Elements of Mathematical Foundations for Numerical Approaches for Weakly Random Homogenization Problems

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Abstract. This work is a follow-up to our previous work [2]. It extends and complements, both theoretically and experimentally, the results presented there. Under consideration is the homogenization of a model of a weakly random heterogeneous material. The material consists of a reference periodic material randomly perturbed by another periodic material, so that its homogenized behavior is close to that of the reference material. We consider laws for the random perturbations more general than in [2]. We prove the validity of an asymptotic expansion in a certain class of settings. We also extend the formal approach introduced in [2]. Our perturbative approach shares common features with a defect-type theory of solid state physics. The computational efficiency of the approach is demonstrated.

1 Introduction

Our purpose is to follow up on our previous study [2]. Let us recall, for consistency, that we consider homogenization for the following elliptic problem

$$\begin{cases} -\text{div}\left(\left(A_{per}\left(\dfrac{x}{\epsilon}\right)+b_{\eta}\left(\dfrac{x}{\epsilon},\omega\right)C_{per}\left(\dfrac{x}{\epsilon}\right)\right)\nabla u_{\epsilon}\right)=f(x), & \text{in } \mathcal{D}\subset\mathbb{R}^d, \\ u_{\epsilon}=0, & \text{on } \partial\mathcal{D}, \end{cases} \quad (1.1)$$

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where the tensor A_{per} models a reference $\mathbb{Z}^d$-periodic material which is randomly perturbed by the $\mathbb{Z}^d$-periodic tensor C_{per}, the stochastic nature of the problem being encoded in the stationary ergodic scalar field b_η (the latter getting small when η vanishes). We have studied in [2] the case of a perturbation that has a Bernoulli law with parameter η, meaning that b_η is equal to 1 with probability η and 0 with probability $1-\eta$. In the present work, we address more general laws. The common setting is that all the perturbations we consider are, to some extent, rare events which, although rare, modify the homogenized properties of the material. Our approach is a perturbative approach, and consists in approximating the stochastic homogenization problem for

$$A_\eta(x,\omega) = A_{per}(x) + b_\eta(x,\omega)C_{per}$$

using the periodic homogenization problem for A_{per}. In short, let us say that our main contribution is to derive an expansion

$$A_\eta^* = A_{per}^* + \eta \bar{A}_1^* + \eta^2 \bar{A}_2^* + o(\eta^2), \tag{1.2}$$

where A_η^* and A_{per}^* are the homogenized tensors associated with A_η and A_{per} respectively, and the first and second-order corrections $\bar{A}_1^*$ and $\bar{A}_2^*$ can be, loosely speaking, computed in terms of the microscopic properties of A_{per} and C_{per} and the statistics of second order of the random field b_η. The formulation is made precise in [2] and in Sections 2 and 3 below.

Motivations behind setting (1.1), as well as a review of the mathematical literature on similar issues and a comprehensive bibliography, can be found in [2]. We complement our study of the perturbative approach introduced with [2] in two different directions.

In Section 2, we rigorously establish an asymptotic expansion of the homogenized tensor in a mathematical setting where our input parameter (the field b_η in (1.1)) enjoys appropriate weak convergence properties, as η vanishes, in a reflexive Banach space, namely a Lebesgue space $L^\infty(\mathcal{D}, L^p(\Omega))$ (with $p > 1$). In such a setting, we are in position to rigorously prove a first order asymptotic expansion (announced in [3] and precisely stated in [3, Theorem 2.1] and Theorem 2.1 below) for the homogenization of A_η, using simple functional analysis techniques very similar to those exposed in [4]. In our Corollaries 2.1 and 2.2, the expansion is pushed to second order under additional assumptions.

Our aim in Section 3 is to further extend our formal theory of [2]. Recall that this formal theory, rather than manipulating the random field b_η itself, consists in focusing on its law. We indeed assume that the image measure (the law) corresponding to the perturbation admits an expansion (see (3.3) below) with respect to η in the sense of distributions. While [2] has only addressed the specific case of a Bernoulli law, we consider here more general laws and proceed with the same formal derivations. These derivations lead to a first-order correction $\bar{A}_1^*$ in (1.2) obtained as the limit when $N \to \infty$ of a sequence of tensors $A_1^{*,N}$ computed on the supercell $[-N/2, N/2]^d$. It is the purpose of Proposition 3.1 to prove the convergence of $A_1^{*,N}$. The second-order term $\bar{A}_2^*$ is likewise defined, in Proposition 3.2, as the limit of a sequence of tensors $A_2^{*,N}$ when $N \to \infty$. The proof of

convergence is not given here for it involves long and technical computations. We refer the reader to [1] for the details. As in [2], our approach in this Section exhibits close ties with classical defect-type theories used in solid state physics.

We emphasize that, in sharp contrast to the exact stochastic homogenization of A_η, the determination of the first and second-order terms in (1.2) relies on entirely deterministic computations, albeit of very different kind, for both approaches of Sections 2 and 3.

Finally, a comprehensive series of numerical tests in Section 4 show, beyond those contained in [2], that the two approaches exposed here are efficient and quite robust: the computational workload induced by the perturbative approach is light compared to the direct homogenization of [2], and expansion (1.2) proves to be accurate for not so small perturbations.

We complement the text by a long appendix. The reader less interested in theoretical issues can easily omit the reading of this appendix. Besides providing, in Section A.1 and for consistency, some theoretical results useful in the body of the text, the purpose of this appendix is two-fold. We examine in details in Section A.2 the one-dimensional setting, and we show that, expectedly, all our formal expansions can be made rigorous through explicit computations. We next demonstrate, in Section A.3, that our two modes of derivation coincide in a particular setting appropriate for both the theoretical results of Section 2 and the formal results of Section 3. This final section therefore provides a *proof* of our formal manipulations of Section 3, in a setting-we concede it-that is not the setting the approach was designed to specifically address. Definite conclusions on the theoretical validity of the approach developed in Section 3 are yet to be obtained, even though applicability and efficiency are beyond doubt.

Throughout this paper, and unless otherwise mentioned, C denotes a constant that depends at most on the ambient dimension d, and on the tensors A_{per} and C_{per}. We write $C(\gamma)$ when C depends on γ and possibly on d, A_{per} and C_{per}. The indices i and j denote indices in $[\![1,d]\!]$.

2　A model of a weakly randomly perturbed material

For consistency, we first recall the general setting of our related work [2].

Throughout this article $(\Omega,\mathcal{F},\mathbb{P})$ denotes a probability space with $\mathbb{P}$ the probability measure and $\omega\in\Omega$ an event. We denote by $\mathbb{E}(X)$ the expectation of a random variable X and $Var(X)$ its variance.

We assume that the group $(\mathbb{Z}^d,+)$ acts on Ω and denote by $\tau_k, k\in\mathbb{Z}^d$, the group action. We also assume that this action is measure-preserving, that is,

$$\forall\mathcal{A}\in\mathcal{F},\ \forall k\in\mathbb{Z}^d,\quad \mathbb{P}(\mathcal{A})=\mathbb{P}(\tau_k\mathcal{A}),$$

and ergodic:

$$\forall\mathcal{A}\in\mathcal{F},\ (\forall k\in\mathbb{Z}^d,\ \mathcal{A}=\tau_k\mathcal{A})\Longrightarrow(\mathbb{P}(\mathcal{A})=0\ \text{or}\ \mathbb{P}(\mathcal{A})=1).$$

We call $F \in L^1_{loc}(\mathbb{R}^d, L^1(\Omega))$ stationary if

$$\forall k \in \mathbb{Z}^d, \ F(x+k, \omega) = F(x, \tau_k \omega) \text{ almost everywhere in } x \in \mathbb{R}^d \text{ and } \omega \in \Omega. \qquad (2.1)$$

Notice that if F is deterministic, the notion of stationarity used here reduces to $\mathbb{Z}^d$-periodicity, that is,

$$\forall k \in \mathbb{Z}^d, \ F(x+k) = F(x) \text{ almost everywhere in } x \in \mathbb{R}^d. \qquad (2.2)$$

We then consider the tensor field from $\mathbb{R}^d \times \Omega$ to $\mathbb{R}^{d \times d}$:

$$A_\eta(x, \omega) = A_{per}(x) + b_\eta(x, \omega) C_{per}(x), \qquad (2.3)$$

where A_{per} and C_{per} are two deterministic $\mathbb{Z}^d$-periodic tensor fields and b_η a stationary ergodic scalar field. The matrix A_{per} models the reference periodic material, perturbed by C_{per}. This perturbation is random, thus the presence of b_η. We refer the reader to [4] for a more detailed presentation of the stationary ergodic setting in a similar weakly random framework.

We make the following assumptions on the random field b_η:

$$\exists M > 0, \ \forall \eta > 0, \ \|b_\eta\|_{L^\infty(Q \times \Omega)} \leq M, \qquad (2.4a)$$

$$\|b_\eta\|_{L^\infty(Q; L^2(\Omega))} \xrightarrow[\eta \to 0^+]{} 0, \qquad (2.4b)$$

where Q is the unit cell $[-1/2, 1/2]^d$.

Assumption (2.4b) encodes that the perturbation for small η is a rare event. Still, it is able to significantly modify the local structure of the material when it happens, for we do not require it to be small in $L^\infty(Q \times \Omega)$ as $\eta \to 0$.

We additionally assume that there exist $0 < \alpha \leq \beta$ such that for all $\xi \in \mathbb{R}^d$, for almost all $x \in \mathbb{R}^d$ and for all $s \in [-M, M]$,

$$\alpha |\xi|^2 \leq A_{per}(x)\xi \cdot \xi, \qquad \alpha |\xi|^2 \leq (A_{per} + sC_{per})(x)\xi \cdot \xi, \qquad (2.5a)$$

$$A_{per}(x)\xi \leq \beta |\xi|, \qquad |(A_{per} + sC_{per})(x)\xi| \leq \beta |\xi|. \qquad (2.5b)$$

We can therefore use the classical stochastic homogenization results (see for instance [7] for a comprehensive review or [2] for a concise presentation). The cell problems associated with (2.3) read

$$\begin{cases} -\mathrm{div}\left(A_\eta(\nabla w_i^\eta + e_i)\right) = 0, & \text{in } \mathbb{R}^d, \\ \nabla w_i^\eta \text{ stationary}, \ \mathbb{E}\left(\int_Q \nabla w_i^\eta\right) = 0. \end{cases} \qquad (2.6)$$

Problem (2.6) has a solution unique up to the addition of a random constant. The function w_i^η is called the i-th corrector or cell solution.

The homogenized tensor A_η^* is given by

$$\forall i \in [\![1,d]\!], \quad A_\eta^* e_i = \mathbb{E}\left(\int_Q A_\eta (\nabla w_i^\eta + e_i)\right). \tag{2.7}$$

Throughout the rest of this paper we will denote by w_i^0 the i-th cell solution associated with A_{per}, defined up to an additive constant by

$$\begin{cases} -\mathrm{div}\left(A_{per}(\nabla w_i^0 + e_i)\right) = 0, & \text{in } Q, \\ w_i^0 \ \mathbb{Z}^d\text{-periodic.} \end{cases} \tag{2.8}$$

The periodic homogenized tensor is then given by

$$\forall i \in [\![1,d]\!], \quad A_{per}^* e_i = \int_Q A_{per}(\nabla w_i^0 + e_i). \tag{2.9}$$

Due to the specific form of A_η, the following zero-order result can be easily proved. The proof is actually the same as that in Lemma 1 of [2], which relies on the fact that $\|b_\eta\|_{L^\infty(Q;L^2(\Omega))}$ converges to 0 as η tends to 0.

Lemma 2.1. *When $\eta \to 0$, $A_\eta^* \to A_{per}^*$.*

Our goal is to find an asymptotic expansion for A_η with respect to η, and a first answer is given by the following theorem announced as Theorem 1 in [3]:

Theorem 2.1 (Theorem 1, [3]). *Assume that b_η satisfies (2.4a) and (2.4b), and denote by $m_\eta = \|b_\eta\|_{L^\infty(Q;L^2(\Omega))}$. There exists a subsequence of η, still denoted η for the sake of simplicity, such that b_η/m_η converges weakly-* in $L^\infty(Q;L^2(\Omega))$ to a limit field denoted by $\bar{b}_0$ when $n \to 0$. Then*

- *for all $i \in [\![1,d]\!]$, the following expansion*

$$\nabla w_i^\eta = \nabla w_i^0 + m_\eta \nabla v_i^0 + o(m_\eta) \tag{2.10}$$

  holds weakly in $L^2(Q;L^2(\Omega))$, where w_i^0 is the solution to the i-th periodic cell problem and v_i^0 is solution to

$$\begin{cases} -\mathrm{div}(A_{per}\nabla v_i^0) = \mathrm{div}\left(\bar{b}_0 C_{per}(\nabla w^0 + e_i)\right), & \text{in } \mathbb{R}^d, \\ \nabla v_i^0 \text{ stationary}, \quad \mathbb{E}\left(\int_Q \nabla v_i^0\right) = 0. \end{cases} \tag{2.11}$$

- *A_η^* can be expanded up to first order as*

$$A_\eta^* = A_{per}^* + m_\eta \widetilde{A_1^*} + o(m_\eta), \tag{2.12}$$

  where

$$\forall i \in [\![1,d]\!], \quad \widetilde{A_1^*} e_i = \int_Q \mathbb{E}(\bar{b}_0) C_{per}(\nabla w_i^0 + e_i) + \int_Q A_{per}\nabla \mathbb{E}(v_i^0). \tag{2.13}$$

Proof. We fix $i \in [\![1, d]\!]$ and define $v_i^\eta = (w_i^\eta - w_i^0)/m_\eta$. v_i^η is solution to

$$
\begin{cases}
-\mathrm{div}\left(A_\eta \nabla v_i^\eta\right) = \mathrm{div}\left(\dfrac{b_\eta}{m_\eta} C_{per}(\nabla w_i^0 + e_i)\right), & \text{in } \mathbb{R}^d, \\[2mm]
\nabla v_i^\eta \text{ stationary,} \quad \mathbb{E}\left(\displaystyle\int_Q \nabla v_i^\eta\right) = 0.
\end{cases}
\tag{2.14}
$$

Using an argument similar to that used in the proof of Lemma 1 in [2], we have

$$
\forall \eta > 0, \quad \|\nabla v_i^\eta\|_{L^2(Q \times \Omega)} \leq \frac{1}{\alpha} \|C_{per}(\nabla w_i^0 + e_i)\|_{L^2(Q)},
$$

where α is defined in (2.5a).

