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# An Energy Regularization Method for the Backward Diffusion Problem and its Applications to Image Deblurring

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**Abstract.** For the backward diffusion equation, a stable discrete energy regularization algorithm is proposed. Existence and uniqueness of the numerical solution are given. Moreover, the error between the solution of the given backward diffusion equation and the numerical solution via the regularization method can be estimated. Some numerical experiments illustrate the efficiency of the method, and its application in image deblurring.

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**Key words**: Energy regularization method, inverse problem, heat equation, backward diffusion equation, image deblurring, error estimate, ill-posed, well-posed.

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## 1 Introduction

Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ and let $\partial\Omega$ be its boundary. Then $\Sigma=\Omega\times(0,T)$ is a bounded domain in $\mathbf{R}^{n+1}$. We are interested in finding the numerical solution of the following backward diffusion problem:

$$
\begin{aligned}
&\frac{\partial u}{\partial t}=\sum_{k,l=1}^{n}\frac{\partial}{\partial x_k}\left(a^{kl}(x)\frac{\partial u}{\partial x_l}\right)-c(x)u, \quad \text{in } \Sigma,\\
&u=0 \quad \left( or\,\frac{\partial u}{\partial \nu}=0\right), \quad \text{on } \partial\Omega\times[0,T),\\
&u(x,T)=g(x), \qquad x\in\Omega,
\end{aligned}
\tag{1.1}
$$

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where $c(x)$ is a given non-negative smooth function on $\overline{\Omega}$, $g(x)$ defines homogeneous boundary conditions on $\overline{\Omega}$, i.e.,

$$g(x)=0 \quad \text{or} \quad \frac{\partial g}{\partial \nu}=0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Moreover,

$$\frac{\partial u}{\partial \nu}=\sum_{k,l=1}^{n} a^{kl}(x)\frac{\partial u}{\partial x_l}n_k, \tag{1.3}$$

where $\{n_k\}$ are the components of the unit normal vector on the boundary $\partial\Omega$ and $\{a^{kl}(x)\}$ is smooth on $\overline{\Omega}$ satisfying, for all $x\in\overline{\Omega}$,

$$\begin{aligned} &a^{kl}(x)=a^{lk}(x), \qquad 1\le k,\ l\le n,\\ &\alpha_0\sum_{k=1}^{n}\zeta_k^2\le \sum_{k,l=1}^{n} a^{kl}(x)\zeta_k\zeta_l\le \alpha_1\sum_{k=1}^{n}\zeta_k^2, \quad \forall\zeta=(\zeta_1,\cdots,\zeta_n)\in\mathbf{R}^n, \end{aligned} \tag{1.4}$$

where $0<\alpha_0<\alpha_1$ are two constants.

The problem (1.1) is reduced to the isotropic heat diffusion problem if we let $a^{kl}=c_0\delta_{kl}$, where $c_0$ is a positive constant and $\delta_{kl}$ is the Kronecker delta defined by

$$\delta_{kl}=\begin{cases} 1, & \text{when } k=l,\\ 0, & \text{when } k\neq l. \end{cases} \tag{1.5}$$

The backward diffusion problem (1.1) is a typical ill-posed problem in the sense of Hadamard [9,16]. The uniqueness of the given problem (1.1) can be found in [16], but the solution of problem (1.1) does not depend continuously on the given final data $g(x)$, and in general for any given function $g(x)$ with the vanishing boundary condition (1.2), there is no solution satisfying (1.1). In 1935, Tikhonov [1] obtained the backward diffusion problem by a geophysical interpretation, namely recovering the geothermal prehistory from contemporary data.

The problem (1.1) has been considered by many authors since the last century. After adding a priori information about the solution of the problem, such as smoothness or bounds on the solution in a given norm, we can restore stability and construct efficient numerical algorithms. Regularization methods are used by most authors to construct a solution of the ill-posed Cauchy problem for the backward diffusion equation. The main idea of most algorithms is solving a well-posed problem which is perturbed from the ill-posed one, and approximating the solution of the original problem with the solution of the well-posed one. A number of perturbations have been proposed, including the method of quasi-reversibility [3], pseudo-parabolic regularization [4], hyperbolic regularization [15]. Only the differential equation is perturbed in these methods. In [8], Showalter perturbed the initial condition rather than the differential equation, which has a better stability estimate than the previous ones.

Regularization techniques have been well developed for numerically solving the backward diffusion problem [2, 5, 6, 13, 14, 17]. The difference of all these approaches lies in the functional selected to be minimized or the perturbation. In [17], we used an energy bounded solution as a regularization, and presented a possible formulation for the backward diffusion equation. The effectiveness is shown by examples in [17], while no order of convergence is proved in [17], and the present work may be considered as a discrete version of it. Similarly to [17], a given energy functional is minimized in order to obtain the regularizing approximation to solution of the original problem. The numerical examples in [17] demonstrate that the approach is very well suited to numerically solve the ill-posed problem. Furthermore, the error between the solution of the original initial boundary problem and the discrete regularizing solution can be estimated.

The image deblurring is a important subject in image reconstruction [19–22], and the backward diffusion equation can be applied to image deblurring. As is well known, image blurring is regarded as an image degrading procedure which can be described by convolutions. Therein Gaussian convolution, also known as Gaussian blur, is the most frequent. The Gaussian blur of an image $u$ can be viewed as the solution of the linear heat equation with $u$ as the initial value [18]. In image deblurring manner, one is to find the true image before degrading from the blur one. This is equivalent to solving the backward diffusion equation, particularly the backward heat equation for Gaussian blur. In fact, the backward heat equation has been widely investigated in [7, 10, 12, 19, 21, 22] for image deblurring and enhancement etc.. In this paper we consider the application of a general backward diffusion equation based on its energy regularization method.

The outline of the paper is as follows. In the next section, we add a priori information on the solution of the original problem, and obtain some stability properties in discrete form on the solution. In Section 3, we propose a stable discrete energy regularization method for the backward diffusion equation, existence and uniqueness of the method are given. The error estimates between the numerical solution and the solution of the original problem are given in this section also. In Section 4, we provide some numerical experiments, which show the efficiency of the given method and its use in image deblurring. Finally, we end this paper with a short concluding section.

