefficient, T_{01}, is

$$T_{01}=\frac{2c_1\rho_1}{c_1\rho_1+c_0\rho}=\frac{2\rho_1}{\rho_1+\rho_0}=\frac{\frac{2\rho_0}{1-\beta}}{\frac{\rho_0}{1-\beta}+\rho_0}=\frac{2}{2-\beta}=\frac{1}{1-\frac{\beta}{2}}.$$

Hence, the transmission coefficient can be represented as a geometrical series

$$\frac{1}{1-\frac{\beta}{2}}=1+\frac{\beta}{2}+\frac{\beta^2}{4}+\frac{\beta^3}{8}+\frac{\beta^4}{16}+\cdots, \tag{3.11}$$

which is convergent for $|\beta/2|<1$. In terms of density, this series requires that $\rho_0/\rho<3$ for convergence. Comparing the geometrical series in Eq. (3.11) with the sum of terms in Eq. (3.9), we see that the forward series predicts the correct transmitted wave field in a constant velocity and density varying acoustic medium with a limited contrast condition of $\rho_0<3\rho$ for convergence.

3.1.2 Reflection case

Now, we consider the case where we locate the receiver above the perturbation. Hence, we will obtain a reflected wave field which has interacted with the perturbation and a direct wave which has propagated directly from the source to the receiver without interacting with the perturbation. In this case, the Green's functions in Eq. (2.9) propagating from the source to the first scattering potential and the from the last scatterer back to the receiver will not have absolute values since $z_g < a$ and $z_s < a$. After performing the integrals for this source-receiver configuration, the forward series yields the same amplitude coefficients as in Eq. (3.8). However, there are differences in the phase of the direct wave compared to the scattered wave. The direct wave, G_0, forms an event by itself; it has only traveled in the reference medium and is therefore correct.

The reflection coefficient is formed from the scattered waves, the part of the wave that has interacted with the perturbation. Hence, we have two events, the direct arrival plus the reflected wave

$$P(z_g < z|z_s;k) = \frac{\rho_0 e^{ik(z_g - z_s)}}{2ik} + \left(\frac{\beta}{2} + \frac{\beta^2}{4} + \frac{\beta^3}{8} + \cdots\right) \frac{\rho_0 e^{2ika} e^{ik(-z_g - z_s)}}{2ik}. \tag{3.12}$$

In order to compare the forward series solution of the wave equation with the analytical solution, we expand the reflection coefficient in a Taylor series,

$$\begin{aligned} R_{01} &= \frac{c_1\rho_1 - c_0\rho_0}{c_1\rho_1 + c_0\rho_0} = \frac{\rho_1 - \rho_0}{\rho_1 + \rho_0} \\ &= \frac{\frac{\rho_0}{1-\beta} - \rho_0}{1-\beta}\left(\frac{\rho_0}{1-\beta} + \rho_0\right)^{-1} = \frac{\rho_0\beta}{2\rho_0 - \rho_0\beta} = \frac{\beta}{2-\beta} \\ &= \frac{\beta}{2} + \frac{\beta^2}{4} + \frac{\beta^3}{8} + \frac{\beta^4}{16} + \cdots. \end{aligned} \tag{3.13}$$

Comparing the Taylor series expansion of the reflection coefficient, R_{01}, with the coefficient series in front of the second term in Eq. (3.12), we find that the forward series predicts the actual reflected wave field recorded above the perturbation. Equation (3.13) has a limited contrast condition since it is only convergent when $|\beta/2| < 1$ or $\rho_0/\rho < 3$.

The convergence for transmission and reflection experiments in the density only acoustic case of the forward series requires that $\rho_0 < 3\rho$. This limited contrast convergence condition of the series is consistent with the results found in [7] and [9] for the velocity varying and constant density case.

3.2 Two parameters, both density and velocity changes

How will the forward series act when we allow the two acoustic parameters to change? In this section we will study the model in Fig. 2 which involves the same 1D model structure studied in the previous section, but now we will let both velocity and density change over the interface. We will calculate the transmitted wave field below the interface.