The sequence ∇v_i^η is bounded in $L^2(Q \times \Omega)$ and therefore, up to extraction, weakly converges in $L^2(Q \times \Omega)$ to some limit which is necessarily a gradient and which we denote ∇v_i^0. Since b_η converges strongly to 0 in $L^2(Q \times \Omega)$, $b_\eta \nabla v_i^\eta$ converges to 0 in $\mathcal{D}'(Q \times \Omega)$. It is then easy to pass to the limit $\eta \to 0$ in (2.14) and to deduce that v_i^0 is solution to

$$
\begin{cases}
-\mathrm{div}\left(A_{per} \nabla v_i^0\right) = \mathrm{div}\left(\bar{b}_0 C_{per}(\nabla w_i^0 + e_i)\right), & \text{in } \mathbb{R}^d, \\[2mm]
\nabla v_i^0 \text{ stationary,} \quad \mathbb{E}\left(\displaystyle\int_Q \nabla v_i^0\right) = 0.
\end{cases}
$$

Thus $(\nabla w_i^\eta - \nabla w_i^0)/m_\eta$ converges, up to extraction, weakly to ∇v_i^0 in $L^2(Q \times \Omega)$. This amounts to say that we have the following first-order expansion:

$$
\nabla w_i^\eta = \nabla w_i^0 + m_\eta \nabla v_i^0 + o(m_\eta), \quad \text{in } L^2(Q \times \Omega) \text{ weak.}
$$

Inserting this expansion in (2.7), we obtain

$$
A_\eta^* e_i = A_{per}^* e_i + m_\eta \int_Q \mathbb{E}(\bar{b}_0) C_{per}(\nabla w_i^0 + e_i) + m_\eta \int_Q A_{per} \nabla \mathbb{E}(v_i^0) + o(m_\eta),
$$

which concludes the proof. $\qquad\square$

Remark 2.1. Notice that taking the expectation of both sides of (2.11), $\mathbb{E}(v_i^0)$ is actually the $\mathbb{Z}^d$-periodic function that is the unique solution (up to an additive constant) to

$$
\begin{cases}
-\mathrm{div}\left(A_{per} \nabla \mathbb{E}(v_i^0)\right) = \mathrm{div}\left(\mathbb{E}(\bar{b}_0) C_{per}(\nabla w_i^0 + e_i)\right), & \text{in } Q, \\[2mm]
\mathbb{E}(v_i^0) \ \mathbb{Z}^d\text{-periodic.}
\end{cases}
\tag{2.15}
$$

The computation of A_η^* up to the first order in m_η only requires solving $2d$ deterministic problems, namely (2.8) and (2.15), in the unit cell Q.

In fact, the situation is even more advantageous when A_{per} is a symmetric matrix, as shown by our next remark.

Remark 2.2. Defining the adjoint problems to the cell problems (2.8),

$$\begin{cases} -\operatorname{div}\left(A_{per}^{T}(\nabla\widetilde{w}_{i}^{0}+e_{i})\right)=0, & \text{in } Q, \\ \widetilde{w}_{i}^{0} \ \mathbb{Z}^{d}\text{-periodic}, \end{cases} \tag{2.16}$$

where we have denoted by A_{per}^{T} the transposed matrix of A_{per}, allows to write the first-order correction (2.13) in a slightly different form. Indeed, multiplying (2.15) by $\widetilde{w}_{j}^{0}$ and integrating by parts, we obtain

$$\int_{Q} A_{per}\nabla\mathbb{E}(v_{i}^{0})\cdot\nabla\widetilde{w}_{j}^{0}=-\int_{Q}\mathbb{E}(\bar{b}_{0})C_{per}\left(\nabla w_{i}^{0}+e_{i}\right)\cdot\nabla\widetilde{w}_{j}^{0}.$$

Likewise, multiplying (2.16) by $\nabla\mathbb{E}(v_{i}^{0})$ and integrating by parts yields

$$\int_{Q} A_{per}\nabla\mathbb{E}(v_{i}^{0})\cdot\left(\nabla\widetilde{w}_{j}^{0}+e_{j}\right)=0.$$

Combining these equalities gives

$$\int_{Q} A_{per}\nabla\mathbb{E}(v_{i}^{0})\cdot e_{j}=\int_{Q}\mathbb{E}(\bar{b}_{0})C_{per}\left(\nabla w_{i}^{0}+e_{i}\right)\cdot\nabla\widetilde{w}_{j}^{0}$$

and thus (2.13) may be equivalently phrased as

$$\forall(i,j)\in[\![1,d]\!]^{2}, \quad \widetilde{A_{1}^{*}}e_{i}\cdot e_{j}=\int_{Q}\mathbb{E}(\bar{b}_{0})C_{per}(\nabla w_{i}^{0}+e_{i})\cdot(\nabla\widetilde{w}_{j}^{0}+e_{j}). \tag{2.17}$$

When A_{per} is symmetric, $\widetilde{w}_{j}^{0}=w_{j}^{0}$, and solving the periodic cell problems (2.8) suffices to determine A_{η}^{*} up to the first order in m_{η}.

Pushing expansion (2.12) to second order requires more information on b_{η}:

Corollary 2.1. Assume in addition to (2.4a) and (2.4b) that

$$b_{\eta}=\eta\bar{b}_{0}+\eta^{2}\bar{r}_{0}+o(\eta^{2}) \text{ weakly}-^{*} \text{ in } L^{\infty}(Q;L^{2}(\Omega)). \tag{2.18}$$

Then

- for all $i\in[\![1,d]\!]$, the following expansion

$$\nabla w_{i}^{\eta}=\nabla w_{i}^{0}+\eta\nabla v_{i}^{0}+\eta^{2}\nabla z_{i}^{0}+o(\eta^{2}) \tag{2.19}$$

holds weakly in $L^{2}(Q;L^{2}(\Omega))$, where z_{i}^{0} is solution to

$$\begin{cases} -\operatorname{div}(A_{per}\nabla z_{i}^{0})=\operatorname{div}\left(\bar{r}_{0}C_{per}(\nabla w_{i}^{0}+e_{i})\right)+\operatorname{div}\left(\bar{b}_{0}C_{per}\nabla v_{i}^{0}\right), & \text{in } \mathbb{R}^{d}, \\ \nabla z_{i}^{0} \text{ stationary}, \quad \mathbb{E}\left(\int_{Q}\nabla z_{i}^{0}\right)=0. \end{cases} \tag{2.20}$$

- A_η^* can be expanded up to second order as

$$A_\eta^* = A_{per}^* + \eta \widetilde{A_1^*} + \eta^2 \widetilde{A_2^*} + o(\eta^2),\qquad(2.21)$$

where $\widetilde{A_1^*}$ is defined by (2.13) and for all $i \in [\![1,d]\!]$,

$$\widetilde{A_2^*} e_i = \int_Q \mathbb{E}(\bar{r}_0) C_{per} (\nabla w_i^0 + e_i) + \eta^2 \int_Q C_{per} \mathbb{E}(\bar{b}_0 \nabla v_i^0) + \int_Q A_{per} \nabla \mathbb{E}(z_i^0),\qquad(2.22)$$

or equivalently, for all $(i,j) \in [\![1,d]\!]^2$,

$$\widetilde{A_2^*} e_i \cdot e_j = \int_Q \mathbb{E}(\bar{r}_0) C_{per} (\nabla w_i^0 + e_i) \cdot (\nabla \widetilde{w}_j^0 + e_j) + \int_Q C_{per} \mathbb{E}(\bar{b}_0 \nabla v_i^0) \cdot (\nabla \widetilde{w}_j^0 + e_j).\qquad(2.23)$$

Proof. The proof follows the same pattern as that of Theorem 2.1. The computation of the second order relies on the fact that (2.18) implies that b_η / η converges strongly to $\bar{b}_0$ in $L^\infty(Q; L^2(\Omega))$, whereas the convergence was weak in Theorem 2.1. Likewise, the expansion of the cell solution, namely (2.19), implies that $(\nabla w_i^\eta - \nabla w_i^0)/\eta$ converges strongly to ∇v_i^0 in $L^2(Q; L^2(\Omega))$. We then obtain (2.21) and (2.22) by inserting (2.19) in (2.7) and deduce (2.23) from (2.22) as in Remark 2.2. $\qquad\square$

The computation of A_η^* up to the order η^2 is much more intricate than that up to the order η, for it requires determining $\mathbb{E}(\bar{b}_0 \nabla v_i^0)$. Computing the periodic deterministic function $\mathbb{E}(v_i^0)$ solution to the simpler problem (2.15) is not sufficient in general. We have to determine the stationary random field v_i^0 solution to (2.11) in $\mathbb{R}^d$.

It turns out that in a particular, practically relevant setting, we may still avoid solving the random problem (2.11). This setting presents the additional advantage to provide insight on the influence of spatial correlation.

Corollary 2.2. Assume that b_η is uniform in each cell of $\mathbb{Z}^d$ and writes

$$b_\eta(x,\omega) = \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{Q+k}(x) B_\eta(\tau_k \omega),\qquad(2.24)$$

where B_η satisfies

$$\forall \eta > 0, \ \|B_\eta\|_{L^\infty(\Omega)} \leq M,\qquad(2.25a)$$
$$B_\eta = \eta \bar{B}_0 + \eta^2 \bar{R}_0 + o(\eta^2) \text{ weakly in } L^2(\Omega).\qquad(2.25b)$$

Assume also that

$$\sum_{k \in \mathbb{Z}^d} \left| cov(\bar{B}_0, \bar{B}_0(\tau_k \cdot)) \right| < \infty.\qquad(2.26)$$

Then the second-order term (2.23) can be rewritten

$$\widetilde{A_2^*}e_i \cdot e_j = \mathbb{E}(\bar{R}_0) \int_Q C_{per}(\nabla w_i^0 + e_i) \cdot (\nabla \widetilde{w}_j^0 + e_j) + Var(\bar{B}_0) \int_Q C_{per} \nabla t_i \cdot (\nabla \widetilde{w}_j^0 + e_j)$$

$$+ (\mathbb{E}(\bar{B}_0))^2 \int_Q C_{per} \nabla s_i \cdot (\nabla \widetilde{w}_j^0 + e_j)$$

$$+ \sum_{k \in \mathbb{Z}^d, k \neq 0} cov(\bar{B}_0, \bar{B}_0(\tau_k \cdot)) \int_Q C_{per} \nabla t_i(\cdot - k) \cdot (\nabla \widetilde{w}_j^0 + e_j), \tag{2.27}$$

where t_i is a $L_{loc}^2(\mathbb{R}^d)$ function solving

$$\begin{cases} -\mathrm{div}(A_{per} \nabla t_i) = \mathrm{div}(C_{per} \mathbb{1}_Q(\nabla w_i^0 + e_i)), & \text{in } \mathbb{R}^d, \\ \nabla t_i \in L^2(\mathbb{R}^d), \end{cases} \tag{2.28}$$

and s_i solves

$$\begin{cases} -\mathrm{div}(A_{per} \nabla s_i) = \mathrm{div}(C_{per}(\nabla w_i^0 + e_i)), & \text{in } Q, \\ s_i \ \mathbb{Z}^d\text{-periodic}. \end{cases} \tag{2.29}$$

Proof. We notice first that the specific form (2.24) of b_η considered implies that $\bar{b}_0$ and $\bar{r}_0$ defined in (2.18) here write

$$\bar{b}_0(x,\omega) = \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{Q+k}(x) \bar{B}_0(\tau_k \omega), \tag{2.30a}$$

$$\bar{r}_0(x,\omega) = \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{Q+k}(x) \bar{R}_0(\tau_k \omega). \tag{2.30b}$$

The rest of the proof mainly consists in showing that in this particular setting, ∇v_i^0 and the product $\bar{b}_0 \nabla v_i^0$ can be written using the deterministic functions t_i and s_i. The existence of t_i and its uniqueness up to an additive constant come from Lemmas 6 and 7 in [2].

We start by proving that the sum

$$\sum_{k \in \mathbb{Z}^d} (\bar{B}_0(\tau_k \omega) - \mathbb{E}(\bar{B}_0)) \nabla t_i(x-k) \tag{2.31}$$

is a convergent series in $L^2(Q \times \Omega)$.

To this end, we compute the norm of the remainder of this series:

$$\left\| \sum_{|k| \geq N} (\bar{B}_0(\tau_k \cdot) - \mathbb{E}(\bar{B}_0)) \nabla t_i(\cdot - k) \right\|_{L^2(Q \times \Omega)}^2$$

$$= \sum_{|k| \geq N} \sum_{|l| \geq N} cov(\bar{B}_0(\tau_k \cdot), \bar{B}_0(\tau_l \cdot)) \int_Q \nabla t_i(\cdot - k) \nabla t_i(\cdot - l)$$

$$\leq \frac{1}{2} \sum_{|k| \geq N} \sum_{|l| \geq N} |cov(\bar{B}_0(\tau_k \cdot), \bar{B}_0(\tau_l \cdot))| (\|\nabla t_i(\cdot - k)\|_{L^2(Q)}^2 + \|\nabla t_i(\cdot - l)\|_{L^2(Q)}^2)$$

$$\leq \sum_{|k|\geq N}\sum_{|l|\geq N} |cov(\bar{B}_0(\tau_k\cdot),\bar{B}_0(\tau_l\cdot))| \, \|\nabla t_i(\cdot-k)\|^2_{L^2(Q)}$$

$$\leq \sum_{|k|\geq N}\left(\|\nabla t_i(\cdot-k)\|^2_{L^2(Q)}\sum_{|l|\geq N}|cov(\bar{B}_0(\tau_k\cdot),\bar{B}_0(\tau_l\cdot))|\right)$$

$$\leq \sum_{|k|\geq N}\left(\|\nabla t_i(\cdot-k)\|^2_{L^2(Q)}\sum_{|l|\geq N}|cov(\bar{B}_0,\bar{B}_0(\tau_{l-k}\cdot))|\right)$$

$$\leq \sum_{|k|\geq N}\|\nabla t_i(\cdot-k)\|^2_{L^2(Q)}\sum_{k\in\mathbb{Z}^d}|cov(\bar{B}_0,\bar{B}_0(\tau_k\cdot))|.$$

Using (2.26), we obtain

$$\left\|\sum_{|k|\geq N}\left(\bar{B}_0(\tau_k\cdot)-\mathbb{E}(\bar{B}_0)\right)\nabla t_i(\cdot-k)\right\|^2_{L^2(Q\times\Omega)}\leq C\sum_{|k|\geq N}\|\nabla t_i(\cdot-k)\|^2_{L^2(Q)}. \tag{2.32}$$

Since $\nabla t_i\in L^2(\mathbb{R}^d)$, the right-hand side of (2.32) converges to zero when N goes to infinity.