## 2 A finite difference scheme and its stability analysis

For the sake of simplicity, in the problem (1.1), we take $n=2$, $T=1$, $\Omega=(0,1)\times(0,1)$, $\Sigma=\Omega\times(0,1)$. Then problem (1.1) is reduced to the following 2-D backward diffusion problem:

$$\begin{aligned} &\frac{\partial u}{\partial t}=\sum_{k,l=1}^{2}\frac{\partial}{\partial x_k}\left(a^{kl}(x)\frac{\partial u}{\partial x_l}\right)-c(x)u, \quad \text{in } \Sigma, \\ &u=0 \quad (\text{or } \frac{\partial u}{\partial \nu}=0), \quad \text{on } \partial\Omega\times[0,1), \\ &u(x,1)=g(x), \qquad x\in\Omega. \end{aligned} \tag{2.1}$$

We will discuss only the vanishing Dirichlet boundary condition in this paper; for the other boundary condition, we can get similar results without any difficulty.

Suppose that problem (2.1) has an unique solution $u(x,t)$. The main concern in this paper is to find the numerical approximation of $u(x,t)$, the solution of problem (2.1). We now construct a finite difference scheme for problem (2.1). Let $I$, $J$, and $N$ be three positive integers and let $h_1=1/I, h_2=1/J, \tau=1/N$ be the three mesh sizes. We introduce the following notations:

$$\Omega_h=\{(x_1^i,x_2^j)\,\big|\, x_1^i=ih_1,\quad x_2^j=jh_2,\quad 0\le i\le I,\quad 0\le j\le J\},$$
$$\Sigma_{h,\tau}=\{(x_1^i,x_2^j,t^n)\,\big|\,(x_1^i,x_2^j)\in\Omega_h,\quad t^n=n\tau,\quad 0\le n\le N\}.$$

For any given mesh function $w=\{w_{i,j}^n,\ 0\le i\le I,\ 0\le j\le J,\ 0\le n\le N\}$ on $\Sigma_{h,\tau}$, we define:

$$D_1^- w_{i,j}^n=(w_{i,j}^n-w_{i-1,j}^n)\frac{1}{h_1},\qquad D_1^+ w_{i,j}^n=(w_{i+1,j}^n-w_{i,j}^n)\frac{1}{h_1},$$
$$D_2^- w_{i,j}^n=(w_{i,j}^n-w_{i,j-1}^n)\frac{1}{h_2},\qquad D_2^+ w_{i,j}^n=(w_{i,j+1}^n-w_{i,j}^n)\frac{1}{h_2}.$$

Furthermore, we have the following definitions of corresponding discrete functions of $g(x), a^{kl}(x)$ and $c(x)$:

$$g_{i,j}=g(ih_1,jh_2),\quad c_{i,j}=c(ih_1,jh_2),\quad a_{i,j}^{kl}=a^{kl}(ih_1,jh_2).$$

Having the above definitions, we can obtain the following finite difference scheme for backward diffusion problem(2.1):

$$\begin{aligned}&\frac{u_{i,j}^{n+1}-u_{i,j}^n}{\tau}=\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\frac{u_{i,j}^n+u_{i,j}^{n+1}}{2}\right)-c_{i,j}\frac{u_{i,j}^n+u_{i,j}^{n+1}}{2},\\ &\qquad 1\le i\le I-1,\ 1\le j\le J-1,\ 0\le n\le N-1,\\ &u_{0,j}^n=u_{I,j}^n=u_{i,0}^n=u_{i,J}^n=0,\quad 0\le i\le I,\ 0\le j\le J,\ 0\le n\le N-1,\\ &u_{i,j}^N=g_{i,j},\quad 0\le i\le I,\ 0\le j\le J.\end{aligned}\tag{2.2}$$

It is easy to see that the finite difference scheme (2.2) is unstable, and the solution $\{u_{i,j}^0\}$ is not continuously dependent on the final data $\{g_{i,j}\}$.

Let us define

$$V_{i,j}^n:=\frac{u_{i,j}^{n+1}-u_{i,j}^n}{\tau},\quad \overline{W}_{ij}^n:=\frac{W_{ij}^{n+1}+W_{ij}^n}{2}.\tag{2.3}$$

We can easily get the following finite difference scheme for $\{V_{i,j}^n\}$:

$$\begin{aligned}&\frac{V_{i,j}^{n+1}-V_{i,j}^n}{\tau}=\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{V}_{i,j}^n\right)-c_{i,j}\overline{V}_{i,j}^n,\\ &\qquad 1\le i\le I-1,\ 1\le j\le J-1,\ 0\le n\le N-2,\\ &V_{0,j}^n=V_{I,j}^n=V_{i,0}^n=V_{i,J}^n=0,\quad 0\le i\le I,\ 0\le j\le J,\ 0\le n\le N-1.\end{aligned}\tag{2.4}$$

Furthermore, for a given integer $m$ $(1\le m\le N-1)$, take integer $\widetilde{m}$ $(0<\widetilde{m}\le\min\{m,N-m\})$, and we define

$$\varphi_{i,j}^n=u_{i,j}^{2m-1-n},\qquad m-\widetilde{m}\le n\le m.$$

Then we get the following finite difference scheme for $\{\varphi_{i,j}^n\}$:

$$\begin{aligned}
&-\frac{\varphi_{i,j}^n-\varphi_{i,j}^{n-1}}{\tau}=\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{\varphi}_{i,j}^{n-1}\right)-c_{i,j}\overline{\varphi}_{i,j}^{n-1},\\
&\qquad 1\le i\le I-1,\ 1\le j\le J-1,\ m-\widetilde{m}+1\le n\le m,\\
&\varphi_{0,j}^n=\varphi_{I,j}^n=\varphi_{i,0}^n=\varphi_{i,J}^n=0,\quad 0\le i\le I,\ 0\le j\le J,\ m-\widetilde{m}\le n\le m.
\end{aligned}\tag{2.5}$$

**Theorem 2.1.** *If the energy norm is defined by*

$$|u^m|_*^2=\sum_{i=0,j=0}^{I-1,J-1}\left(\sum_{k,l=1}^{2}a_{i,j}^{kl}(D_l^+u_{i,j}^m)(D_k^+u_{i,j}^m)+c_{i,j}u_{i,j}^mu_{i,j}^m\right)h_1h_2,\tag{2.6}$$