We use the 1D version of the perturbation given in Eq. (2.5). In the case for a single interface model, in a 1D medium, the perturbation in Eq. (2.5) can be written as

$$V = \frac{\omega^2}{\rho_0 c_0^2}\left[\alpha H(z-a) + \beta H(z-a) + \alpha\beta H(z-a)\right] + \frac{\beta}{\rho_0}\frac{\partial}{\partial z} H(z-a)\frac{\partial}{\partial z}. \tag{3.14}$$

For mathematical convenience, we will introduce a new parameter, χ, defined as $\chi = \alpha(1-\beta)$. Using this definition together with $k^2 = \omega^2/c_0^2$ we obtain

$$V = \frac{k^2(\beta+\chi)H(z-a)}{\rho_0} + \frac{\beta}{\rho_0}\frac{\partial}{\partial z}H(z-a)\frac{\partial}{\partial z}. \tag{3.15}$$

The selection of these parameters is very important for an easier interpretation of each term and its task in the forward series. If we had chosen to introduce the bulk modulus or the Lamé parameter $\lambda = \rho c^2$ which involves both the density and the velocity, the forward series could not be divided into task specific subseries. An analogous choice of parameters, density and velocity, in the inverse scattering series for parameter estimation has shown to be a convenient and transparent selection which eliminates the problem of linear "leaking" between linear property change predictions. The special parameters, density and velocity, were identified in the inverse series in [21].

In the 2-parameters acoustic wave equation, velocity and density are independent of each other. We found that these two parameters lead to a clear and transparent understanding of the different tasks performed by the forward series. The introduction of the parameter χ is convenient, because χ contains the portion of the perturbation associated with velocity which only acts as an isotropic part of the perturbation. Hence, χ represents an isotropic only part of the two parameter acoustic perturbation. On the other hand, β appears in both parts (isotropic and anisotropic) of the perturbation, and it is exactly the same perturbation as the one in the density-only case as seen in Eq. (3.2). The fact that we can separate the density-only part of the perturbation from the velocity dependent part, allows us to use the results, analysis and conclusions we obtained in the previous sections to understand and make inferences about the more general 2-parameter acoustic case currently studied. The first order approximation to the transmitted field is easily calculated using Eq. (3.7),

$$G_0VG_0 = \rho_0\frac{e^{ik(z_g-z_s)}}{2ik}\left(\frac{\chi}{4} - \frac{2ik\chi(z_g-a)}{4} + \frac{\beta}{2}\right). \tag{3.16}$$

The parameterization that we chose allows us to infer that the forward series is going to decide of whether the purpose of a given computation is overall necessary or not in the same clear way as it did in the density-only perturbation case. Equation (3.16), gives a transmitted-type wave field multiplied by the coefficient

$$\left(\frac{\chi}{4} - \frac{2ik\chi(z_g-a)}{4} + \frac{\beta}{2}\right). \tag{3.17}$$

The first and third term in this coefficient have a similar form, while the second term is multiplied by the depth difference between the position where the perturbation started (at the interface in the model) and the scatterer. The isotropic part of the perturbation contains χ which in turn contains the depth of the velocity change and the value of that change. It will be integrated with a factor (z_g-a), which is dependent of depth. The depth difference is the best estimate of the correct arrival time that the first order approximation of the forward series can make. The factor (z_g-a) is the depth that the wave traveled in the actual medium. The knowledge of this depth will create the correct time. The term $2ik\chi(z_g-a)/4$ in Eq. (3.17) is the linear term in a Taylor series expansion for an exponential function which is responsible for correcting the travel time of the reference wave towards the travel time of the actual wave field. Hence, the second term on the right hand side of Eq. (3.17) corresponds to the output of a time corrector, while the first and third terms are amplitude correctors.