Consequently, (2.31) defines a vector T in $L^2(Q\times\Omega)$. It is clear from (2.31) that $\partial T_p/\partial x_n = \partial T_n/\partial x_p$ for all $(n,p)\in [\![1,d]\!]^2$. Thus T is a gradient, and there exists a function $\tilde{v}_i$ such that

$$\nabla \tilde{v}_i = T + \mathbb{E}(\bar{B}_0)\nabla s_i = \sum_{k\in\mathbb{Z}^d}\left(\bar{B}_0(\tau_k\cdot)-\mathbb{E}(\bar{B}_0)\right)\nabla t_i(x-k)+\mathbb{E}(\bar{B}_0)\nabla s_i. \tag{2.33}$$

Since s_i is $\mathbb{Z}^d$-periodic, we deduce from (2.33) that

$$\nabla \tilde{v}_i \text{ is stationary and } \mathbb{E}\left(\int_Q\nabla\tilde{v}_i\right)=0. \tag{2.34}$$

We then compute, using (2.28) and (2.29),

$$-\text{div}(A_{per}\nabla\tilde{v}_i) = \sum_{k\in\mathbb{Z}^d}-\text{div}(A_{per}\nabla t_i(\cdot-k))\left(\bar{B}_0(\tau_k\cdot)-\mathbb{E}(\bar{B}_0)\right)-\text{div}(A_{per}\nabla s_i)\mathbb{E}(\bar{B}_0)$$

$$= \sum_{k\in\mathbb{Z}^d}\text{div}\left(C_{per}\mathbb{1}_{Q+k}(\nabla w_i^0+e_i)\right)\left(\bar{B}_0(\tau_k\cdot)-\mathbb{E}(\bar{B}_0)\right)+\text{div}\left(C_{per}(\nabla w_i^0+e_i)\right)\mathbb{E}(\bar{B}_0)$$

$$= \sum_{k\in\mathbb{Z}^d}\text{div}\left(C_{per}\mathbb{1}_{Q+k}\bar{B}_0(\tau_k\cdot)(\nabla w_i^0+e_i)\right). \tag{2.35}$$

Because of (2.30a), (2.35) implies

$$-\text{div}(A_{per}\nabla\tilde{v}_i)=\text{div}\left(\bar{b}_0C_{per}(\nabla w^0+e_i)\right). \tag{2.36}$$

It follows from (2.34) and (2.36) that $\tilde{v}_i$ solves (2.11). As (2.11) has a solution unique up to the addition of a random constant, we obtain

$$\nabla v_i^0 = \nabla\tilde{v}_i = \sum_{k\in\mathbb{Z}^d}\left(\bar{B}_0(\tau_k\cdot)-\mathbb{E}(\bar{B}_0)\right)\nabla t_i(x-k)+\mathbb{E}(\bar{B}_0)\nabla s_i. \tag{2.37}$$

We deduce from (2.30a) and (2.37) that

$$\mathbb{E}(\bar{b}_0 \nabla v_i^0) = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \mathbb{1}_{Q+l}\mathbb{E}(\bar{B}_0(\tau_l\cdot)(\bar{B}_0(\tau_k\cdot)-\mathbb{E}(\bar{B}_0)))\nabla t_i(\cdot-k)+(\mathbb{E}(\bar{B}_0))^2\sum_{l\in\mathbb{Z}^d}\mathbb{1}_{Q+l}\nabla s_i$$

$$= \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \mathbb{1}_{Q+l}cov(\bar{B}_0(\tau_k\cdot),\bar{B}_0(\tau_l\cdot))\nabla t_i(\cdot-k)+(\mathbb{E}(\bar{B}_0))^2\sum_{l\in\mathbb{Z}^d}\mathbb{1}_{Q+l}\nabla s_i,$$

and then that

$$\mathbb{1}_Q\mathbb{E}(\bar{b}_0\nabla v_i^0)=Var(\bar{B}_0)\nabla t_i+\sum_{k\in\mathbb{Z}^d,k\neq 0}cov(\bar{B}_0(\cdot),\bar{B}_0(\tau_k\cdot))\nabla t_i(\cdot-k)+(\mathbb{E}(\bar{B}_0))^2\nabla s_i. \quad (2.38)$$

We conclude by inserting (2.30b) and (2.38) in (2.23).   □

Theorem 2.1 (and its two corollaries) are only of interest if $\mathbb{E}(\bar{b}_0)\neq 0$. Indeed, if $\mathbb{E}(\bar{b}_0)=0$ it only states that $A_\eta^*=A_{per}^*+o(m_\eta)$.

The prototypical case where Theorem 2.1 does not provide valuable information is the case studied in [2]: $b_\eta(x,\omega)=\sum_{k\in\mathbb{Z}^d}\mathbb{1}_{Q+k}(x)B_\eta^k(\omega)$, where the B_η^k are independent identically distributed variables that have Bernoulli law with parameter η, i.e., are equal to 1 with probability η and to 0 with probability $1-\eta$. Then, using the notation of Theorem 2.1, $b_\eta^2=b_\eta$, $m_\eta=\sqrt{\eta}$ and $\bar{b}_0=0$, and we only get $A_\eta^*=A_{per}^*+o(\sqrt{\eta})$ (while Appendix 6.1 of [2] shows that there exists a tensor $\bar{A}_1^*$ such that $A_1^*=A_{per}^*+\eta\bar{A}_1^*+o(\eta)$ at least in dimension one). Omitting the dependence on the space variables since b_η is uniform in each cell of $\mathbb{Z}^d$ in this particular setting, a suitable functional space F on Ω to obtain a non trivial weak limit of $b_\eta/\|b_\eta\|_F$ would be $L^1(\Omega)$ for the norm of each B_η^k in $L^1(\Omega)$ is equal to η. The Dunford-Petti weak compactness criterion in that space is however not satisfied by $b_\eta/\|b_\eta\|_{L^1(\Omega)}$. The reason is of course that $b_\eta/\|b_\eta\|_{L^1(\Omega)}$ converges in the set of bounded measures to a Dirac mass. The techniques used in the proof of Theorem 2.1 and its two corollaries thus do not work in this setting.

The above considerations somehow suggest that an alternative viewpoint might be useful. Because of (2.4b), the image measure dP_η^x of $b_\eta(x,\cdot)$ converges to a Dirac mass in the sense of distributions. Our alternate approach, related to our work [2], consists in working out an expansion of the image measure (or of the law), rather than an expansion of the random variable. Like in [2], our manipulations are mostly formal. Some rigorous foundations, in specific settings, are provided in the appendix.

3 A formal approach

3.1 A new assumption on the image measure

For simplicity, we assume as in Corollary 2.2 that b_η is uniform in each cell of $\mathbb{Z}^d$, and is of the form

$$b_\eta(x,\omega)=\sum_{k\in\mathbb{Z}^d}\mathbb{1}_{Q+k}(x)B_\eta^k(\omega), \quad (3.1)$$

where the B_η^k are independent identically distributed random variables, the distribution of which is given by a "mother variable" B_η. For convenience we slightly modify (2.25a) and require

$$\exists \epsilon > 0, \ \forall \eta > 0, \ \|B_\eta\|_{L^\infty(\Omega)} \leq M - \epsilon, \tag{3.2a}$$

$$\|B_\eta\|_{L^2(\Omega)} \xrightarrow[\eta \to 0]{} 0. \tag{3.2b}$$

Assumption (3.2a) is a technical assumption which implies in particular that for every $\eta > 0$, the image measure dP_η of B_η is a distribution with compact support contained in the *open* set $[-M, M]$. Of course the specific values of M and ϵ have no particular significance. Throughout the sequel we denote by $\mathcal{E}'([-M, M])$ the space of distributions on $\mathbb{R}$ with compact support in $[-M, M]$, and by $\langle T, \varphi \rangle$ the action of a distribution $T \in \mathcal{E}'([-M, M])$ on a test function $\varphi \in \mathcal{C}^\infty([-M, M])$ (basic elements of distribution theory can be found for instance in [6]).

Because of Assumption (3.2b) and Lebesgue dominated convergence theorem, it is clear that for every $\varphi \in \mathcal{C}^\infty([-M, M])$,

$$\mathbb{E}(\varphi(B_\eta)) \xrightarrow[\eta \to 0^+]{} \varphi(0).$$

Since $\mathbb{E}(\varphi(B_\eta)) = \langle dP_\eta, \varphi \rangle$ and $\varphi(0) = \langle \delta_0, \varphi \rangle$ where δ_0 is the Dirac mass at 0, dP_η converges to δ_0 in $\mathcal{E}'([-M, M])$.

This leads us to assume that dP_η satisfies

$$dP_\eta = \delta_0 + \eta d\bar{P}_1 + \eta^2 d\bar{P}_2 + o(\eta^2), \quad \text{in } \mathcal{E}'([-M, M]), \tag{3.3}$$

which is equivalent to

$$\forall \varphi \in \mathcal{C}^\infty([-M, M]), \quad \mathbb{E}(\varphi(B_\eta)) = \langle dP_\eta, \varphi \rangle = \varphi(0) + \eta \langle d\bar{P}_1, \varphi \rangle + \eta^2 \langle d\bar{P}_2, \varphi \rangle + o(\eta^2).$$

Of course $d\bar{P}_1$ and $d\bar{P}_2$ also have a compact support contained in $[-M, M]$: for every test function φ with compact support in $\mathbb{R} \setminus [-M + \epsilon, M - \epsilon]$, it holds for all $\eta > 0$

$$\langle dP_\eta, \varphi \rangle = \mathbb{E}(\varphi(B_\eta)) = 0 = \eta \langle d\bar{P}_1, \varphi \rangle + \eta^2 \langle d\bar{P}_2, \varphi \rangle + o(\eta^2),$$

which yields $\langle d\bar{P}_1, \varphi \rangle = \langle d\bar{P}_2, \varphi \rangle = 0$. Then the supports of $d\bar{P}_1$ and $d\bar{P}_2$ are contained in $[-M + \epsilon, M - \epsilon] \subset [-M, M]$.

Denoting by $M' = M - \epsilon/2$, we deduce from classical distribution theory that there exist a constant $C > 0$ and integers p_1 and p_2 (namely the orders of $d\bar{P}_1$ and $d\bar{P}_2$ respectively) such that

$$\forall \varphi \in \mathcal{C}^\infty([-M, M]), \quad |\langle d\bar{P}_1, \varphi \rangle| \leq C \sup_{s \in [-M', M']} \sup_{0 \leq n \leq p_1} \left| \frac{d^n}{ds^n} \varphi(s) \right|, \tag{3.4a}$$

$$\forall \varphi \in \mathcal{C}^\infty([-M, M]), \quad |\langle d\bar{P}_2, \varphi \rangle| \leq C \sup_{x \in [-M', M']} \sup_{0 \leq n \leq p_2} \left| \frac{d^n}{ds^n} \varphi(s) \right|. \tag{3.4b}$$

Let us now give some additional motivations underlying Assumption (3.3).

The first motivation is related to our work presented in [2] in which B_η has Bernoulli law with parameter η, meaning that it is equal to 1 with probability η and 0 with probability $1-\eta$. Then the image measure dP_η is equal to $\delta_0+\eta(\delta_1-\delta_0)$, so that it satisfies (3.3) exactly at order 1 with $d\bar{P}_1=\delta_1-\delta_0$.

The second motivation comes from the following result, which shows that there is an easy way, used in our numerical experiments, to build perturbations satisfying (3.3).

Lemma 3.1. *Consider B a random variable in $L^3(\Omega)$. Let K be a positive real, and define $B_\eta = \eta B \mathbb{1}_{|\eta B|\leq K}$. Then B_η, which obviously satisfies (3.2a) and (3.2b), also satisfies (3.3) with*

$$dP_\eta = \delta_0 - \eta\mathbb{E}(B)\delta_0' + \frac{\eta^2}{2}\mathbb{E}(B^2)\delta_0'' + \mathcal{O}(\eta^3), \quad in \ \mathcal{E}'(\mathbb{R}). \tag{3.5}$$

Proof. Let us denote by dP the image measure of B, and consider $\varphi \in \mathcal{D}(\mathbb{R})$ (i.e., $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ and has compact support). Then

$$\langle dP_\eta, \varphi \rangle = \int_{|\eta s|\leq K} \varphi(\eta s)dP + \varphi(0)\int_{|\eta s|\geq K} dP \tag{3.6}$$

$$= \int_\mathbb{R} \varphi(\eta s)dP + \int_{|\eta s|\geq K}(\varphi(0)-\varphi(\eta s))dP. \tag{3.7}$$

Since B is in $L^3(\Omega)$,

$$\int_{|\eta s|\geq K} dP = \mathcal{O}(\eta^3),$$

and thus, φ being a bounded function,

$$\langle dP_\eta, \varphi \rangle = \int_\mathbb{R} \varphi(\eta s)dP + \mathcal{O}(\eta^3).$$

Then, since $\varphi \in \mathcal{D}(\mathbb{R})$, there exists $C>0$ such that

$$\forall s \in \mathbb{R}, \quad \left|\frac{\varphi(\eta s)-\varphi(0)-\eta s\varphi'(0)-\frac{\eta^2}{2}s^2\varphi''(0)}{\eta^2}\right| \leq C\eta|s|^3.$$

Again using $B \in L^3(\Omega)$, this implies that

$$\int_\mathbb{R}\left(\frac{\varphi(\eta s)-\varphi(0)-\eta s\varphi'(0)-\frac{\eta^2}{2}s^2\varphi''(0)}{\eta^2}\right)dP \underset{\eta\to 0}{\longrightarrow} 0,$$

which is just a rewriting of (3.5) since

$$\int dP = 1, \quad \int s\,dP = \mathbb{E}(B) \quad and \quad \int s^2 dP = \mathbb{E}(B^2).$$

So, the lemma is proved. □

Before exposing our approach in this new setting, we prove the following elementary result which we will often use in the sequel:

Lemma 3.2. *It holds* $\langle d\bar{P}_1,1\rangle=0$ *and* $\langle d\bar{P}_2,1\rangle=0$.