*then we have the following estimate:*

$$|u^m|_*^2\le|u^{m-1}|_*|u^{m+1}|_*,\qquad 1\le m\le N-1.\tag{2.7}$$

*Proof.* By the definition of $\varphi_{i,j}^n$ and $V_{i,j}^n$, one has

$$\begin{aligned}
&\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\overline{\varphi}_{i,j}^n\frac{V_{i,j}^{n+1}-V_{i,j}^n}{\tau}h_1h_2\right)\\
=&\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\overline{\varphi}_{i,j}^n\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{V}_{i,j}^n\right)-c_{i,j}\overline{V}_{i,j}^n\right)h_1h_2\right)\\
=&\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\overline{V}_{i,j}^n\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{\varphi}_{i,j}^n\right)-c_{i,j}\overline{\varphi}_{i,j}^n\right)h_1h_2\right)\\
=&-\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\overline{V}_{i,j}^n\frac{\varphi_{i,j}^{n+1}-\varphi_{i,j}^n}{\tau}h_1h_2\right),
\end{aligned}\tag{2.8}$$

where the summation for $i,j$ is for $1\le i\le I-1$, $1\le j\le J-1$. Moving the right-hand side terms to the left-hand side, we get

$$\begin{aligned}
0=&\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\overline{\varphi}_{i,j}^n\frac{V_{i,j}^{n+1}-V_{i,j}^n}{\tau}h_1h_2\right)+\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\overline{V}_{i,j}^n\frac{\varphi_{i,j}^{n+1}-\varphi_{i,j}^n}{\tau}h_1h_2\right)\\
=&\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\frac{\varphi_{i,j}^{n+1}V_{i,j}^{n+1}-\varphi_{i,j}^nV_{i,j}^n}{\tau}h_1h_2\right).
\end{aligned}\tag{2.9}$$

If we let $\widetilde{m}=1$, we can get

$$\begin{aligned}&\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\frac{\varphi_{i,j}^{n+1}V_{i,j}^{n+1}-\varphi_{i,j}^{n}V_{i,j}^{n}}{\tau}h_1h_2\right)\\&=\sum_{ij}(\varphi_{i,j}^{m}V_{i,j}^{m}-\varphi_{i,j}^{m-1}V_{i,j}^{m-1})h_1h_2=\sum_{ij}(u_{i,j}^{m-1}V_{i,j}^{m}-u_{i,j}^{m}V_{i,j}^{m-1})h_1h_2\\&=\sum_{ij}u_{i,j}^{m-1}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{u}_{i,j}^{m}\right)-c_{i,j}\overline{u}_{i,j}^{m}\right)h_1h_2\\&\quad-\sum_{ij}u_{i,j}^{m}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{u}_{i,j}^{m-1}\right)-c_{i,j}\overline{u}_{i,j}^{m-1}\right)h_1h_2\end{aligned}$$

Rearranging the right-hand side gives

$$\begin{aligned}&\sum_{n=m-\widetilde{m}}^{m-1}\tau\left(\sum_{ij}\frac{\varphi_{i,j}^{n+1}V_{i,j}^{n+1}-\varphi_{i,j}^{n}V_{i,j}^{n}}{\tau}h_1h_2\right)\\&=\sum_{ij}\overline{u}_{i,j}^{m}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+u_{i,j}^{m-1}\right)-c_{i,j}u_{i,j}^{m-1}\right)h_1h_2\\&\quad-\sum_{ij}u_{i,j}^{m}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{u}_{i,j}^{m-1}\right)-c_{i,j}\overline{u}_{i,j}^{m-1}\right)h_1h_2\\&=\sum_{ij}\frac{u_{i,j}^{m+1}}{2}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+u_{i,j}^{m-1}\right)-c_{i,j}u_{i,j}^{m-1}\right)h_1h_2\\&\quad-\sum_{ij}u_{i,j}^{m}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\frac{u_{i,j}^{m}}{2}\right)-c_{i,j}\frac{u_{i,j}^{m}}{2}\right)h_1h_2\\&=-\frac{1}{2}\sum_{i=0,j=0}^{I-1,J-1}\left(\sum_{k,l=1}^{2}a_{i,j}^{kl}(D_l^+u_{i,j}^{m-1})(D_k^+u_{i,j}^{m+1})+c_{i,j}u_{i,j}^{m-1}u_{i,j}^{m+1}\right)h_1h_2\\&\quad+\frac{1}{2}\sum_{i=0,j=0}^{I-1,J-1}\left(\sum_{k,l=1}^{2}a_{i,j}^{kl}(D_l^+u_{i,j}^{m})(D_k^+u_{i,j}^{m})+c_{i,j}u_{i,j}^{m}u_{i,j}^{m}\right)h_1h_2.\end{aligned}\tag{2.10}$$

Using the definition of the energy norm (2.6) and the Cauchy-Schwarz inequality gives the desired estimate (2.7). □

Following some analysis we can get the following Hölder-type estimate:

**Theorem 2.2.** *With the solution $u$ of* (2.2) *and the functional $|\cdot|_*$ defined in* (2.6)*, the following estimates holds:*

$$|u^n|_*\le|u^N|_*^{n/N}|u^0|_*^{1-n/N},\qquad n=0,\cdots,N.\tag{2.11}$$

*Proof.* We prove it in three steps.

*Step 1.* In the case $|u^0|_* = 0$, from (2.7) we have

$$|u^n|_*^2 \le |u^{n-1}|_* |u^{n+1}|_*, \quad \forall 1 \le n \le N-1.$$

So we obtain $|u^n|_*^2 \le 0$ for all $n$, which implies that $|u^n|_* = 0$. Consequently, the estimate (2.11) holds.

*Step 2.* In the case $|u^0|_* \neq 0$, we define

$$G_n := \ln\{|u^n|_* / |u^0|_*\}, \qquad n = 0, \cdots, N. \tag{2.12}$$

Obviously $G_0 = 0$ and we shall prove that

$$G_n \le \frac{t}{s+t} G_{n+s} + \frac{s}{s+t} G_{n-t} \tag{2.13}$$

for integers $s$, $t$ and $0 < s \le N-n$, $0 < t \le n$. If the estimate (2.13) is proved, we choose $s = N-n$ and $t = n$ in the inequality (2.13), to get

$$G_n \le \frac{n}{N} G_N + \frac{N-n}{N} G_0 = \frac{n}{N} G_N. \tag{2.14}$$

The above equality in (2.14) is from the fact that $G_0 = 0$. Substituting the definition of $G_n$ in (2.14), we obtain

$$\begin{aligned} & \ln \frac{|u^n|_*}{|u^0|_*} \le \frac{n}{N} \ln \frac{|u^N|_*}{|u^0|_*} = \ln \left[ \left(\frac{|u^N|_*}{|u^0|_*}\right)^{n/N} \right] \\ \Longleftrightarrow \quad & \frac{|u^n|_*}{|u^0|_*} \le \left( \frac{|u^N|_*}{|u^0|_*} \right)^{n/N} \\ \Longleftrightarrow \quad & |u^n|_* \le |u^N|_*^{n/N} |u^0|_*^{1-n/N}. \end{aligned} \tag{2.15}$$