The second order approximation is given by

$$\begin{aligned}
G_0VG_0VG_0=&\int_{-\infty}^{\infty}\frac{\rho_0 e^{ik|z_g-z|}}{2ik}\left(\frac{\omega^2(\chi+\beta)H(z-a)}{\rho_0 c_0^2}+\frac{\beta}{\rho_0}\frac{\partial}{\partial z}H(z-a)\frac{\partial}{\partial z}\right)\\
&\times\rho_0\frac{e^{ik(z-z_s)}}{2ik}\left(\frac{\chi}{4}-\frac{2ik\chi(z-a)}{4}+\frac{\beta}{2}\right)dz,\\
G_0VG_0VG_0=&\int_a^{\infty}\frac{\rho_0 e^{ik|z_g-z|}}{-4}e^{ikz}e^{-ikz_s}(\chi+\beta)\left(\frac{\chi+2\beta}{4}-\frac{2ik\chi(z-a)}{4}\right)dz\\
&+\int_a^{\infty}\frac{\rho_0 e^{ik|z_g-z|}}{-4k^2}e^{ikz}e^{-ikz_s}\left(sgn(z_g-z)ik\beta\right)\\
&\times\left(ik\left(\frac{\chi+2\beta}{4}-2ik(z-a)\frac{\chi}{4}\right)-2ik\frac{\chi}{4}\right)dz.
\end{aligned}\tag{3.18}$$

Expanding the absolute values of the Green's functions propagating from the source to a scatterer, we obtain

$$\begin{aligned}
G_0VG_0VG_0=&\int_a^{z_g}\left[(\chi+\beta)\frac{\chi+2\beta-2ik\chi(z-a)}{4}-\beta\frac{-\chi+2\beta-2ik\chi(z-a)}{4}\right]\\
&\times\frac{-\rho_0 e^{ik(z_g-z_s)}}{4}dz\\
&+\int_{z_g}^{\infty}\left[(\chi+\beta)\frac{\chi+2\beta-2ik\chi(z-a)}{4}+\beta\frac{-\chi+2\beta-2ik\chi(z-a)}{4}\right]\\
&\times\frac{-\rho_0}{4}e^{2ikz}e^{ik(-z_g-z_s)}dz.
\end{aligned}\tag{3.19}$$

The integral in Eq. (3.19) where $a<z<z_g$ is the time corrector. The second integral corresponds to an amplitude corrector. In the first integral, the terms containing only β cancel out, while all the terms containing the velocity perturbation χ give a contribution,

$$\begin{aligned}
&G_0VG_0VG_0\\
=&\left[\frac{\chi^2}{8}+\frac{\beta\chi}{4}+\frac{\beta^2}{4}-2ik\left(\frac{\chi^2}{8}+\frac{3\beta\chi}{8}\right)(z_g-a)+(ik)^2\frac{\chi^2}{8}(z_g-a)^2\right]\times\frac{\rho_0}{2ik}e^{ik(z_g-z_s)}.
\end{aligned}\tag{3.20}$$

The density-only part of the perturbation does not give any contribution to time correctors. From Eqs. (3.16) and (3.20) we see that all terms involving the density-only part of the perturbation, β, multiplied by the time correction factor, or depth difference, $(z_g - a)$, have vanished. This is the same effect as seen in the density-only case studied in Section (3.1.1) which showed that the isotropic and anisotropic parts containing the time correctors cancel for each term in the series. The only parts that survive, are the terms responsible for amplitude corrections and the nonlinear terms coupled with the velocity term, χ, which are responsible of correcting the travel time and amplitudes.

The higher order terms of the forward series are calculated in the same manner as shown for the first two terms. Summing all terms in the forward series yields