Proof. It holds on the one hand $\langle dP_\eta,1\rangle=1$ since dP_η is a probability measure, and on the other hand

$$\langle dP_\eta,1\rangle=\langle\delta_0,1\rangle+\eta\langle d\bar{P}_1,1\rangle+\eta^2\langle d\bar{P}_2,1\rangle+o(\eta^2)$$
$$=1+\eta\langle d\bar{P}_1,1\rangle+\eta^2\langle d\bar{P}_2,1\rangle+o(\eta^2),$$

so that the conclusion follows. $\square$

3.2 An ergodic approximation of the homogenized tensor

Let us consider a specific realization $\widetilde{\omega}\in\Omega$ of A_η in $I_N=[-N/2,N/2]^d$, N being for simplicity an odd integer, and solve the following "supercell" problem:

$$\begin{cases} -\operatorname{div}\big(A_\eta(x,\widetilde{\omega})(\nabla w_i^{\eta,N,\widetilde{\omega}}+e_i)\big)=0, & \text{in } I_N, \\ w_i^{\eta,N,\widetilde{\omega}} \ (N\mathbb{Z})^d\text{-periodic}. \end{cases} \tag{3.8}$$

Then we have

$$\forall i\in[\![1,d]\!], \quad A_\eta^*e_i=\lim_{N\to+\infty}\frac{1}{N^d}\mathbb{E}\Big(\int_{I_N}A_\eta(x,\omega)(\nabla w_i^{\eta,N,\omega}(x)+e_i)\Big)dx. \tag{3.9}$$

The proof of (3.9) is given in [2]. We only outline it here for convenience. We know from Theorem 1 in [5] that

$$\frac{1}{N^d}\int_{I_N}A_\eta(x,\widetilde{\omega})(\nabla w_i^{\eta,N,\widetilde{\omega}}(x)+e_i)dx \text{ converges to } A_\eta^*e_i \text{ almost surely in } \widetilde{\omega}\in\Omega. \tag{3.10}$$

Since $(N^d)^{-1}\int_{I_N}A_\eta(x,\widetilde{\omega})(\nabla w_i^{\eta,N,\widetilde{\omega}}(x)+e_i)dx$ is the periodic homogenization of $A_\eta(x,\widetilde{\omega})$ on I_N, it is also well known that for all $(i,j)\in[\![1,d]\!]^2$,

$$\frac{1}{N^d}\Big(\int_{I_N}A_\eta^{-1}(x,\widetilde{\omega})dx\Big)^{-1}e_i\cdot e_j\leq\frac{1}{N^d}\int_{I_N}A_\eta(x,\widetilde{\omega})\Big(\nabla w_i^{\eta,N,\widetilde{\omega}}(x)+e_i\Big)\cdot e_jdx$$
$$\leq\frac{1}{N^d}\Big(\int_{I_N}A_\eta(x,\widetilde{\omega})dx\Big)e_i\cdot e_j, \tag{3.11}$$

so that for all $N\in 2\mathbb{N}+1$, for all $\eta>0$ and for almost all $\widetilde{\omega}\in\Omega$,

$$\Big|\frac{1}{N^d}\int_{I_N}A_\eta(x,\widetilde{\omega})(\nabla w_i^{\eta,N,\widetilde{\omega}}(x)+e_i)\cdot e_jdx\Big|\leq\beta, \tag{3.12}$$

where β is defined by (2.5b). Using (3.12) and the Lebesgue dominated convergence theorem, we can take the expectation in (3.10) and get (3.9).

Remark 3.1. The same result holds for homogeneous Dirichlet and Neumann boundary conditions instead of periodic conditions in the definition of $w_i^{\eta,N,\widetilde{\omega}}$ (see [5] for more details).

For convenience, we label the unit cells of I_N from 1 to N^d. The k-th cell is denoted by Q_k, for $1 \le k \le N^d$. A given realization $A_\eta(x,\widetilde{\omega})$ can then be rewritten

$$A_\eta(x,\widetilde{\omega}) = A_{per}(x) + \sum_{k=1}^{N^d} \mathbb{1}_{Q_k}(x) s_k C_{per}(x),$$

with $s_k = B_\eta^k(\widetilde{\omega})$ for all $k \in [\![1,N^d]\!]$. The $B_\eta^k(\widetilde{\omega})$ being independent random variables, the joint probability of the N^d-uplet $(s_1,\cdots,s_{N^d})$ is simply the product $\prod_{k=1}^{N^d} dP_\eta(s_k)$.

Remark 3.2. The approach exposed in the sequel works also, with minor changes, for random variables which are not independent but correlated with a finite length of correlation. We present it in the independent setting for simplicity.

We now define

$$A^{s_1,\cdots,s_{N^d}} = A_{per} + \sum_{k=1}^{N^d} \mathbb{1}_{Q_k} s_k C_{per} \quad \text{for } (s_1,\cdots,s_{N^d}) \in [-M,M]^{N^d}.$$

We denote by $w_i^{s_1,\cdots,s_{N^d}}$ the solution of the i-th cell problem for the periodic homogenization of $A^{s_1,\cdots,s_{N^d}}$ on I_N, that is

$$\begin{cases} -\text{div}\left(A^{s_1,\cdots,s_{N^d}}(\nabla w_i^{s_1,\cdots,s_{N^d}} + e_i)\right) = 0, & \text{in } I_N, \\ w_i^{s_1,\cdots,s_{N^d}} \ (N\mathbb{Z})^d\text{-periodic.} \end{cases} \tag{3.13}$$

Then, defining

$$A_\eta^{*,N} e_i = \frac{1}{N^d} \mathbb{E}\left(\int_{I_N} A_\eta(x,\omega)(\nabla w_i^{\eta,N,\omega}(x) + e_i)\right) dx, \tag{3.14}$$

we have

$$A_\eta^{*,N} e_i = \frac{1}{N^d} \int_{\mathbb{R}^{N^d}} \left(\int_{I_N} A^{s_1,\cdots,s_{N^d}}(\nabla w_i^{s_1,\cdots,s_{N^d}} + e_i)\right) \prod_{k=1}^{N^d} dP_\eta(s_k). \tag{3.15}$$

It is proved in Lemma A.1 of the Appendix that $\nabla w_i^{s_1,\cdots,s_{N^d}}$ is a $\mathcal{C}^\infty$ function of $(s_1,\cdots,s_{N^d})$ in $[-M,M]^{N^d}$. Thus, since $d\bar{P}_1$ and $d\bar{P}_2$ have compact support in $[-M,M]$ (as well as δ_0 of course), we can make these distributions act on $A^{s_1,\cdots,s_{N^d}}$ and $\nabla w_i^{s_1,\cdots,s_{N^d}}$ as functions of $(s_1,\cdots,s_{N^d})$.

It follows from (3.3) that

$$\prod_{k=1}^{N^d} dP_\eta(s_k) = \prod_{k=1}^{N^d} \delta_0(s_k) + \eta \sum_{l=1}^{N^d} d\bar{P}_1(s_l) \prod_{k=1,k\neq l}^{N^d} \delta_0(s_k) + \frac{\eta^2}{2} \sum_{l=1}^{N^d} \sum_{m=1}^{N^d} d\bar{P}_1(s_l) d\bar{P}_1(s_m) \prod_{k=1,k\neq\{l,m\}}^{N^d} \delta_0(s_k)$$

$$+ \eta^2 \sum_{l=1}^{N^d} d\bar{P}_2(s_l) \prod_{k=1,k\neq l}^{N^d} \delta_0(s_k) + o_N(\eta^2), \quad \text{in } \mathcal{E}'([-M,M]^{N^d}). \tag{3.16}$$

We stress that the remainder $o_N(\eta^2)$ in (3.16) depends on N, hence the notation.

Moreover the products (3.16) are to be understood as tensorized products: we work in $\mathcal{E}'([-M,M]) \otimes_1 \mathcal{E}'([-M,M]) \otimes_2 \cdots \otimes_{N^d-1} \mathcal{E}'([-M,M]) \subset \mathcal{E}'([-M,M]^{N^d})$.

Inserting (3.16) in (3.15), we obtain the following second-order expansion

$$A_\eta^{*,N} = A_0^{*,N} + \eta A_1^{*,N} + \eta^2 A_2^{*,N} + o_N(\eta^2). \tag{3.17}$$

Before making the first three orders in (3.17) precise, note that (3.9), (3.14) and (3.17) imply

$$A_\eta^* = \lim_{N\to\infty} \left(A_0^{*,N} + \eta A_1^{*,N} + \eta^2 A_2^{*,N} + o_N(\eta^2) \right). \tag{3.18}$$

In the sequel we exchange in (3.18) the limit in N and the series in η in order to guess a second-order expansion of A_η^* depending only on η. Since we are not able to justify this permutation, our approach is formal.

We now detail the first three orders in (3.17).

First, we notice that for $i \in [\![1,d]\!]$,

$$A_0^{*,N} e_i = \frac{1}{N^d} \left\langle \prod_{k=1}^{N^d} \delta_0(s_k), \int_{I_N} A^{s_1,\cdots,s_{N^d}} (\nabla w_i^{s_1,\cdots,s_{N^d}} + e_i) \right\rangle$$

$$= \frac{1}{N^d} \int_{I_N} A^{0,\cdots,0} (\nabla w_i^{0,\cdots,0} + e_i)$$

$$= \frac{1}{N^d} \int_{I_N} A_{per} \left(\nabla w_i^0 + e_i \right)$$

$$= A_{per}^* e_i,$$

which obviously gives the zero-order term expected for A_η^*. Then

$$A_1^{*,N} e_i = \frac{1}{N^d} \sum_{l=1}^{N^d} \left\langle d\bar{P}_1(s_l) \prod_{k=1,k\neq l}^{N^d} \delta_0(s_k), \int_{I_N} A^{s_1,\cdots,s_{N^d}} (\nabla w_i^{s_1,\cdots,s_{N^d}} + e_i) \right\rangle. \tag{3.19}$$

It is easy to see that, by $(N\mathbb{Z})^d$-periodicity of $w_i^{s_1,\cdots,s_{N^d}}$,

$$\left\langle d\bar{P}_1(s_l) \prod_{k=1,k\neq l}^{N^d} \delta_0(s_k), \int_{I_N} A^{s_1,\cdots,s_{N^d}} (\nabla w_i^{s_1,\cdots,s_{N^d}} + e_i) \right\rangle$$

does not depend on l. The expression (3.19) can then be rewritten

$$A_1^{*,N} e_i = \left\langle d\bar{P}_1(s), \int_{I_N} A^{s,0\cdots,0}(\nabla w_i^{s,0,\cdots,0} + e_i) \right\rangle. \tag{3.20}$$

We change the notations for convenience, and define, for $s \in [-M,M]$,

$$A_1^{s,0} = A^{s,0\cdots,0} = A_{per} + s\mathbb{1}_Q C_{per}, \tag{3.21}$$

and $w_i^{1,s,0,N} = w_i^{s,0,\cdots,0}$ solution to

$$\begin{cases} -\mathrm{div}\left(A_1^{s,0}(\nabla w_i^{1,s,0,N} + e_i)\right) = 0, & \text{in } I_N, \\ w_i^{1,s,0,N} \ (N\mathbb{Z})^d\text{-periodic.} \end{cases} \tag{3.22}$$

The matrix $A_1^{s,0}$ corresponds to the periodic material with a defect of amplitude s located in Q (i.e., at a position $0 \in \mathbb{Z}^d$ in I_N), and $w_i^{1,s,0,N}$ is the i-th cell solution for the periodic homogenization of $A_1^{s,0}$ in I_N. Since $w_i^{1,s,0,N} = w_i^{s,0,\cdots,0}$, it is of course a C^∞ function of $s \in [-M,M]$.

With these notations, we find that

$$A_1^{*,N} e_i = \left\langle d\bar{P}_1(s), \int_{I_N} A_1^{s,0}(\nabla w_i^{1,s,0,N} + e_i) \right\rangle. \tag{3.23}$$

For the second-order term, we first define the set

$$\mathcal{T}_N = \left\{ k \in \mathbb{Z}^d, Q+k \subset I_N \right\} = \left[\!\left[-\frac{N-1}{2}, \frac{N-1}{2} \right]\!\right]^d. \tag{3.24}$$

The cardinal of $\mathcal{T}_N$ is of course N^d and $\bigcup_{k \in \mathcal{T}_N} \{Q+k\} = I_N$.

For $(s,t) \in [-M,M]^2$ and $k \in \mathcal{T}_N$, we define

$$A_2^{s,t,0,k} = A_{per} + s\mathbb{1}_Q C_{per} + t\mathbb{1}_{Q+k} C_{per}, \tag{3.25}$$

and $w_i^{2,s,t,0,k,N}$ solution to

$$\begin{cases} -\mathrm{div}\left(A_2^{s,t,0,k}(\nabla w_i^{2,s,t,0,k,N} + e_i)\right) = 0, & \text{in } I_N, \\ w_i^{2,s,t,0,k,N} \ (N\mathbb{Z})^d\text{-periodic.} \end{cases} \tag{3.26}$$

The matrix $A_2^{s,t,0,k}$ corresponds to the periodic material with two defects of amplitude s and t located in Q and $Q+k$ (i.e., at positions $0 \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$ in I_N) respectively. The function $w_i^{2,s,t,0,k,N}$ is the i-th cell solution for the periodic homogenization of $A_2^{s,t,0,k}$ in I_N. It is a C^∞ function of $(s,t) \in [-M,M]^2$.
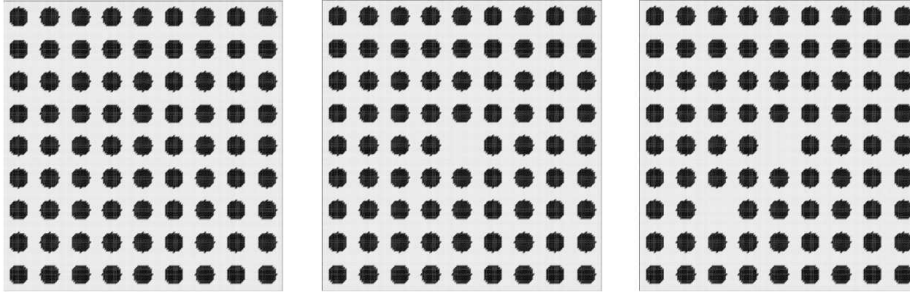
Figure 1: From left to right: zero defect, one defect and two defects.

Then computations similar to that presented for the first order yield

$$
A_2^{*,N} e_i = \frac{1}{2} \sum_{k \in \mathcal{T}_N, k \neq 0} \left\langle d\bar{P}_1(s) d\bar{P}_1(t), \int_{I_N} A_2^{s,t,0,k} (\nabla w_i^{2,s,t,0,k,N} + e_i) \right\rangle
$$
$$
+ \left\langle d\bar{P}_2(s), \int_{I_N} A_1^{s,0} (\nabla w_i^{1,s,0,N} + e_i) \right\rangle. \tag{3.27}
$$

A setting with zero, one and two defects is shown in Fig. 1 in the two-dimensional case of a reference material A_{per} consisting of a periodic lattice of circular inclusions.