*Step 3.* The proof of (2.13). In order to prove (2.13), we use the method of mathematical induction. We first observe that it is satisfied when $s = t = 1$:

$$G_n \le \frac{1}{1+1} G_{n+1} + \frac{1}{1+1} G_{n-1}. \tag{2.16}$$

This follows directly from (2.7),

$$|u^n|_*^2 \le |u^{n-1}|_* |u^{n+1}|_*, \tag{2.17}$$

and therefore,

$$\begin{aligned} G_n &= \ln \frac{|u^n|_*}{|u^0|_*} \le \frac{1}{2} \ln \frac{|u^{n-1}|_* |u^{n+1}|_*}{|u^0|_*^2} \\ &= \frac{1}{2} \ln \frac{|u^{n+1}|_*}{|u^0|_*} + \frac{1}{2} \ln \frac{|u^{n-1}|_*}{|u^0|_*} = \frac{1}{2} G_{n+1} + \frac{1}{2} G_{n-1}. \end{aligned} \tag{2.18}$$

Then let us suppose that (2.13) holds for all $(s,t)$ such that $s+t\le m$, $s>0$, $t>0$, where $m$ is a positive integer. We have to prove that it holds for all $(s+1,t)$ and $(s,t+1)$. From (2.13),

$$\begin{aligned} G_n &\le \frac{t}{s+t}G_{n+s}+\frac{s}{s+t}G_{n-t} \\ &\le \frac{t}{s+t}\left(\frac{s}{s+1}G_{n+s+1}+\frac{1}{s+1}G_n\right)+\frac{s}{s+t}G_{n-t}. \end{aligned} \tag{2.19}$$

Therefore,

$$\left(1-\frac{t}{s+t}\frac{1}{s+1}\right)G_n \le \frac{t}{s+t}\frac{s}{s+1}G_{n+s+1}+\frac{s}{s+t}G_{n-t},$$

which gives that

$$\begin{aligned} G_n &\le \frac{(s+t)(s+1)}{s^2+st+s}\left(\frac{t}{s+t}\frac{s}{s+1}G_{n+s+1}+\frac{s}{s+t}G_{n-t}\right) \\ &= \frac{t}{s+t+1}G_{n+s+1}+\frac{s+1}{s+t+1}G_{n-t}. \end{aligned} \tag{2.20}$$

Similarly we can get the following inequality for the case $(s,t+1)$:

$$G_n \le \frac{t+1}{s+t+1}G_{n+s}+\frac{s}{s+t+1}G_{n-t-1}. \tag{2.21}$$

Based on what is proved above, we finally obtain

$$G_n \le \frac{t}{s+t}G_{n+s}+\frac{s}{s+t}G_{n-t}, \qquad s,\ t\ge 0,\ 0<t\le n,\ 0<s\le N-n. \tag{2.22}$$

This completes the proof of (2.11). □

Based on the definition of the energy norm, we have the following lemma.

**Lemma 2.1.** *The solution of* (2.2) *satisfies:*

$$|u^N|_*^2 \le |u^0|_*^2. \tag{2.23}$$

*Proof.* Using the definition of the functional $|\cdot|_*$, we get that

$$\begin{aligned} |u^N|_*^2-|u^0|_*^2 &= \sum_{n=0}^{N-1}\left(|u^{n+1}|_*^2-|u^n|_*^2\right) \\ &= \sum_{n=0}^{N-1}\left\{\sum_{i=0,j=0}^{I-1,J-1}\left(\sum_{k,l=1}^{2}a_{i,j}^{kl}(D_l^+u_{i,j}^{n+1})(D_k^+u_{i,j}^{n+1})+c_{i,j}u_{i,j}^{n+1}u_{i,j}^{n+1}\right)h_1h_2\right. \\ &\quad \left. -\sum_{i=0,j=0}^{I-1,J-1}\left(\sum_{k,l=1}^{2}a_{i,j}^{kl}(D_l^+u_{i,j}^{n})(D_k^+u_{i,j}^{n})+c_{i,j}u_{i,j}^{n}u_{i,j}^{n}\right)h_1h_2\right\}. \end{aligned}$$

Rearranging the right-hand side with some standard tricks gives

$$\begin{aligned}|u^N|_*^2-|u^0|_*^2=&-2\sum_{n=0}^{N-1}\left\{\sum_{ij}u_{i,j}^{n+1}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\frac{u_{i,j}^{n+1}}{2}\right)-c_{i,j}\frac{u_{i,j}^{n+1}}{2}\right)h_1h_2\right.\\&\left.-\sum_{ij}u_{i,j}^{n}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\frac{u_{i,j}^{n}}{2}\right)-c_{i,j}\frac{u_{i,j}^{n}}{2}\right)h_1h_2\right\}\\=&-2\sum_{n=0}^{N-1}\left\{\sum_{ij}u_{i,j}^{n+1}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{u}_{i,j}^{n}\right)-c_{i,j}\overline{u}_{i,j}^{n}\right)h_1h_2\right.\\&\left.-\sum_{ij}u_{i,j}^{n}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{u}_{i,j}^{n}\right)-c_{i,j}\overline{u}_{i,j}^{n}\right)h_1h_2\right\}\\=&-2\sum_{n=0}^{N-1}\left(\sum_{ij}(u_{ij}^{n+1}V_{ij}^n-u_{ij}^nV_{ij}^n)h_1h_2\right)=-2\sum_{n=0}^{N-1}\tau\left(\sum_{ij}V_{ij}^nV_{ij}^nh_1h_2\right)\le 0.\end{aligned}\tag{2.24}$$

This completes the proof of this lemma. □

**Theorem 2.3.** *If the solution $u$ of* (2.2) *also satisfies $|u^0|_*^2\le M$, where $M$ is a constant greater than 0, and we define two functionals*

$$|u|_{1,*}^2=\tau\sum_{n=1}^{N}|u^n|_*^2,\qquad |u^n|_0^2=\sum_{ij}(u_{i,j}^n)^2h_1h_2,\tag{2.25}$$

*then we have the following stability estimates for the solution in these two given functionals:*

$$|u|_{1,*}^2\le\frac{M-\varepsilon_1}{\ln M-\ln\varepsilon_1},\tag{2.26}$$

$$|u^n|_0^2\le\varepsilon_0+2\frac{M-\varepsilon_1}{\ln M-\ln\varepsilon_1},\qquad n=1,\cdots,N-1\tag{2.27}$$

$$|u^0|_0^2\le\varepsilon_0+2\frac{M-\varepsilon_1}{\ln M-\ln\varepsilon_1}+\tau M,\tag{2.28}$$