$$\begin{aligned} P(z_g > a|z_s;k) = \Bigg(&\left[1 + \frac{\chi}{4} + \frac{\beta}{2} + \frac{\chi^2}{8} + \frac{\beta^2}{4} + \frac{\beta\chi}{4} + \frac{5\chi^3}{64} + \frac{\beta^3}{8} + \frac{7\beta\chi^2}{32} + \cdots\right] \\ &- 2ik(z_g - a)\left[\frac{\chi}{4} + \frac{\chi^2}{8} + \frac{5\chi^3}{64} + \frac{3\beta\chi}{8} + \frac{9\beta\chi^2}{32} + \frac{7\beta^2\chi}{16} + \cdots\right] \\ &- k^2(z_g - a)^2\left[\frac{\chi^2}{8} + \frac{3\chi^3}{32} + \frac{5\chi^2\beta}{16} + \frac{9\chi^4}{128} + \frac{5\chi^3\beta}{16} + \cdots\right] \\ &+ ik^3(z_g - a)^3\left[\frac{\chi^3}{48} + \frac{\chi^4}{48} + \frac{7\chi^3\beta}{96} + \frac{7\chi^5}{384} + \cdots\right]\Bigg) e^{ik(z_g - z_s)}. \end{aligned} \tag{3.21}$$

The transmission coefficient for this model written in terms of β and χ is

$$T_{01} = \frac{2c_1\rho_1}{c_1\rho_1 + c_0\rho} = \frac{2}{1 + (1-\beta)\sqrt{1 - \frac{\chi}{1-\beta}}}; \tag{3.22}$$

it can be expanded with a double Taylor series to obtain

$$T_{01} = 1 + \frac{\chi}{4} + \frac{\beta}{2} + \frac{\chi^2}{8} + \frac{\beta^2}{4} + \frac{\beta\chi}{4} + \frac{5\chi^3}{64} + \frac{\beta^3}{8} + \frac{7\beta\chi^2}{32} + \cdots.$$

Note that the Taylor expansion of the 2-parameters acoustic transmission coefficient in terms of β and χ corresponds to the first term given by the forward series, which is shown in Eq. (3.21).

Let us define γ as the quotient between the vertical wavenumbers $k_0 = \omega / c_0$ and $k_1 = \omega / c_1$, and write it in terms of β and γ;

$$\gamma = \frac{k_0}{k_1} = \sqrt{1-\alpha} = \sqrt{1 - \frac{\chi}{1-\beta}}. \tag{3.23}$$

By performing Taylor expansions for the transmission coefficient, T_{01}, times powers of

$(1-\gamma)$ we can collapse the factors multiplying powers of $[ik(z_g-a)]$ in Eq. (3.21):

$$
\begin{aligned}
T_{01}(1-\gamma) &= 2\left[\frac{\chi}{4}+\frac{\chi^2}{8}+\frac{5\chi^3}{64}+\frac{3\beta\chi}{8}+\frac{9\beta\chi^2}{32}+\frac{7\beta^2\chi}{16}+\cdots\right],\\
T_{01}\frac{(1-\gamma)^2}{2} &= \left[\frac{\chi^2}{8}+\frac{3\chi^3}{32}+\frac{5\chi^2\beta}{16}+\frac{9\chi^4}{128}+\frac{5\chi^3\beta}{16}+\cdots\right],\\
T_{01}\frac{(1-\gamma)^3}{6} &= \left[\frac{\chi^3}{48}+\frac{\chi^4}{48}+\frac{7\chi^3\beta}{96}+\frac{7\chi^5}{384}+\cdots\right].
\end{aligned} \tag{3.24}
$$

With these definitions, the result in Eq. (3.21) reduces to

$$
\begin{aligned}
P(z_g>a|z_s;k) = T_{01}\,\frac{\rho_0}{2ik}e^{ik(z_g-z_s)}\Big[&1-2ik(1-\gamma)(z_g-a)\\
&-\frac{1}{2}k^2(1-\gamma)^2(z_g-a)^2+\frac{1}{6}ik^3(1-\gamma)^3(z_g-a)^3+\cdots\Big],
\end{aligned} \tag{3.25}
$$

where the factor in squared brackets contains all the depth factors $(z_g-a)^n$ multiplied by powers of $(1-\gamma)$. This term is a Taylor expansion of an exponential which corrects the phase (time) of the predicted scattered field