Remark 3.3. It is illustrative to consider the particular case where the random variable B_η has a Bernoulli law. This is the case treated in [2]. Then, expansion (3.3) holds exactly with $d\bar{P}_1 = \delta_1 - \delta_0$. The distribution $d\bar{P}_2$ and all other terms of higher order identically vanish. The expressions (3.23) and (3.27) then coincide with (3.17) and (3.18) in [2].

In the next section we prove that $A_1^{*,N}$ converges to a finite limit when $N \to \infty$. The case of the second-order term $A_2^{*,N}$, which is also shown to converge, is discussed in Section 3.4.

3.3   Convergence of the first-order term

We study here the convergence as N goes to infinity of $A_1^{*,N}$ defined by (3.23).

Proposition 3.1. The sequence $A_1^{*,N}$ converges in $\mathbb{R}^{d \times d}$ to a finite limit $\bar{A}_1^*$ when $N \to \infty$.

Proof. We fix $(i,j) \in [\![1,d]\!]^2$ and study the convergence of $A_1^{*,N} e_i \cdot e_j$.

Using (3.22) and the adjoint problems defined by (2.16), we first obtain, for all $s \in [-M,M]$,

$$
\int_{I_N} A_1^{s,0} (\nabla w_i^{1,s,0,N} + e_i) \cdot e_j = \int_{I_N} A_1^{s,0} (\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0).
$$

Then, letting the distribution $d\bar{P}_1$ act on the left and right-hand sides, and using (3.23), we find that

$$A_1^{*,N} e_i \cdot e_j = \left\langle d\bar{P}_1(s), \int_{I_N} A_1^{s,0}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) \right\rangle. \tag{3.28}$$

Because of the definition of $A_1^{s,0}$,

$$\int_{I_N} A_1^{s,0}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0)$$
$$= \int_{I_N} A_{per}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) + \int_Q s C_{per}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0). \tag{3.29}$$

Next, using (2.16),

$$\int_{I_N} A_{per}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) = \int_{I_N} (\nabla w_i^{1,s,0,N} + e_i) \cdot A_{per}^T(e_j + \nabla \widetilde{w}_j^0)$$
$$= \int_{I_N} e_i \cdot A_{per}^T(e_j + \nabla \widetilde{w}_j^0). \tag{3.30}$$

We know from Lemma 3.2 that $\langle d\bar{P}_1, 1 \rangle = 0$. Thus

$$\left\langle d\bar{P}_1(s), \int_{I_N} e_i \cdot A_{per}^T(e_j + \nabla \widetilde{w}_j^0) \right\rangle = 0. \tag{3.31}$$

Collecting (3.28), (3.29), (3.30) and (3.31), we get

$$A_1^{*,N} e_i \cdot e_j = \left\langle d\bar{P}_1(s), \int_Q s C_{per}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) \right\rangle. \tag{3.32}$$

We now define

$$q_i^{1,s,0,N} = w_i^{1,s,0,N} - w_i^0. \tag{3.33}$$

$q_i^{1,s,0,N}$ solves

$$\begin{cases} -\text{div}\left(A_1^{s,0} \nabla q_i^{1,s,0,N}\right) = \text{div}(s \mathbb{1}_Q C_{per}(\nabla w_i^0 + e_i)), & \text{in } I_N, \\ q_i^{1,s,0,N} \ (N\mathbb{Z})^d\text{-periodic.} \end{cases} \tag{3.34}$$

Using (3.33) in (3.32), we rewrite

$$A_1^{*,N} e_i \cdot e_j = \left\langle s d\bar{P}_1(s), \int_Q C_{per}(\nabla w_i^0 + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) \right\rangle$$
$$+ \left\langle d\bar{P}_1(s), \int_Q s C_{per}(\nabla q_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) \right\rangle. \tag{3.35}$$

The rest of the proof consists in showing that

$$\Big\langle d\bar{P}_1(s), \int_Q s C_{per}(\nabla q_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\Big\rangle,$$

which is of course equal to

$$\Big\langle s d\bar{P}_1(s), \int_Q C_{per}(\nabla q_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\Big\rangle$$

converges to a finite limit when $N\to\infty$.

More precisely, defining

$$\forall s\in[-M,M], \ \forall N\in 2\mathbb{N}+1, \ f^N(s)=\int_Q C_{per}(\nabla q_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0),$$

we will prove that the sequence f^N and its derivatives converge uniformly, when N goes to infinity, to a limit function f^∞ and its derivatives.

Applying Lemma A.2 of the appendix to (3.34), we obtain that for all $s\in[-M,M]$, $\nabla q_i^{1,s,0,N}$ converges in $L^2(Q)$, when $N\to\infty$, to $\nabla q_i^{1,s,0,\infty}$, where $q_i^{1,s,0,\infty}$ is a $L^2_{loc}(\mathbb{R}^d)$ function solving

$$\begin{cases} -\text{div}\big(A_1^{s,0}\nabla q_i^{1,s,0,\infty}\big)=\text{div}(s\mathbb{1}_Q C_{per}(\nabla w_i^0+e_i)), & \text{in } \mathbb{R}^d, \\ \nabla q_i^{1,s,0,\infty}\in L^2(\mathbb{R}^d). \end{cases} \tag{3.36}$$

Moreover, arguing as in the proof of Lemma A.2 (given in our previous work [2]), it is easy to see that for all $k\in\mathbb{N}$ and all $s\in[-M,M]$, $\nabla\partial_s^k q_i^{1,s,0,N}$ converges in $L^2(Q)$ to $\nabla\partial_s^k q_i^{1,s,0,\infty}$.

We then define f^∞ by

$$\forall s\in[-M,M], \quad f^\infty(s)=\int_Q C_{per}(\nabla q_i^{1,s,0,\infty}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0).$$

Because of (A.6a) and (A.6b) in Lemma A.3 of the appendix, and using a classical result of differentiation under the integral sign, it is clear that

$$\forall k\in\mathbb{N}, \ \forall s\in[-M,M], \ \frac{d^k}{ds^k}f^N(s)=\int_Q C_{per}(\nabla\partial_s^k q_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0),$$

and

$$\forall k\in\mathbb{N}, \ \forall s\in[-M,M], \ \frac{d^k}{ds^k}f^\infty(s)=\int_Q C_{per}(\nabla\partial_s^k q_i^{1,s,0,\infty}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0).$$

The convergence of $\nabla\partial_s^k q_i^{1,s,0,N}$ to $\nabla\partial_s^k q_i^{1,s,0,\infty}$ in $L^2(Q)$ for every $k\in\mathbb{N}$ thus yields

$$\forall k\in\mathbb{N}, \ \forall s\in[-M,M], \ \lim_{N\to+\infty}\frac{d^k}{ds^k}f^N(s)=\frac{d^k}{ds^k}f^\infty(s). \tag{3.37}$$

On the other hand, we deduce from Lemma A.4 that there exists a constant $C(p_1,M)$ (recall that p_1 is the order of $d\bar{P}_1(s)$) such that for all $k\in[\![0,p_1]\!]$,

$$\forall(s,s')\in[-M,M]^2, \ \ \forall N\in2\mathbb{N}+1, \ \ \left|\frac{d^k}{ds^k}f^N(s)-\frac{d^k}{ds^k}f^N(s')\right|\leq C(p_1,M)|s-s'|. \quad (3.38)$$

It is straightforward to see that (3.37) and (3.38) imply that

$$\forall 0\leq k\leq p_1, \ \ \frac{d^k}{ds^k}f^N \text{ converges uniformly to } \frac{d^k}{ds^k}f^\infty, \ \ \text{in } [-M,M]. \quad (3.39)$$

It follows from (3.4a) and (3.39) that

$$\langle sd\bar{P}_1(s),f^N(s)\rangle\to\langle sd\bar{P}_1(s),f^\infty(s)\rangle,$$

and then

$$\left\langle d\bar{P}_1(s),\int_Q sC_{per}(\nabla q_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle$$
$$\xrightarrow[N\to\infty]{}\left\langle d\bar{P}_1(s),\int_Q sC_{per}(\nabla q_i^{1,s,0,\infty}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle. \quad (3.40)$$

Collecting (3.35) and (3.40), we conclude that $A_1^{*,N}$ converges to a limit tensor $\bar{A}_1^*$ defined by

$$\forall(i,j)\in[\![1,d]\!]^2, \ \ \bar{A}_1^*e_i\cdot e_j=\left\langle sd\bar{P}_1(s),\int_Q C_{per}(\nabla w_i^0+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle$$
$$+\left\langle d\bar{P}_1(s),\int_Q sC_{per}(\nabla q_i^{1,s,0,\infty}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle. \quad (3.41)$$

Thus, the proof is completed. $\qquad\square$

3.4 Convergence of the second-order term

We state here the result concerning the second-order term $A_2^{*,N}$ defined by (3.27). The proof being rather long and technical, we do not present it here for conciseness. We refer the reader to [1] for the details.

Proposition 3.2. The sequence $A_2^{*,N}$ converges to a finite limit $\bar{A}_2^*$ in $\mathbb{R}^{d\times d}$ when $N\to\infty$.

4 Numerical experiments

The purpose of this section is to assess the numerical relevance of the approaches of Sections 2 and 3. To this end we build and homogenize stochastic composite materials using laws that satisfy the assumptions of these sections. Our motivations are not strictly identical for the two approaches. In contrast to the first approach which relies on a rigorous

proof, our second approach is formal and we thus need to demonstrate its correctness experimentally (note that the tests performed in [2] in the Bernoulli case are already to be considered as a component of the validation of the approach). We wish to check that the expansions derived in Sections 2 and 3 provide an accurate and efficient approximation to the direct stochastic computation. The limited computational facilities we have access to impose that we restrict ourselves to the two-dimensional setting. We first explain our general methodology, which is the same as that presented in [2], and then make precise the specific settings.

4.1 Methodology

We mainly consider as in [2] a reference material A_{per} that consists of a constant background reinforced by a periodic lattice of circular inclusions, that is

$$A_{per}(x_1,x_2) = 20 \times Id + 100 \sum_{k \in \mathbb{Z}^2} \mathbb{1}_{B(k,0.3)}(x_1,x_2) \times Id,$$

where $B(k,0.3)$ is the ball of center k and radius 1. Loosely speaking, the role of the perturbation is to randomly eliminate some fibers:

$$C_{per}(x_1,x_2) = -100 \sum_{k \in \mathbb{Z}^2} \mathbb{1}_{B(k,0.3)}(x_1,x_2) \times Id.$$

We will also, in our last test, consider a laminate

$$A_{per}(x_1,x_2) = 5 + 10 \sum_{l \in \mathbb{Z}} \mathbb{1}_{l \le x_1 \le l+1}(x_1,x_2),$$

with the perturbation yielding an error in the lamination direction:

$$C_{per}(x_1,x_2) = 10 \sum_{l \in \mathbb{Z}} \mathbb{1}_{l \le x_2 \le l+1}(x_1,x_2) \times Id - 10 \sum_{l \in \mathbb{Z}} \mathbb{1}_{l \le x_1 \le l+1}(x_1,x_2) \times Id.$$

For both materials (shown in Fig. 2), we have chosen the values of the coefficients in order to have a high contrast between A_{per} and $A_{per} + C_{per}$ and thus for the perturbation to have an important impact on the microscopic structure. The specific value of these coefficients has no other significance.

We will consider different perturbations b_η, all of which satisfy (3.1) with the B_η^k independent and identically distributed.

Our goal is to compare A_η^* with its approximation $A_{per}^* + \eta A_1^{*,N} + \eta^2 A_2^{*,N}$. A major computational difficulty is the Monte-Carlo computation of the "exact" matrix A_η^* given by formula (2.7). It ideally requires to solve the stochastic cell problems (2.6) on $\mathbb{R}^d$. To this end we first use ergodicity and formula (3.9), and actually compute, for a given realization ω and a domain I_N chosen here to be $[0,N]^2$ for convenience, $A_\eta^{*,N}(\omega)$ defined by

$$A_\eta^{*,N}(\omega)e_i = \frac{1}{N^d} \int_{I_N} A_\eta(x,\omega)\big(\nabla w_i^{\eta,N,\omega}(x) + e_i\big)dx. \tag{4.1}$$
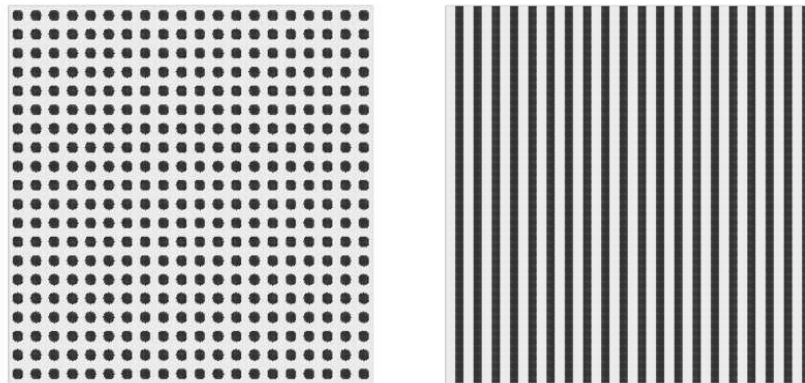
Figure 2: Left: a periodic lattice of circular inclusions. Right: a one-dimensional laminate.

In a second step, we take averages over the realizations ω.

For each ω, we use the finite element software FreeFem++[†] to solve the boundary value problems (3.8) and compute the integrals (4.1). We work with standard $P1$ finite elements on a triangular mesh such that there are 10 degrees of freedom on each edge of the unit cell Q.

We define an approximate value $A_\eta^{*,N}$ as the average of $A_\eta^{*,N}(\omega)$ over 40 realizations ω. Our numerical experiments indeed show that the number 40 is sufficiently large for the convergence of the Monte-Carlo computation. We then let N grow from 5 to 80 by steps of 5. We observe that $A_\eta^{*,N}$ stabilizes at a fixed value around $N=80$ and thus take $A_\eta^{*,80}$ as the reference value for A_η^* in our subsequent tests.