*where*

$$\varepsilon_0:=|u^N|_0^2,\quad \varepsilon_1:=|u^N|_*^2.$$

*Proof.* Summing up (2.11), we obtain

$$\begin{aligned}\tau\sum_{n=1}^{N}|u^n|_*^2\le&\tau\sum_{n=1}^{N}(|u^N|_*^2)^{n/N}(|u^0|_*^2)^{1-n/N}\\=&|u^0|_*^2\tau\sum_{n=1}^{N}\left(\frac{|u^N|_*^2}{|u^0|_*^2}\right)^{n/N}=|u^0|_*^2\tau\sum_{n=1}^{N}e^{n\ln\tilde{a}/N},\end{aligned}$$

where

$$\widetilde{a}:=|u^N|_*^2/|u^0|_*^2<1, \quad \ln\widetilde{a}<0.$$

Using the geometrical meaning of the definite integrals gives

$$\begin{aligned}\tau\sum_{n=1}^{N}|u^n|_*^2 &\le |u^0|_*^2\int_0^1 e^{t\ln\widetilde{a}}dt\\ &=|u^0|_*^2\frac{\widetilde{a}-1}{\ln\widetilde{a}}=\frac{|u^N|_*^2-|u^0|_*^2}{\ln|u^N|_*^2-\ln|u^0|_*^2}\\ &=\int_0^1|u^N|_*^{2t}|u^0|_*^{2-2t}dt\le\int_0^1|u^N|_*^{2t}M^{1-t}dt\\ &=\frac{|u^N|_*^2-M}{\ln|u^N|_*^2-\ln M}=\frac{M-\varepsilon_0}{\ln M-\ln\varepsilon_0}.\end{aligned} \tag{2.29}$$

So the stability estimate (2.26) is proved. Next, we have to prove the second stability estimate. For $k=0,\cdots,N-1$, we have

$$\begin{aligned}&|u^N|_0^2-|u^k|_0^2=\sum_{n=k}^{N-1}(|u^{n+1}|_0^2-|u^n|_0^2)\\ =&\sum_{n=k}^{N-1}\sum_{ij}\Big((u_{i,j}^{n+1})^2-(u_{i,j}^n)^2\Big)h_1h_2=\sum_{n=k}^{N-1}\sum_{ij}\Big(u_{i,j}^{n+1}-u_{i,j}^n\Big)\Big(u_{i,j}^{n+1}+u_{i,j}^n\Big)h_1h_2.\end{aligned}$$

Using the definition of the difference scheme in (2.2), we get

$$\begin{aligned}&|u^N|_0^2-|u^k|_0^2\\ =&\tau\sum_{n=k}^{N-1}\sum_{ij}\left(\sum_{k,l=1}^{2}D_k^-\left(a_{i,j}^{kl}D_l^+\overline{u}_{i,j}^n\right)-c_{i,j}\overline{u}_{i,j}^n\right)\Big(u_{i,j}^{n+1}+u_{i,j}^n\Big)h_1h_2\\ =&-\frac{1}{2}\tau\sum_{n=k}^{N-1}|u^n+u^{n+1}|_*^2\ge-\tau\sum_{n=k}^{N-1}(|u^n|_*^2+|u^{n+1}|_*^2).\end{aligned}$$

Furthermore, we obtain

$$|u^k|_0^2\le|u^N|_0^2+\tau\sum_{n=k}^{N-1}(|u^n|_*^2+|u^{n+1}|_*^2). \tag{2.30}$$

So that, for $k=1,\cdots,N-1$, we can get

$$\begin{aligned}|u^k|_0^2&\le|u^N|_0^2+2\tau\sum_{n=0}^{N-1}(|u^{n+1}|_*^2)=\varepsilon_1+2|u|_{1,*}^2\\ &\le\varepsilon_1+2\frac{M-\varepsilon_0}{\ln M-\ln\varepsilon_0}.\end{aligned} \tag{2.31}$$

Moreover, for $k=0$, we have

$$\begin{aligned}|u^0|_0^2 \le |u^N|_0^2 + 2\tau \sum_{n=0}^{N-1} (|u^{n+1}|_*^2) + \tau |u^0|_*^2 = \varepsilon_1 + 2|u|_{1,*}^2 + \tau |u^0|_*^2 \\ \le \varepsilon_1 + 2\frac{M-\varepsilon_0}{\ln M - \ln\varepsilon_0} + \tau M. \end{aligned} \tag{2.32}$$

This completes the proof of this theorem. □

If there is a bound on $|u^0|_*^2$ for the solution, then we can see that the solution $\{u_{i,j}^n\}$ is continuously dependent on the given data $\{g_{i,j}\}$ for $n=1,\cdots,N$.

## 3 An energy regularization method and the error estimates

Based on the stability estimates (2.26)-(2.28) proved in the last section, we will propose an energy regularization method for the numerical solution of the problem (2.1) which is an ill-posed problem and no classical numerical method in partial differential equations can be used to get a numerical approximation of it.

Instead of considering the backward diffusion problem, let us focus on the following finite difference scheme for the forward diffusion problem:

$$\begin{aligned}&\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\tau} = \sum_{k,l=1}^{2} D_k^- \left( a_{i,j}^{kl} D_l^+ \overline{v}_{i,j}^n \right) - c_{i,j} \overline{v}_{i,j}^n, \\ &\quad 1 \le i \le I-1,\ 1 \le j \le J-1,\ 0 \le n \le N-1, \\ &v_{0,j}^n = v_{I,j}^n = v_{i,0}^n = v_{i,J}^n = 0, \quad 0 \le i \le I,\ 0 \le j \le J,\ 1 \le n \le N, \\ &v_{i,j}^0 \text{ given}, \qquad 0 \le i \le I,\ 0 \le j \le J.\end{aligned} \tag{3.1}$$

For any given grid function

$$v_h^0 = \{v_{i,j}^0 : v_{0,j}^0 = v_{I,j}^0 = v_{i,0}^0 = v_{i,J}^0 = 0,\ 0 \le i \le I,\ 0 \le j \le J\},$$

the problem (3.1) has an unique solution $\{v_{i,j}^n,\ 0 \le i \le I,\ 0 \le j \le J,\ 0 \le n \le N\}$. Let $v_h^N$ denote the grid function $\{v_{i,j}^N,\ 0 \le i \le I,\ 0 \le j \le J\}$. Then we obtain a mapping $B$,

$$B:\ Bv_h^0 = v_h^N. \tag{3.2}$$

It is easy to see that the operator $B$ is bounded and linear, while the inverse problem is unstable. To overcome this difficulty, we will introduce an energy regularization method for it, finding a solution in a small set in which we have some compactness.