$$
e^{-iX} = 1-iX-\frac{X^2}{2}+\frac{iX^3}{6}+\cdots, \tag{3.26}
$$

where $X=k(1-\gamma)(zg-a)$. Thus the Born series collapses to

$$
\begin{aligned}
P(z_g>a|z_s;k) &= T_{01}\frac{\rho_0}{2ik}e^{ik(z_g-z_s)}e^{-ik(z_g-a)(1-\gamma)}\\
&= T_{01}\frac{\rho_0}{2ik}e^{ik(z_g-z_s)}e^{ik(a-z_g)}e^{ik(zg-a)\gamma},
\end{aligned}
$$

and substitute $\gamma=k_1/k_0$ to obtain

$$
\begin{aligned}
P(z_g>a|z_s;k) &= T_{01}\frac{\rho_0}{2ik}e^{ik_1(z_g-a)}e^{ik(a-z_s)}\\
&= T_{01}\frac{\rho_0}{2ik}e^{i\omega(z_g-a)/c_1}e^{i\omega(a-z_s)/c_0}.
\end{aligned}
$$

Thus,

$$
P(z_g>a|z_s;\omega) = T_{01}\frac{\rho_0}{2ik}e^{i\omega\left(\frac{zg-a}{c_1}+\frac{a-zs}{c_0}\right)}
$$

is the wave field that travels with velocity c_0 from a point at z_s to the reflector at a, it is then transmitted into the perturbation and travels with velocity c_1 from the reflector to the measurement point at z_g. Again, the forward series predicts the actual wave field recorded at a received located inside the perturbation. The convergence of this exponential depends entirely on the convergence of the series expansion for $(1-\gamma)$ and T_{01}.

The factor $(1-\gamma)$ converges if $\alpha<1$ which implies that $c_0<\sqrt{2}c_1$, in accordance with the results discussed in [7, 9] in the case where the Born series is derived for a model with constant density and velocity variations. The fact that c_0 needs to be less than $(\sqrt{2}c_1)$ to satisfy the convergence requirements of $(1-\gamma)$ is not surprising since this factor helps building the correct arrival time of the wave field predicted by the Born series. The time correction is only dependent on the velocity perturbation, just as it was in the analysis performed in [7, 9]. On the other hand, the convergence of T_{01} is now dependent on velocity and density, hence, it is different from the one found for a perturbation with only velocity variations. The factor T_{01} converges for

$$|(1-\beta)\sqrt{1-\frac{\chi}{1-\beta}}|<1$$

which is satisfied by $\rho_0 c_0<\rho_1 c_1$.

4 Analysis

In the history of scattering series in exploration seismology, the study of the forward series has created a framework to build analogies and symmetries with the inverse scattering series [9, 19]. However, the forward and the inverse series have very different objectives; the forward series uses the reference wave field and a perturbation to build the actual wave field, while the inverse series constructs the perturbation using the reference and the measured values of the actual wave field. In other words, the forward series produces the wave field order by order in the perturbation, while the inverse series (and its task specific subseries) does not annihilate the wave field, it uses order by order the measured values of the actual wave field together with the reference wave field to predict the perturbation from the earth that created it, the actual earth. The inverse scattering series currently provides a comprehensive multidimensional method for inversion that allows achieving different objectives, e.g., free-surface and internal multiple elimination, and depth structure maps and parameter estimation or non-linear AVO, all achieved sequentially with distinct algorithms corresponding to task specific subseries.

In a series approach, like this, a reasonable question to ask is how many terms would be required in practice to achieve an appropriate level of effectiveness towards the construction on the wave field with the forward series or the specific task associated with the inverse scattering subseries [18, 19]. As we showed in the previous sections, the forward series takes a decision of whether the purpose of a given computation is overall necessary or not. The forward series gives a clear signal of this decision by not attempting to solve an issue that doesn't exist. That decision occurs at the first approximate step to address that specific issue. For the two parameter acoustic case we have identified special parameters in such a way that we are able to divide the forward scattering series into two task specific subseries, where one subseries is responsible for generating the time of the actual wave field, and the other is responsible for generating the correct amplitudes.