The next step is to compute the zero-order term A_{per}^*, and the first-order and second-order deterministic corrections. Using the same mesh and finite elements as for our reference computation above, we compute A_{per}^* using (2.8) and (2.9). The computation of the next orders depends on the setting:

- in the setting of Section 2, the first-order correction is given by (2.13) in Theorem 2.1 and is thus independent of N; since b_η is of the form (2.24), we use formula (2.27) in Corollary 2.2 for the second-order correction which depends on N through the term t_i defined on $\mathbb{R}^d$ by (2.28), and which has to be approximated on I_N; we let N grow from 5 to 80 by steps of 5;

- in the setting of Section 3, the corrections $A_1^{*,N}$ and $A_2^{*,N}$ are respectively given by (3.23) and (3.27); we let N grow from 5 to 80 by steps of 5 for $A_1^{*,N}$; the computation of $A_2^{*,N}$ being far more expensive (there is not only an integral over I_N but also a sum over the N^2 cells in (3.27)), we have to limit ourselves to $N=25$ and approximate the value for N larger than 25 by the value obtained for $N=25$.

We stress that there are three distinct sources of error in these computations:

[†]available at `www.freefem.org`.

- the finite elements discretization error;

- the truncation error due to the replacement of $\mathbb{R}^d$ with I_N, in the computation of the stochastic cell problems (2.6) that are replaced with (3.8), as well as in the computation of the integrals (4.1);

- the stochastic error arising from the approximation of the expectation value by an empirical mean.

Detailed comments on these various errors and the way we deal with them are provided in [2]. We just emphasize, in the setting of Section 3, that it is not our purpose to prove through our tests that

$$A_\eta^* = A_{per}^* + \eta \bar{A}_1^* + \eta^2 \bar{A}_2^* + o(\eta^2)$$

with a $o(\eta^2)$ which would be independent of N, of the number of realizations and of the size of the mesh. We only wish to demonstrate that the second-order expansion is an approximation to A_η^* sufficiently good for all practical purposes. We will observe that both $A_1^{*,N}$ and $A_2^{*,N}$ converge to their respective limits faster than $A_\eta^{*,N}$ to A_η^* (which is expected since the former quantities are deterministic and contain less information). We will also observe that $A_{per}^* + \eta A_1^{*,N}$ is closer to A_η^* than A_{per}^* and that the inclusion of the second order improves the situation for $A_{per}^* + \eta A_1^{*,N} + \eta^2 A_2^{*,N}$ is even closer.

To present our numerical results, we choose the first diagonal entry (1,1) of all the matrices considered. Other coefficients in the matrices behave qualitatively similarly. We illustrate a practical interval of confidence for our Monte-Carlo computation of A_η^* by showing, for each N, the minimum and maximum values of $A_\eta^{*,N}(\omega)$ achieved over the 40 realizations ω.

We will use the following legend in the graphs:

- periodic: gives the value of the periodic homogenized tensor A_{per}^*;

- first-order: gives the value of the first-order expansion;

- second-order: gives the value of the second-order expansion;

- stochastic mean, minima and maxima: respectively give the values of $A_\eta^{*,N}$ and the extrema obtained in the computation of the empirical mean.

Finally, the results are given for various values of η which serve the purpose of testing our approach in a diversity of situations, and in particular for perturbations that are "not so small".

4.2 An example of setting for our theory in Section 2 (and 3)

Consider $B_\eta = \eta G \mathbb{1}_{0 \leq \eta G \leq 1}$ where G is a normalized centered Gaussian random variable. It is easy to check that

$$B_\eta = \eta G \mathbb{1}_{0 \leq G \leq +\infty} + o(\eta^2), \quad \text{in } L^2(\Omega),$$
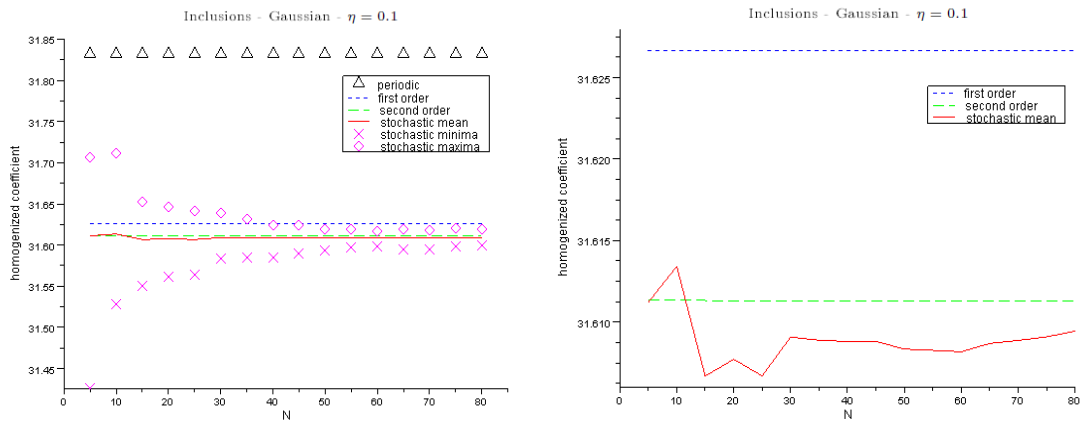
Figure 3: Inclusions – results for a Gaussian perturbation and $\eta = 0.1$. Left: complete results. Right: close-up on $A_\eta^{*,N}$ and the first and second-order corrections.
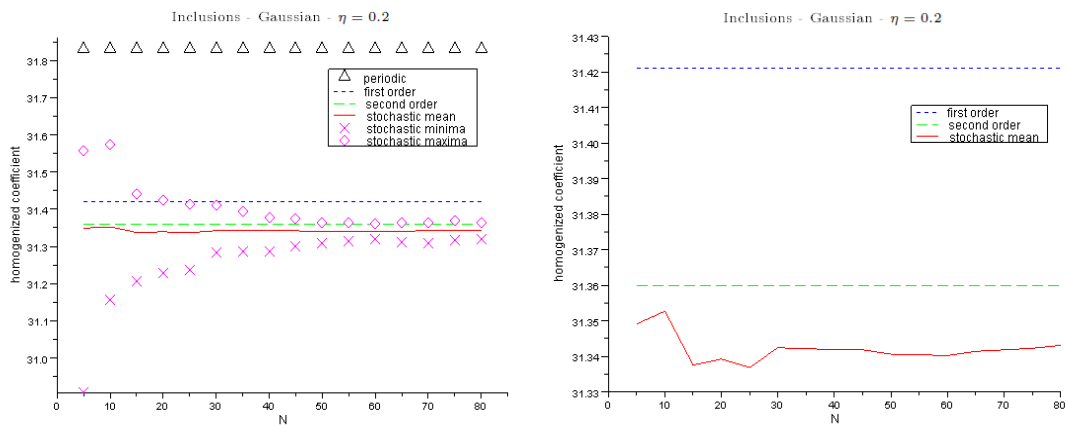

Figure 4: Inclusions – results for a Gaussian perturbation and $\eta = 0.2$. Left: complete results. Right: close-up on $A_\eta^{*,N}$ and the first and second-order corrections.

so that Corollary 2.2 of Section 2 applies. Alternatively, we can use Lemma 3.1, which gives

$$dP_\eta = \delta_0 - \eta \frac{1}{\sqrt{2}} \delta_0' + \frac{\eta^2}{4} \delta_0'' + o(\eta^2), \quad \text{in } \mathcal{E}'(\mathbb{R}),$$

to perform our formal approach. We verify in Section A.3 of the appendix that both approaches yield the same results up to second order.

We show results for the lattice of inclusions and for $\eta = 0.1$ and $\eta = 0.2$ (Figs. 3 and 4 respectively).

The results are very satisfying for both values of η. The first-order correction, which does not depend on N, enables to get substantially closer to A_η^*. Moreover, it is clear

(especially from the close-ups) that the second-order correction $A_2^{*,N}$ converges very fast (convergence is already reached at $N=5$), and in particular much faster than the stochastic computation $A_\eta^{*,N}$. It also provides excellent accuracy.

4.3   A first example of setting for our formal approach of Section 3

Consider R_η a random variable having Bernoulli law with parameter η, and G a normalized centered Gaussian random variable independent of R_η. We define the product random variable

$$B_\eta = R_\eta \times \eta G \mathbb{1}_{|\eta G| \leq 1}.$$

Then

$$
\begin{aligned}
\mathbb{E}(\varphi(B_\eta)) &= \mathbb{E}(\varphi(R_\eta \times \eta G \mathbb{1}_{|\eta G| \leq 1})) \\
&= \eta \mathbb{E}(\varphi(\eta G \mathbb{1}_{|\eta G| \leq 1})) + (1-\eta)\varphi(0) \\
&= \eta\left(\varphi(0) + \eta \mathbb{E}(G)\varphi'(0) + \frac{\eta^2}{2}\varphi''(0) + o(\eta^2)\right) + (1-\eta)\varphi(0) \\
&= \varphi(0) + \frac{\eta^3}{2}\varphi''(0) + o(\eta^3).
\end{aligned}
$$

This implies

$$dP_\eta = \delta_0 + \frac{\eta^3}{2}\delta_0'' + o(\eta^3), \quad \text{in } \mathcal{E}'(\mathbb{R}). \tag{4.2}$$

In this case we only consider the first-order correction since the dominant order in (4.2) is already tiny. We present the results in the case of the lattice of inclusions, for $\eta = 0.2$, $\eta = 0.3$ and $\eta = 0.5$ (Figs. 5, 6, 7 respectively).

Once again, our approach converges rapidly and allows for an accurate approximate value of A_η^* even for η as large as 0.5.

4.4   A second example of setting for our formal approach of Section 3

Consider R_η a random variable having Bernoulli law with parameter η, and U a uniform variable on $[0,1]$ independent of R_η. We define $B_\eta = R_\eta - \eta U$. Then

$$
\begin{aligned}
\mathbb{E}(\varphi(B_\eta)) &= \mathbb{E}(\varphi(R_\eta - \eta U)) \\
&= \eta \mathbb{E}(\varphi(1 - \eta U)) + (1-\eta)\mathbb{E}(\varphi(-\eta U)) \\
&= \eta\left(\varphi(1) - \eta \mathbb{E}(U)\varphi'(1) + o(\eta)\right) \\
&\quad + (1-\eta)\left(\varphi(0) - \eta \mathbb{E}(U)\varphi'(0) + \frac{\eta^2}{2}\mathbb{E}(U^2)\varphi''(0) + o(\eta^2)\right) \\
&= \varphi(0) + \eta\left(-\mathbb{E}(U)\varphi'(0) + \varphi(1) - \varphi(0)\right) \\
&\quad + \eta^2\left(-\mathbb{E}(U)(\varphi'(1) - \varphi'(0)) + \frac{1}{2}\mathbb{E}(U^2)\varphi''(0)\right) + o(\eta^2),
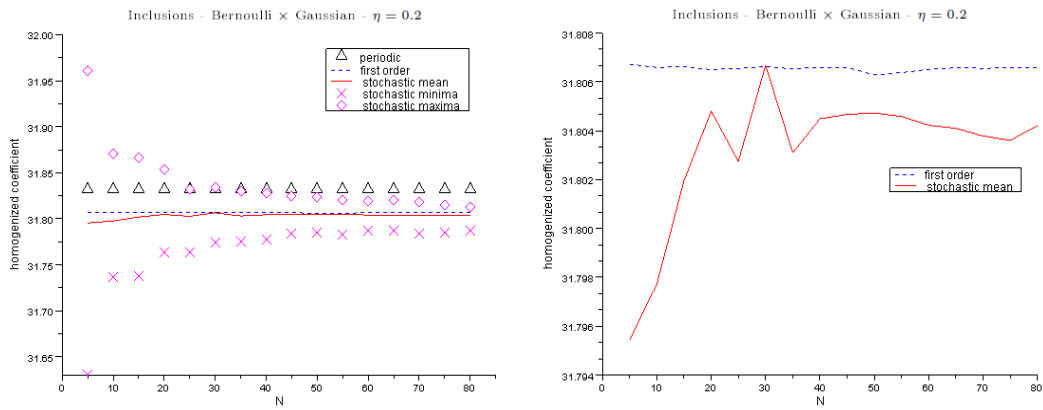\end{aligned}
$$

Figure 5: Inclusions – results for perturbation (4.2) and $\eta = 0.2$. Left: complete results. Right: close-up on $A_\eta^{*,N}$ and the first-order correction.
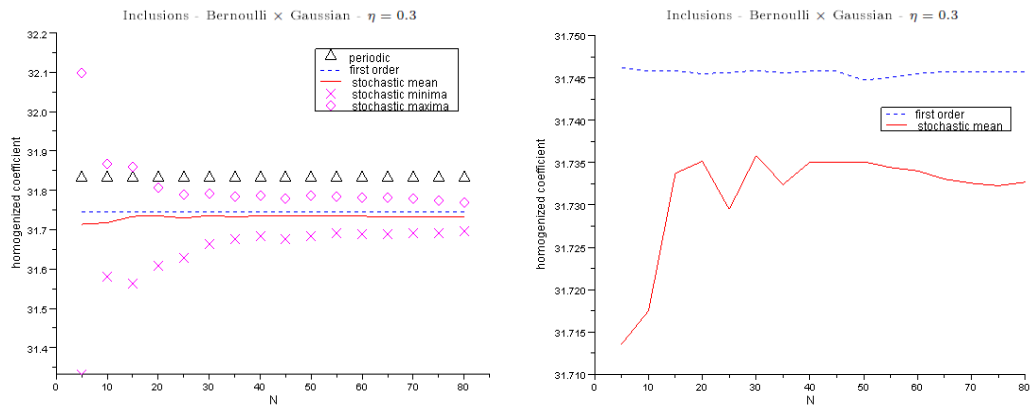

Figure 6: Inclusions – results for perturbation (4.2) and $\eta = 0.3$. Left: complete results. Right: close-up on $A_\eta^{*,N}$ and the first-order correction.
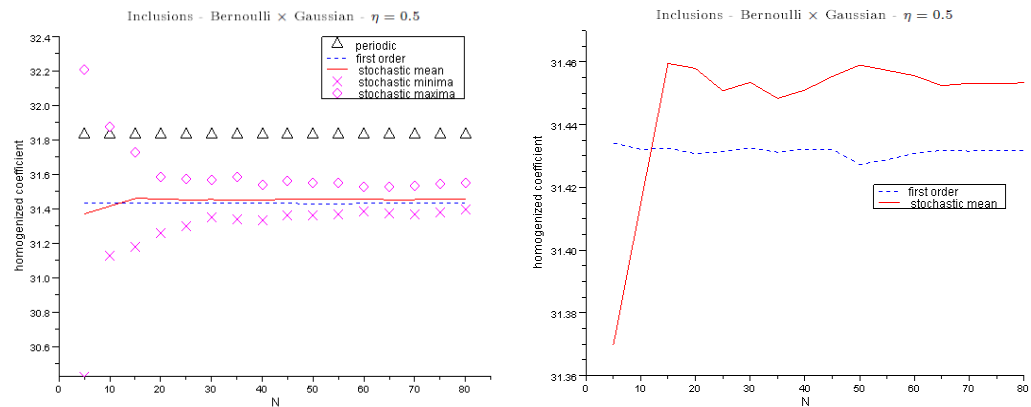