Suppose that the smooth function $u^*(x,t) \in C^{4,2}(\overline{\Omega}\times[0,1])$ is the unique solution of the continuous parabolic equation (2.1). Then $u^*(x,t)$ satisfies:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \sum_{k,l=1}^{2} \frac{\partial}{\partial x_k}\left(a^{kl}(x)\frac{\partial u}{\partial x_l}\right) - c(x)u, \qquad \forall (x,t)\in\Sigma,\\
&u=0, \quad \text{on } \partial\Omega\times[0,1]\\
&u\big|_{t=0} = u^*(x,0).
\end{aligned} \tag{3.3}$$

If $u^*(x,0)$ is given, the problem (3.3) is a well-posed problem.

Let $\{(u_h^*)_{i,j}^n,\ 0\le i\le I,\ 0\le j\le J,\ 0\le n\le N\}$ be the solution of the finite difference problem (3.1) with the initial condition $v_{i,j}^0 = u^*(ih_1,jh_2,0)$. Furthermore, we know that $\{(u_h^*)_{i,j}^n,\ 0\le i\le I,\ 0\le j\le J,\ 0\le n\le N\}$ is a finite difference approximation of $u^*(x,t)$. Using the standard methods, we can get the following error estimates between the grid function $u^* = \{(u^*)_{i,j}^n,\ 0\le i\le I,\ 0\le j\le J,\ 0\le n\le N\}$( here $(u^*)_{i,j}^n = u^*(ih_1,jh_2,n\tau)$) of $u^*(x,t)$ and $u_h^*$:

$$|u^* - u_h^*|_{1,*}^2 = \mathcal{O}(h^2), \tag{3.4}$$

$$|(u^*)^N - (u_h^*)^N|_*^2 = |g_h - (u_h^*)^N|_*^2 = \mathcal{O}(h^2), \tag{3.5}$$

$$|(u^*)^n - (u_h^*)^n|_0^2 = \mathcal{O}(h^4),\ \ n=0,\cdots,N-1, \tag{3.6}$$

where $h=\max(h_1,h_2,\tau)$ and $g_h$ is the discrete grid function of $g$ in (2.2), $g_h = \{g_{i,j},\ 0\le i\le I,\ 0\le j\le J\}$. From (3.5), we can assume that there exists a constant $C_1>0$ such that

$$|g_h - (u_h^*)^N|_*^2 \le C_1(h^2), \quad \forall h>0.$$

Then we consider the finite difference scheme of the backward diffusion problem. Since the inverse problem is unstable and we can not find the exact grid function $v_h^0$ from $v_h^N = g_h$, we have to find an approximation of $v_h^0$ in a small set.

For any given $\varepsilon$ $(C_1h^2 < \varepsilon \ll 1)$, we define the set

$$K_{\varepsilon,h} := \{v_h^0 \,|\, |Bv_h^0 - g_h|_*^2 \le \varepsilon\}, \tag{3.7}$$

which is a non-empty closed convex subset. The set is non-empty because

$$|g_h - (u_h^*)^N|_*^2 \le \varepsilon,$$

and $(u_h^*)^0$ belongs to $K_{\varepsilon,h}$.

Now we consider the following energy regularization problem:

$$\text{Find } u_h^0 \in K_{\varepsilon,h}, \text{ such that} \quad |u_h^0|_* = \min_{v_h^0\in K\varepsilon,h} |v_h^0|_*. \tag{3.8}$$

For fixed $\varepsilon > C_1h^2$, problem (3.8) is a well-posed problem, there exists a unique solution $u_h^0 \in K_{\varepsilon,h}$, and the associated grid functions at all time steps $\{u_h^n,\ 0\le n\le N\}$ are obtained.

We now claim that $u_h^n$ is an approximation of the solution $u^*(x,t)$ of the continuous problem (2.1).

**Theorem 3.1.** *Let $C_1h^2<\varepsilon\ll 1$ be any given constant. Suppose that $u^*(x,t)$ is the solution of the continuous problem* (2.1)*, $u_h^*$ is the solution of the finite difference problem* (3.1) *with the initial condition $v_{i,j}^0=u^*(ih_1,jh_2,0)$, and $u_h^0$ is the solution of the minimization problem* (3.8) *with $\{u_h^n\}$ the grid functions at all time steps. Then the solution $u_h=\{u_h^n\}$ is an approximation of $u_h^*$ and satisfies the following error estimates:*

$$|u^*-u_h|_{1,*}^2\le C_2h^2+8\frac{M-\varepsilon}{\ln M-\ln\varepsilon}, \tag{3.9}$$

$$|(u^*)^n-u_h^n|_0^2\le C_5h^4+16\frac{M-\varepsilon}{\ln M-\ln\varepsilon}+2C_4\varepsilon,\qquad n=1,\cdots,N-1 \tag{3.10}$$

$$|(u^*)^0-u_h^0|_0^2\le C_5h^4+16\frac{M-\varepsilon}{\ln M-\ln\varepsilon}+2C_4\varepsilon+8\tau M, \tag{3.11}$$

*where $|(u_h^*)^0|_*^2=M$ and $C_2$, $C_4$, $C_5$ are constants independent of $h$ and $\varepsilon$.*

*Proof.* For $u_h^0$ and the associated grid functions at all time steps $u_h^n$, one has

$$|u_h^0|_*^2\le|(u_h^*)^0|_*^2=M. \tag{3.12}$$

Set

$$M_F:=|(u_h^*)^0-u_h^0|_*^2,\ \ \varepsilon_F:=|(u_h^*)^N-u_h^N|_*^2. \tag{3.13}$$

Then

$$M_F\le 4M,\quad \varepsilon_F=|((u_h^*)^N-g_h)-(u_h^N-g_h)|_*^2\le 4\varepsilon.$$

Using the inequality (2.26) proved in Section 2, we obtain

$$|u_h^*-u_h|_{1,*}^2\le\frac{4M-\varepsilon_F}{\ln 4M-\ln\varepsilon_F}. \tag{3.14}$$