When there is no velocity difference between the actual and the reference medium, the subseries responsible for time corrections is non-existent. It does not merely add up to zero, but it is zero from the start. The forward series tells us that it is not necessary to calculate this subseries. This is an example of purposeful perturbation. In other words, if a time issue does not exist, the subseries for correcting the time does not exist either and the first term in that subseries signals whether there is or is no issue to be addressed.

The powerful concept of purposeful perturbation was developed in the context of inverse scattering task-specific subseries, where several examples can be found [19]. Among the identified processes, purposeful perturbation occurs for the free-surface and internal multiple elimination series. The free-surface multiple elimination series eliminates an order of free-surface multiples with the corresponding term in the series; it has an understanding of the specific purpose of each term within the overall task [2]. As long as you have source and receivers between a reflector and a free-surface, you will always have free-surface multiples, hence the free-surface multiple elimination series will always have a contribution. It cannot be zero because your data have all orders of multiples, the series will eliminate the free-surface multiples order by order and it will know and reveal what has and has not been accomplished for a given number of terms computed.

In an earth that has only one reflector, the internal multiple attenuation algorithm [1], as well as the elimination series and its leading order closed form algorithm [12], will be computed as zero without the necessity of computing the three or more integrals involved. The intrinsic knowledge of the algorithm will decide that a single reflector cannot create an internal multiple, and it will stop its whole machinery. This is a clear statement of purposeful perturbation. It agrees with the fact that for this hypothetical case there is only one primary or one transmitted event and no multiples can be created or eliminated.

What basically happens in each task-specific subseries is that specific non-linear interactions take place between events in the data as a whole. The data times data communication allow free-surface and internal multiple prediction or accurate depth imaging to take place without an accurate velocity model.

In the subseries for imaging at depth without an accurate velocity [6, 15], the first term is the current standard migration performed with a reference velocity. It places each event exactly where that input reference velocity dictates. The second term in the inverse series, has integral terms represented by the separate diagram and non-integral terms represented in the diagrams by self interactions. The separate diagrams have the task of moving the incorrectly imaged events resulting from the linear migration step towards their correct spatial location. There is a non-linear dependence on the data, allowing non-linear interactions (*e.g.*, multiplication) between primary events from different reflectors. In these interactions, the primaries are able to determine the accuracy of the input velocity. If the reference velocity is not precise for one or more events, then the troubled events will receive information via specific non-linear interactions with the shallower events to help moving the deeper events towards their correct location. When the reference velocity is consistent with the actual velocity, then there is no depth to correct and the first term in the imaging series, represented by a separate diagram, will be zero. The first separated diagram immediately and unambiguously judges the adequacy of the input velocity in an analogy with the transmission-like forward series diagram that is zero when the reference velocity is equal to the actual one.

The behavior of the imaging series has a clear symmetry with the 2-parameter acoustic forward series. When the density is the only parameter changing across the interfaces, there is no time to correct and there are no time corrector or transmission-like diagrams. Furthermore, at each step in the forward series the decision is taken and returns an unambiguous zero for any time corrector diagram, and there are no mixed diagrams allowed, only the amplitude diagrams are computed. This is not the case when the velocity is allowed to change. If the reference velocity is inadequate, an extra part of the perturbation is alive, and the time corrector diagrams are computed order by order in the perturbation giving the possibility to have specific (time or amplitude) and mixed diagrams.

Another important example of purposeful perturbation that can be studied together with the results of this paper is in the 2-parameter inversion subseries for primaries [21]. The term containing

$$\int_0^z dz'(\alpha_1(z')-\beta_1(z')),$$

where α_1 and β_1 are the first order approximation of the relative change in bulk modulus and density, respectively in terms of the measured data, exists to correct depth imaging for incorrect input velocity, but first determines whether its function is required by a nonlinear interaction of all the primaries that outputs the adequacy of the velocity expressed through $\alpha_1(z')-\beta_1(z')$. When the reference velocity is adequate, $\alpha_1(z')-\beta_1(z')$ will be computed as zero. When the velocity is determined to be inadequate, the same term returns a value and sets the whole target identification subseries machinery to work. Furthermore, in our analytic examples, we identified a special parametrization in terms of density and velocity perturbations (β and χ) which helps us in interpreting the tasks of each term in the forward series. This interpretation and separation of the forward series into time and amplitude correction terms would have not been possible if the Lamé parameter $\lambda=\rho c^2$ was selected instead.