Figure 7: Inclusions – results for perturbation (4.2) and $\eta = 0.5$. Left: complete results. (Right: close-up on $A_\eta^{*,N}$ and the first-order correction.
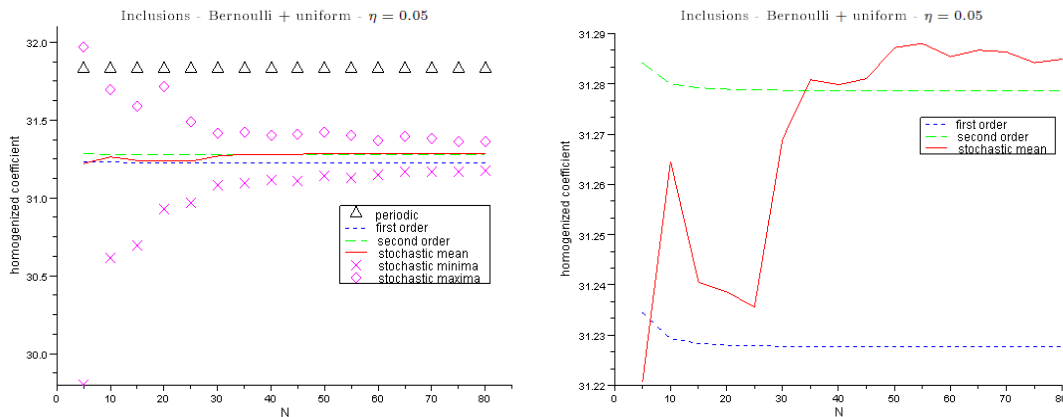
Figure 8: Inclusions – results for perturbation (4.3) and $\eta = 0.05$. Left: complete results. Right: close-up on $A^{*,N}_\eta$ and the first and second-order corrections.
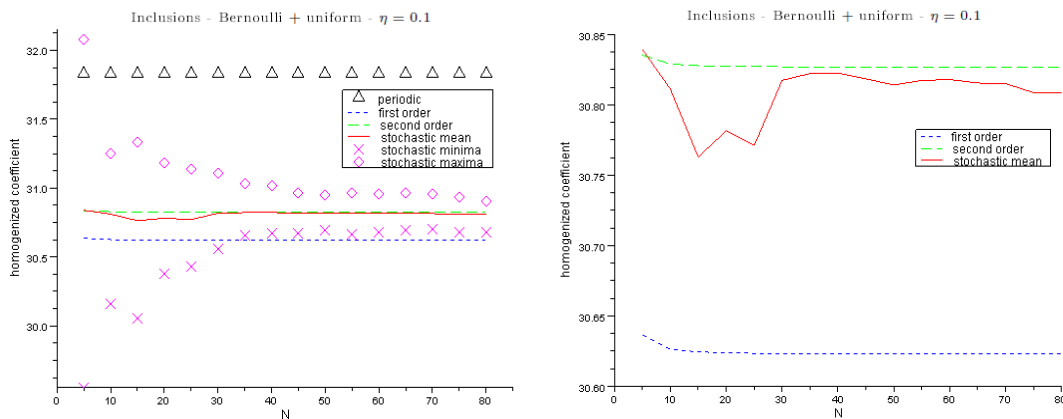

Figure 9: Inclusions – results for perturbation (4.3) and $\eta = 0.1$. Left: complete results. Right: close-up on $A^{*,N}_\eta$ and the first and second-order corrections.

so that

$$dP_\eta = \delta_0 + \eta\left(-\mathbb{E}(U)\delta'_0 + \delta_1 - \delta_0\right)$$
$$+ \eta^2\left(-\mathbb{E}(U)(\delta'_1 - \delta'_0) + \frac{1}{2}\mathbb{E}(U^2)\delta''(0)\right) + o(\eta^2), \quad \text{in } \mathcal{E}'(\mathbb{R}). \tag{4.3}$$

Notice that this complex case is a mixture of Sections 2 and 3. The first-order perturbation is of course only the sum of the first-order perturbations for a Bernoulli law (Section 3 and [2]) and a uniform law (Section 2). The interaction of these laws at order 2, and notably the δ'_1 term, is much more involved and requires the computation of the cross derivatives of $w^{2,s,t,0,k,N}_i$ with respect to s and t at $s = 0$ and $t = 1$.

We give the results in the case of the inclusions and for $\eta = 0.05$, $\eta = 0.1$ and $\eta = 0.2$ (Figs. 8, 9, 10, respectively).
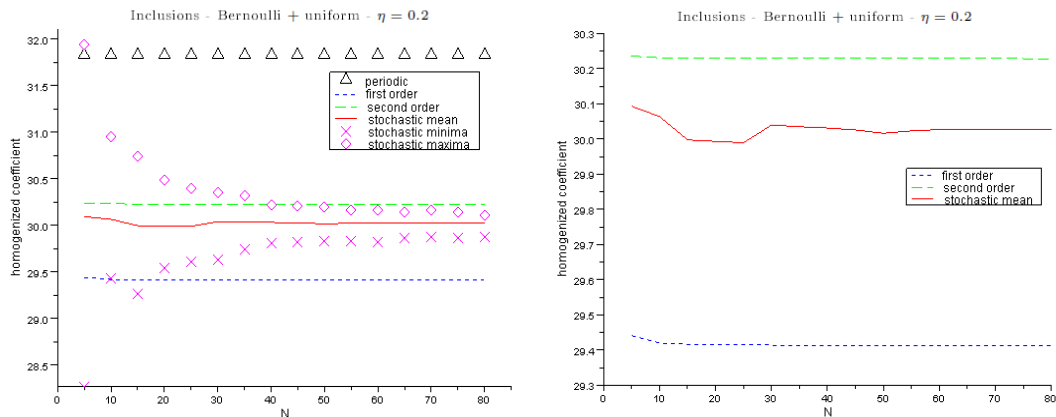
Figure 10: Inclusions – results for perturbation (4.3) and $\eta = 0.2$. Left: complete results. Right): close-up on $A_\eta^{*,N}$ and the first and second-order corrections.
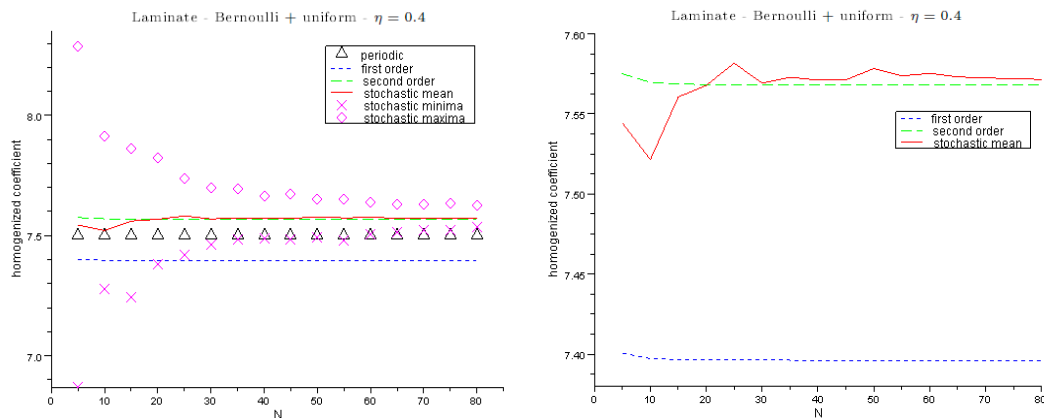

Figure 11: Laminate – results for perturbation (4.3) and $\eta = 0.4$. Left: complete results. Right: close-up on $A_\eta^{*,N}$ and the first and second-order corrections.

For $\eta = 0.05$ and $\eta = 0.1$, the results display the same features as in our previous tests and are very good. The case $\eta = 0.2$ is instructive: the second-order expansion significantly departs from the "exact" value provided by the direct stochastic computation. Our interpretation is that, far from contradicting the validity of our expansion in the limit of small η, it shows the limitations of the approach. The value $\eta = 0.2$ is too large for the expansion to be accurate in the case of a lattice of inclusions with a high contrast between the inclusions and the surrounding phase.

Interestingly, a value of η twice as large (0.4) provides a very accurate approximation for another material, as shown by our final test performed on the laminate (Fig. 11).

Our approach has limitations and deteriorates, like any asymptotic approach, for large values of η. The threshold is case dependent. The approach is however generically robust.

Appendix

The objectives of this appendix are diverse. We first state some results used in Section 3. Next we show that the approach formally derived in Section 3 is rigorous in dimension one. Finally we prove that this approach is also rigorous, in general dimensions, in a specific setting close to that of Theorem 2.1 and Corollary 2.2.

A.1   Some technical results

We state here some technical lemmas used in Section 3. Loosely speaking, these lemmas all deal with the variations of the supercell correctors defined by (3.13), (3.22), and (3.26) with respect to the amplitudes of the defects. For conciseness we refer to [1] for the elementary proofs.

Lemma A.1. *Let $\widetilde{H}^1_{per}(I_N)$ be the set of $(N\mathbb{Z})^d$-periodic functions in $H^1_{loc}(\mathbb{R}^d)$ with zero mean on I_N. The function*

$$F:(s_1,\cdots,s_{N^d})\in[-M,M]^{N^d}\mapsto \bar{w}_i^{s_1,\cdots,s_{N^d}}\in\widetilde{H}^1_{per}(I_N),$$

where $\bar{w}_i^{s_1,\cdots,s_{N^d}}=w_i^{s_1,\cdots,s_{N^d}}-\int_{I_N}w_i^{s_1,\cdots,s_{N^d}}$ and $w_i^{s_1,\cdots,s_{N^d}}$ is defined by (3.13), is C^∞.

Lemma A.2. *Consider $f\in L^2(Q)$, and a tensor field A from $\mathbb{R}^d$ to $\mathbb{R}^{d\times d}$ such that there exist $\lambda>0$ and $\Lambda>0$ such that*

$$\forall\xi\in\mathbb{R}^d,\quad a.e\ in\ x\in\mathbb{R}^d,\ \lambda|\xi|^2\leq A(x)\xi\cdot\xi\ and\ |A(x)\xi|\leq\Lambda|\xi|.$$

Consider q^N solution to

$$\begin{cases} -\mathrm{div}\left(A\nabla q^N\right)=\mathrm{div}(\mathbb{1}_Q f), & in\ I_N, \\ q^N\ (N\mathbb{Z})^d\text{-periodic.} \end{cases} \tag{A.4}$$

Then $\mathbb{1}_{I_N}\nabla q^N$ converges in $L^2(\mathbb{R}^d)$, when N goes to infinity, to ∇q^∞, where q^∞ is a $L^2_{loc}(\mathbb{R}^d)$ function solving

$$\begin{cases} -\mathrm{div}\left(A\nabla q^\infty\right)=\mathrm{div}(\mathbb{1}_Q f), & in\ \mathbb{R}^d, \\ \nabla q^\infty\in L^2(\mathbb{R}^d). \end{cases} \tag{A.5}$$

Lemma A.3. *Consider $q_i^{1,s,0,N}$ and $q_i^{1,s,0,\infty}$ solutions to (3.34) and (3.36) respectively and $k\in\mathbb{N}$. There exists a constant $C(k,M)$, such that*

$$\forall s\in[-M,M],\ \forall N\in 2\mathbb{N}+1,\ \|\nabla\partial_s^k q_i^{1,s,0,N}\|_{L^2(I_N)}\leq C(k,M)\|\nabla w_i^0+e_i\|_{L^2(Q)}, \tag{A.6a}$$

$$\forall s\in[-M,M],\ \|\nabla\partial_s^k q_i^{1,s,0,\infty}\|_{L^2(\mathbb{R}^d)}\leq C(k,M)\|\nabla w_i^0+e_i\|_{L^2(Q)}. \tag{A.6b}$$

The following result is an immediate consequence of Lemma A.3.

Lemma A.4. *Consider $q_i^{1,s,0,N}$ and $q_i^{1,s,0,\infty}$ solutions to (3.34) and (3.36) respectively. For every $k \in \mathbb{N}$, there exists a constant $C(k,M)$ such that for all $(s,s') \in [-M,M]^2$,*

$$\forall N \in 2\mathbb{N}+1, \quad \|\nabla \partial_s^k q_i^{1,s,0,N} - \nabla \partial_s^k q_i^{1,s',0,N}\|_{L^2(I_N)} \leq C(k,M) \|\nabla w_i^0 + e_i\|_{L^2(Q)} |s-s'|, \quad \text{(A.7a)}$$

$$\|\nabla \partial_s^k q_i^{1,s,0,\infty} - \nabla \partial_s^k q_i^{1,s',0,\infty}\|_{L^2(\mathbb{R}^d)} \leq C(k,M) \|\nabla w_i^0 + e_i\|_{L^2(Q)} |s-s'|. \quad \text{(A.7b)}$$

A.2 The one-dimensional case

We address here the one-dimensional context. All the computations are explicit, for the settings of Sections 2 and 3. To stress the fact that we deal with scalar quantities, we use lower-case letters for the tensors. Note also that in this section $Q = [-1/2, 1/2]$ and $I_N = [-N/2, N/2]$.