Using the representation

$$\frac{M-\varepsilon_F}{\ln M-\ln\varepsilon_F}=\int_0^1 M^s\varepsilon_F^{1-s}ds, \tag{3.15}$$

we finally obtain

$$\begin{aligned}|u_h^*-u_h|_{1,*}^2&\le\frac{4M-\varepsilon_F}{\ln 4M-\ln\varepsilon_F}=\int_0^1(4M)^s\varepsilon_F^{1-s}ds\\&\le\int_0^1(4M)^s(4\varepsilon)^{1-s}ds=\frac{4M-4\varepsilon}{\ln 4M-\ln 4\varepsilon}=4\frac{M-\varepsilon}{\ln M-\ln\varepsilon}.\end{aligned} \tag{3.16}$$

Combining the estimate (3.16) with the error estimate (3.4) together with the triangle inequality gives

$$|u^*-u_h|_{1,*}\le|u^*-u_h^*|_{1,*}+|u_h^*-u_h|_{1,*}. \tag{3.17}$$

We obtain the following estimate of the error between the solution and the numerical result:

$$|u^*-u_h|_{1,*}^2 \le C_2h^2+8\frac{M-\varepsilon}{\ln M-\ln\varepsilon}, \tag{3.18}$$

where $C_2$ is a positive constant independent of $h$ and $\varepsilon$. Using the equivalent norm theorem we can find a constant $C_3>0$ such that

$$|(u_h^*)^N-u_h^N|_0^2 \le |(u_h^*)^N-u_h^N|_1^2 \le C_3|(u_h^*)^N-u_h^N|_*^2. \tag{3.19}$$

From the stability estimates proved in Theorem **2.3**, we have the error estimates

$$|(u_h^*)^n-u_h^n|_0^2 \le C_4\varepsilon+8\frac{M-\varepsilon}{\ln M-\ln\varepsilon}, \qquad n=1,\cdots,N-1 \tag{3.20}$$

$$|(u_h^*)^0-u_h^0|_0^2 \le C_4\varepsilon+8\frac{M-\varepsilon}{\ln M-\ln\varepsilon}+4\tau M, \tag{3.21}$$

where $C_4$ is a positive constant independent of $\varepsilon$. Similarly, combining the estimates (3.20)-(3.21) with the estimates (3.6) and using the triangle inequality

$$|(u^*)^n-u_h^n|_0 \le |(u^*)^n-(u_h^*)^n|_0+|(u_h^*)^n-u_h^n|_0, \tag{3.22}$$

we have the following estimates:

$$|(u^*)^n-u_h^n|_0^2 \le C_5h^4+16\frac{M-\varepsilon}{\ln M-\ln\varepsilon}+2C_4\varepsilon, \qquad n=1,\cdots,N-1 \tag{3.23}$$

$$|(u^*)^0-u_h^0|_0^2 \le C_5h^4+16\frac{M-\varepsilon}{\ln M-\ln\varepsilon}+2C_4\varepsilon+8\tau M. \tag{3.24}$$

This completes the proof of this theorem. □

In fact, the upper bound of the error can be also given by

$$|(u^*)^0-u_h^0|_0^2 \le C\left(h^4+\varepsilon+\tau+\frac{M-\varepsilon}{\ln M-\ln\varepsilon}\right), \tag{3.25}$$

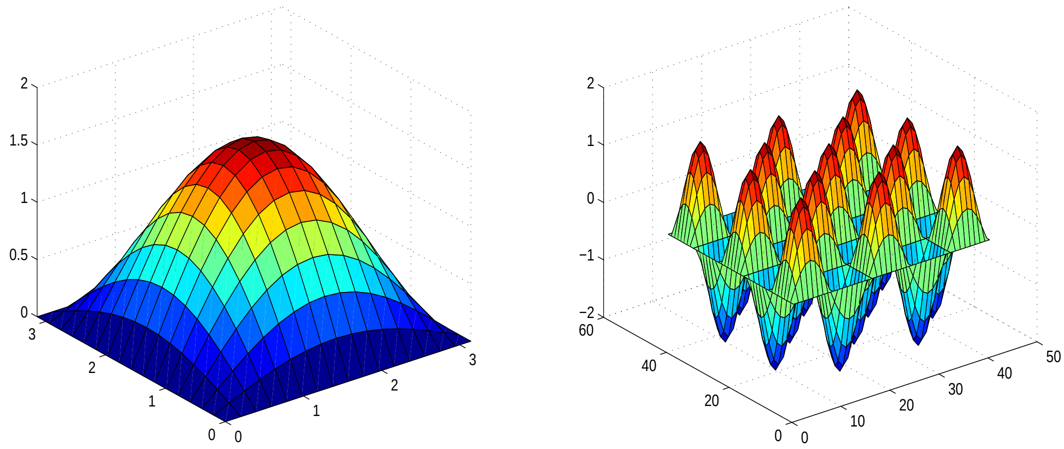
from which we can easily see that the value of $\epsilon$ has a strong influence on the bound of the error. However, as demonstrated in (3.7), the value of $\epsilon$ cannot been arbitrarily small.

# 4 Numerical examples

## 4.1 Test examples

The examples chosen here are such that the solutions are already known. We consider the following problem:

$$\begin{aligned} &u_t=\alpha\Delta u, \qquad (x,y,t)\in(0,\pi)\times(0,\pi)\times(0,1),\\ &u\big|_{x=0}=u|_{x=1}=u|_{y=0}=u|_{y=1}=0,\\ &u\big|_{t=1}=\sin mx\sin my, \qquad (x,y)\in(0,\pi)\times(0,\pi), \end{aligned} \tag{4.1}$$




$m\!=\!1$, $\alpha\!=\!1/4$, $N\!=\!16$, $\epsilon\!=\!10^{-4}$ $m\!=\!5$, $\alpha\!=\!1/100$, $N\!=\!40$, $\epsilon\!=\!10^{-2}$

Figure 1: Numerical result at $t\!=\!0$.

where $\alpha > 0$ is a constant. The unique solution of problem (4.1) is

$$u(x,y,t) = e^{2\alpha(1-t)m^2} \sin mx \sin my. \tag{4.2}$$

For numerical approximations, we discretize the problem by a uniform mesh of size $h = \pi/N$ in the $x$-direction and $y$-direction and a time mesh of size $\tau$, and obtain a numerical solution $u_h^0$ at $t\!=\!0$ to the problem (4.1). The $L^2$ norm errors $|(u^*)^0 - u_h^0|_2$ for $m\!=\!1$, $\alpha\!=\!1/4$ are shown in Table 1. The numerical results demonstrate a convergence rate of $\sqrt{\epsilon}$ as $N$ is sufficiently large.