Using the Lamé parameter and the density as the two parameters in the forward series will not allow for a task independent interpretation. For example, in the density-only model, the forward series will create amplitude and time correctors involving λ and ρ. As it was showed in this paper, the time correctors are not necessary if the velocity doesn't change, and that becomes clear if we use velocity and density as independent parameters. A similar conclusion was reached in [21] in their analysis of the inverse series for parameter estimation, where the common problem of linear "leaking" between linear property change predictions is addressed by an appropriate selection of parameters in the series. Hence, it is important to note that the interpretation and transparency of the tasks which each term in the forward and inverse series does is dependent of the choice of parameters used.

5 Conclusions

The inverse scattering series is a mathematical formalism pursued to process and invert seismic data; it is the only known method for multidimensional direct inversion. A critical concept in the progress of the algorithms based on inverse scattering is the idea of task separation [18,19]. Understanding the behavior of every term in the inverse series benefits the task separation approach. Experience shows that two mappings are required, one associating nonperturbative description of seismic events with their forward scattering series description and a second relating the construction of events in the forward to their treatment in the inverse scattering series [9].

In a multiparameter world, there are more issues in constructing and processing data. Modeling data with the forward series is a mathematical exercise performed with the goal of creating a framework for the processing of data with task specific inverse subseries. The theoretical analysis of the 1-parameter, acoustic, forward series performed originally by [7] gave a mathematical validation to some of the ideas and concepts used and developed to deal with the inverse series and its subseries [17–19]. The forward series is a formalism that creates the actual wave field order by order in an infinite series in terms of G_0 and V. Creating an issue (*i.e.*, propagation in constant velocity and density distribution) in the data through a forward series with G_0 and V has a suggestion of how that issue is addressed in the inverse sense in terms of G_0 and D, which affects directly the imaging and the parameter estimation series.

In this paper we introduced a mathematical description and an analysis of the 2-parameter acoustic forward series. The main result comes from the analysis of a model where only the density was allowed to change. The density only perturbation has two parts in this scattering description. The first part behaves isotropically, its behavior is the same as the one for a model with constant density and changes in velocity [4,7–9]. The second part behaves anisotropically; it has a directionality given by the gradients in its analytic form. These gradients inside the anisotropic part of the perturbation care of the direction from the source, or the last scatterer, and the direction going out from that source or scatterer. This anisotropic behavior cancels out all the time corrector contributions by giving the exact negative of the time corrector output of the isotropic part of the density perturbation. Leaving us with a double contribution of the reflection like diagrams, which build the correct amplitude of the scattered wave field, i.e. the transmission and reflection coefficients. When the velocity was allowed to change, the time corrector diagrams are alive.

The transparency of the task that each diagram has is closely related to the chosen parameters defining the perturbation. In this work, the velocity and density were selected as special parameters showing advantage over other options such as the bulk modulus or the Lamé parameter. This specific parameterization was responsible of the forward series task specific separation into amplitude and time correcting terms and for avoiding the common problem of linear "leaking" between inversion parameters for AVO analysis with the inverse series [21]. Furthermore, the fact that the construction of the correct travel time for the wave field modeled with the forward series is solely dependent on the velocity perturbation gives us a hint of the possibility of achieving a complete algorithm by only using velocity perturbation in the inverse series for depth imaging. That would imply that a model-type independent imaging algorithm is achievable in terms of a velocity-like perturbation in the inverse series [20]. The theory and results for depth imaging without a velocity model using a 1-parameter acoustic model have been published, see, e.g., [5,13,14,19].

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