Fig. 1 shows the numerical results at $t=0$ ($m=1$, $\alpha=1/4$, $N=16$, $\epsilon=10^{-4}$ and $m\!=\!5$, $\alpha\!=\!1/100$, $N\!=\!40$, $\epsilon\!=\!10^{-2}$). These results are found to be in good agreement with the exact solution (4.2).

Table 1: $L^2$ norm errors for problem (4.1) ($m\!=\!1$).

| $\epsilon \backslash N$ | 8 | 16 | 32 | 64 |
|---|---|---|---|---|
| $10^{-1}$ | $3.845\times10^{-1}$ | $3.721\times10^{-1}$ | $3.690\times10^{-1}$ | $3.688\times10^{-1}$ |
| $10^{-2}$ | $1.323\times10^{-1}$ | $1.200\times10^{-1}$ | $1.168\times10^{-1}$ | $1.166\times10^{-1}$ |
| $10^{-3}$ | $5.260\times10^{-2}$ | $4.023\times10^{-2}$ | $3.711\times10^{-2}$ | $3.693\times10^{-2}$ |
| $10^{-4}$ | $2.738\times10^{-2}$ | $1.501\times10^{-2}$ | $1.189\times10^{-2}$ | $1.172\times10^{-2}$ |
| $10^{-5}$ | $1.941\times10^{-2}$ | $7.036\times10^{-3}$ | $3.919\times10^{-3}$ | $3.745\times10^{-3}$ |

In the diffusion equation, the higher the frequency of the signal is, the faster the attenuation. Therefore, if $m$ is large and the diffusion time is long, the given final value will be much smaller than the solution of the backward diffusion equation. This results in that the solution overflows in current computers with finite wordlength. In summery, the computation becomes more and more difficult as the frequency of the final value grows.

## 4.2 Examples in image deblurring

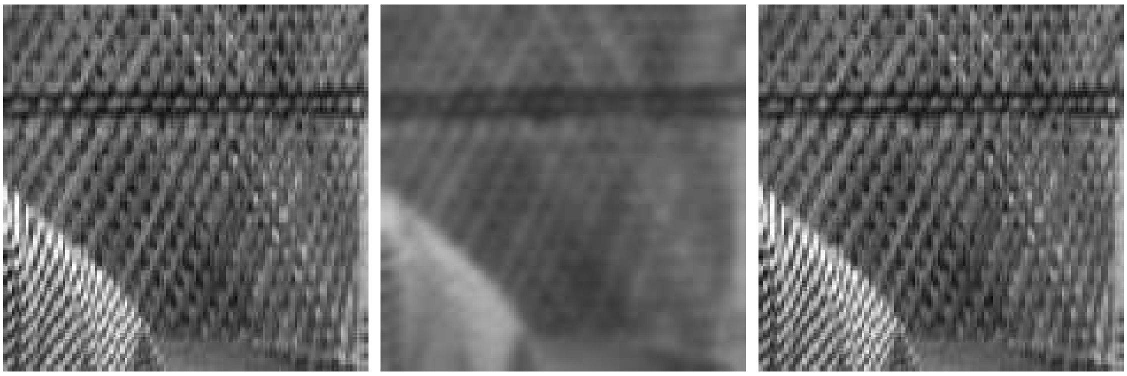
In this subsection, we are concerned with images which are degraded by the diffusion problem (1.1). We assume that the original images are of high quality in focused images. After time $T$, the images are blurred by diffusion (1.1) with the second boundary condition $\frac{\partial u}{\partial \nu}=0$, and we obtained blurred images $g(x)$, which are what we observed at time $T$. The problem is how to get the original images $u_0(x)$ from the known blurred images $g(x)$, which is equivalent to how to obtain the numerical solution of the backward diffusion problem. We use the energy regularization method derived in the last section to reconstruct the images.

In Figs. 2 and 3, we show two examples using the isotropic diffusion equation, which is also known as the heat equation. In each figure, three pictures are given. Therein (a) is the true image; while (b) is the blurred version $u_T(x)$, which is from the original image (a) by the diffusion equation (2.1). The right one (c) is the reconstructed image obtained by solving the backward diffusion equation from the blurred image (b).

(a) (b) (c)

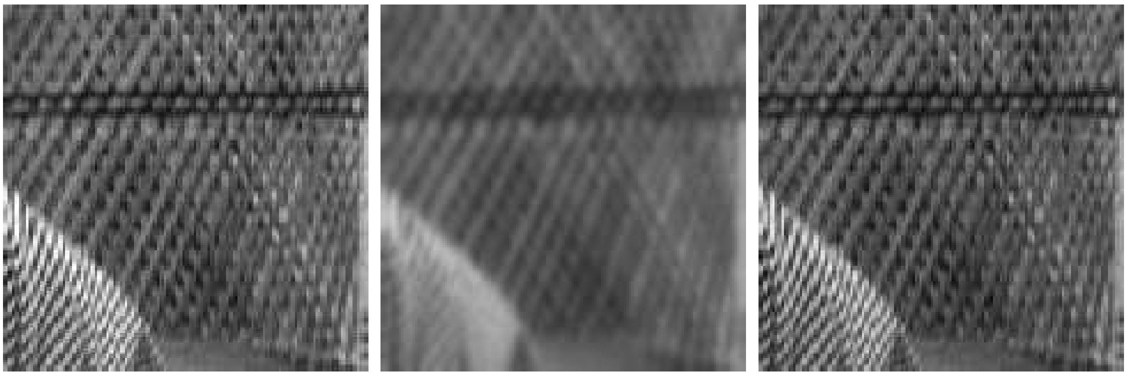
Figure 2: Deblurring using the backward isotropic diffusion equation, left: original image, middle: blurred image, right: recovered image.

In our last example we illustrate the effect of anisotropic blurring and solving the corresponding backward problem. In Fig. 4, (a) is the true image; while (b) is the blurred version which is obtained via anisotropic diffusion equation. Fig. 4 (c) is the recovered image by our method.



(a) (b) (c)

Figure 3: Deblurring using the backward isotropic diffusion equation, left: original image, middle: blurred image, right: recovered image.



(a) (b) (c)

Figure 4: Deblurring using the backward anisotropic diffusion equation, left: original image, middle: blurred image, right: recovered image.

## 5 Conclusion

We proposed a discrete energy regularization method for the backward diffusion equation by finite difference methods. We prove the existence and uniqueness of the solution to this discrete problem. In addition, the error estimates are given. Also, in the example section, we apply this algorithm in image deblurring, and the numerical examples demonstrate the efficiency of this method. Many problems, such as using this method for restoration of noisy blurred image, are still open.

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