A.2.1 An extension of Theorem 2.1

The following theorem extends the result of Theorem 2.1, stated in $L^\infty(Q; L^2(\Omega))$ to $L^\infty(Q; L^p(\Omega))$ for any $p \in [1, \infty]$:

Theorem A.1 (one-dimensional setting). *Assume that $d = 1$, that b_η satisfies (2.4a) and*

$$m_\eta := \|b_\eta\|_{L^\infty\left([-\frac{1}{2}, \frac{1}{2}]; L^p(\Omega)\right)} \underset{\eta \to 0}{\longrightarrow} 0 \quad \text{for some } p > 1.$$

There exists a subsequence of η, still denoted η for simplicity, such that b_η / m_η converges weakly-* in $L^\infty([-\frac{1}{2}, \frac{1}{2}]; L^p(\Omega))$ to a limit field denoted by $\bar{b}_0$ when $\eta \to 0$. Then

- *the expansion*

$$\frac{d}{dx} w^\eta = \frac{d}{dx} w^0 + m_\eta \frac{d}{dx} v^0 + o(m_\eta) \quad \text{(A.8)}$$

holds weakly in $L^2([-\frac{1}{2}, \frac{1}{2}]; L^p(\Omega))$, where w^0 is the periodic corrector and v^0 solves

$$\begin{cases} -\dfrac{d}{dx}\left(a_{per} \dfrac{d}{dx} v^0\right) = \dfrac{d}{dx}\left(\bar{b}_0 c_{per}\left(\dfrac{d}{dx} w^0 + 1\right)\right), & \text{in } \mathbb{R}, \\ \dfrac{d}{dx} v^0 \text{ stationary}, \quad \mathbb{E}\left(\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \dfrac{d}{dx} v^0\right) = 0. \end{cases} \quad \text{(A.9)}$$

- *a_η^* reads*

$$a_\eta^* = a_{per}^* + m_\eta \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{E}(\bar{b}_0) c_{per}\left(\frac{d}{dx} w^0 + 1\right) + m_\eta \int_{-\frac{1}{2}}^{\frac{1}{2}} a_{per} \frac{d}{dx} \mathbb{E}(v^0) + o(m_\eta).$$

Proof. The periodic and stochastic correctors can be computed explicitly. They are respectively given by

$$\frac{d}{dx}w^0 = \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} a_{per}^{-1}\Big)^{-1} a_{per}^{-1} - 1 \quad \text{and} \quad \frac{d}{dx}w^\eta = \Big(\mathbb{E}\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} a_\eta^{-1}\Big)\Big)^{-1} a_\eta^{-1} - 1.$$

Note that w^0 is in $W^{1,\infty}(-\frac{1}{2},\frac{1}{2})$.

We define $v^\eta = (w^\eta - w^0)/m_\eta$. It solves

$$\begin{cases} -\dfrac{d}{dx}\Big(a_\eta \dfrac{d}{dx} v^\eta\Big) = \dfrac{d}{dx}\Big(\dfrac{b_\eta}{\eta} c_{per}\Big(\dfrac{d}{dx} w^0 + 1\Big)\Big), & \text{in } \mathbb{R}, \\[2mm] \dfrac{d}{dx} v^\eta \text{ stationary}, \quad \mathbb{E}\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \dfrac{d}{dx} v^\eta\Big) = 0. \end{cases} \tag{A.10}$$

We deduce from (A.10) that

$$a_\eta \frac{d}{dx} v^\eta = \frac{b_\eta}{m_\eta} c_{per}\Big(\frac{d}{dx} w^0 + 1\Big) + k_\eta, \tag{A.11}$$

where k_η depends only on ω. Since k_η is by construction stationary ergodic, it is constant, and we compute from (A.10) and (A.11):

$$k_\eta = -\frac{1}{m_\eta}\Big(\mathbb{E}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_\eta}\Big)^{-1} \times \Big(\mathbb{E}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{b_\eta}{a_\eta} c_{per}\Big(\frac{d}{dx} w^0 + 1\Big)\Big).$$

Since w^0 is in $W^{1,\infty}(-\frac{1}{2},\frac{1}{2})$, a_η is coercive and c_{per} is bounded, it holds

$$|k_\eta| \le C \frac{\|b_\eta\|_{L^1([-\frac{1}{2},\frac{1}{2}]\times\Omega)}}{m_\eta} \le C \frac{\|b_\eta\|_{L^1([-\frac{1}{2},\frac{1}{2}]\times\Omega)}}{\|b_\eta\|_{L^\infty([-\frac{1}{2},\frac{1}{2}];L^p(\Omega))}}.$$

This implies that k_η is a bounded function of η whatever $p \ge 1$ and thus, using (A.11), that dv^η/dx is bounded in $L^2([-\frac{1}{2},\frac{1}{2}];L^p(\Omega))$ for all $p \ge 1$. As a result, for $p > 1$, dv^η/dx converges weakly and up to extraction in $L^2([-\frac{1}{2},\frac{1}{2}];L^p(\Omega))$ to a limit we denote dv^0/dx.

The random field b_η tends to 0 in $L^2([-\frac{1}{2},\frac{1}{2}];L^p(\Omega))$. Since it is bounded in $L^\infty([-\frac{1}{2},\frac{1}{2}]\times \Omega)$, it converges to 0 in $L^2([-\frac{1}{2},\frac{1}{2}];L^r(\Omega))$ for all $r > p$. By Hölder inequality it also converges to 0 in $L^2([-\frac{1}{2},\frac{1}{2}];L^r(\Omega))$ for all $1 < r < p$. Thus it converges to 0 in $L^2([-\frac{1}{2},\frac{1}{2}];L^q(\Omega))$ where $q = p/(p-1)$.

The space $L^2([-\frac{1}{2},\frac{1}{2}];L^q(\Omega))$ being the dual of $L^2([-\frac{1}{2},\frac{1}{2}];L^p(\Omega))$, we obtain that $b_\eta c_{per} dv^\eta/dx$ tends to 0 in $\mathcal{D}'([-\frac{1}{2},\frac{1}{2}]\times\Omega)$. We can then take the limit $\eta \to 0$ in (A.10) and obtain that v^0 is solution to (A.9).

We have thus proved that $\frac{1}{m_\eta}\Big(\frac{d}{dx}w^\eta - \frac{d}{dx}w^0\Big)$ converges, up to extraction, weakly to dv^0/dx in $L^2([-\frac{1}{2},\frac{1}{2}];L^p(\Omega))$, which is equivalent to (A.8).

The second assertion of Theorem A.1 is obtained by inserting (A.8) into the expression (2.7) of a_η^*. □

Note that the proof of Theorem A.1 depends crucially on the fact that we are able to solve explicitly the cell problems.

Theorem A.1 allows for a better intuitive understanding of Theorem 2.1. In dimension one, the homogenized coefficient is explicitly given by

$$a_\eta^* = \left(\mathbb{E} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + b_\eta c_{per}} \right)^{-1},$$

which, when $b_\eta(x,\omega) = \sum_{k\in\mathbb{Z}} \mathbb{1}_{[k,k+1]}(x) B_\eta(\tau_k\omega)$, may be rewritten as the *formal* series

$$\frac{1}{a_\eta^*} = \sum_{k=0}^{\infty} (-1)^k \mathbb{E}((B_\eta)^k) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{c_{per}}{a_{per}} \right)^k a_{per}^{-1}. \tag{A.12}$$

Assume now that there exists $p>1$ such that $\|B_\eta\|_{L^p(\Omega)} \to 0$ when $\eta\to0$ and $B_\eta / \|B_\eta\|_{L^p(\Omega)}$ converges weakly in $L^p(\Omega)$ to some $\bar{B}_0$ with $\mathbb{E}(\bar{B}_0) \neq 0$. We have in particular

$$\frac{\mathbb{E}(B_\eta)}{\|B_\eta\|_{L^p(\Omega)}} \to \mathbb{E}(\bar{B}_0) \neq 0,$$

which, since $\mathbb{E}(|B_\eta|^p)\to0$, implies

$$\mathbb{E}(|B_\eta|^p) = \underset{\eta\to0^+}{o}\left(\mathbb{E}(B_\eta) \right). \tag{A.13}$$

We now claim that, without loss of generality and up to an extraction in η, we may take $p=2$ in (A.13). Indeed, if $p<2$, then since B_η is bounded in $L^\infty(\Omega)$, (A.13) implies

$$\mathbb{E}(|B_\eta|^2) = \underset{\eta\to0^+}{o}\left(\mathbb{E}(B_\eta) \right).$$

On the other hand, if $p>2$, we consider the normalized sequence $B_\eta / \|B_\eta\|_{L^2(\Omega)}$ in $L^2(\Omega)$. Up to extraction, it weakly converges to $\bar{B}_2 \in L^2(\Omega)$. Since

$$\frac{\mathbb{E}(B_\eta)}{\|B_\eta\|_{L^p(\Omega))}} = \frac{\mathbb{E}(B_\eta)}{\|B_\eta\|_{L^2(\Omega)}} \frac{\|B_\eta\|_{L^2(\Omega)}}{\|B_\eta\|_{L^p(\Omega)}},$$

where the left hand side converges to $\mathbb{E}(\bar{B}_0) \neq 0$ and $\|B_\eta\|_{L^2(\Omega)} / \|B_\eta\|_{L^p(\Omega)}$ is bounded by 1 by Hölder's inequality,

$$\mathbb{E}(\bar{B}_2) = \lim_{\eta\to0} \frac{\mathbb{E}(B_\eta)}{\|B_\eta\|_{L^2(\Omega))}} \neq 0$$

and (A.13) is satisfied with $p=2$.

We then take $p = 2$. Since $\mathbb{E}(|B_\eta|^2) = o_{\eta\to0^+}(\mathbb{E}(B_\eta))$ and B_η is bounded in $L^\infty(\Omega)$, $\mathbb{E}(|B_\eta|^k) = o_{\eta\to0^+}(\mathbb{E}(B_\eta))$ for all $k\geq2$.

This intuitively expresses that all orders higher than or equal to 2 are negligible as compared to the first-order term in the series (A.12), and thus that a kind of "separation

of scales" is satisfied. This is of course formal since one has to check that the remainder term consisting of the *sum* of all terms of order higher than or equal to 2 is $o(\mathbb{E}(B_\eta))$, so that

$$
a_\eta^* = \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per}} \right)^{-1} + \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per}} \right)^{-2} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{E}(B_\eta) \frac{c_{per}}{a_{per}} \right) + o\left(\mathbb{E}(B_\eta) \right)
$$

$$
= \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per}} \right)^{-1} + m_\eta \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per}} \right)^{-2} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{E}(\bar{B}_0) \frac{c_{per}}{a_{per}} \right) + o\left(\mathbb{E}(B_\eta) \right).
$$

But this is the purpose of the proofs of Theorems 2.1 and A.1, using another viewpoint, to show this is indeed the case.

A.2.2   The setting of Section 3 in dimension one

We now prove that our approach of Section 3 is rigorous in dimension one.

Lemma A.5. *In dimension $d = 1$, it holds*

$$
a_\eta^* = a_{per}^* + \eta \bar{a}_1^* + \eta^2 \bar{a}_2^* + o(\eta^2),
$$

where $\bar{a}_1^*$ and $\bar{a}_2^*$ are the limits as $N \to \infty$ of $a_1^{*,N}$ and $a_2^{*,N}$ defined generally by (3.23) and (3.27) respectively.

Proof. Recall that in dimension one, a_η^* is given by the simple explicit expression

$$
a_\eta^* = \left(\mathbb{E} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + b_\eta c_{per}} \right)^{-1} = \left\langle dP_\eta(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle^{-1}.
$$

The proof thus consists in inserting expansion (3.3) in this explicit expression and identifying successively the first three dominant orders.

Using (3.3), we write

$$
(a_\eta^*)^{-1} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per}} + \eta \left\langle d\bar{P}_1(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle + \eta^2 \left\langle d\bar{P}_2(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle + o(\eta^2)
$$

$$
= (a_{per}^*)^{-1} \left(1 + \eta a_{per}^* \left\langle d\bar{P}_1(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle + \eta^2 a_{per}^* \left\langle d\bar{P}_2(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle \right) + o(\eta^2).
$$

This yields the expansion

$$
a_\eta^* = a_{per}^* - \eta (a_{per}^*)^2 \left\langle d\bar{P}_1(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle + \eta^2 (a_{per}^*)^3 \left\langle d\bar{P}_1(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle^2
$$

$$
- \eta^2 (a_{per}^*)^2 \left\langle d\bar{P}_2(s), \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{a_{per} + s c_{per}} \right\rangle + o(\eta^2). \tag{A.14}
$$

We now devote the rest of the proof to verifying that the coefficients of η and η^2 in (A.14) are indeed obtained as the limit as $N \to \infty$ of $a_1^{*,N}$ and $a_2^{*,N}$ defined generally by (3.23) and (3.27) respectively, in this particular one-dimensional setting.

The function $w^{1,s,0,N}$ generally defined by (3.22) satisfies here

$$\begin{cases} -\dfrac{d}{dx}\left(a_1^{s,0}\left(\dfrac{d}{dx}w_i^{1,s,0,N}+1\right)\right)=0, & \text{in } \left[-\dfrac{N}{2},\dfrac{N}{2}\right], \\ w_i^{1,s,0,N} \ \ N\text{-periodic.} \end{cases} \tag{A.15}$$

We easily compute using (A.15):

$$\begin{aligned} a_1^{s,0}\left(\frac{d}{dx}w^{1,s,0,N}+1\right) &= N\left(\int_{-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{a_{per}+s\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}c_{per}}\right)^{-1} \\ &= N\left(N(a_{per}^*)^{-1}-f(s)\right)^{-1} \\ &= a_{per}^* + \frac{(a_{per}^*)^2}{N}f(s) + \frac{(a_{per}^*)^3}{N^2}f(s)^2 + o(N^{-2}), \end{aligned}$$

where

$$f(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{sc_{per}}{a_{per}(a_{per}+sc_{per})}.$$

Thus $a_1^{*,N}$ defined generally by (3.23) takes here the form

$$\begin{aligned} a_1^{*,N} &= \left\langle d\bar{P}_1(s), \int_{-\frac{N}{2}}^{\frac{N}{2}} a_1^{s,0}\left(\frac{d}{dx}w^{1,s,0,N}+1\right)\right\rangle \\ &= Na_{per}^*\left\langle d\bar{P}_1(s),1\right\rangle + (a_{per}^*)^2\left\langle d\bar{P}_1(s),f(s)\right\rangle + o(1). \end{aligned}$$

We know from Lemma 3.2 that $\langle d\bar{P}_1(s),1\rangle = 0$, whence

$$a_1^{*,N} \xrightarrow[N\to\infty]{} \bar{a}_1^* = (a_{per}^*)^2\left\langle d\bar{P}_1(s),f(s)\right\rangle. \tag{A.16}$$

Likewise, we compute from (3.26), for $k \in [\![-\frac{N-1}{2},\frac{N-1}{2}]\!]\setminus\{0\}$,

$$\begin{aligned} a_2^{s,t,0,k}\left(\frac{d}{dx}w^{2,s,t,0,k,N}+1\right) &= N\left(\int_{-\frac{N}{2}}^{\frac{N}{2}} \frac{1}{a_{per}+s\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}c_{per}+t\mathbb{1}_{[k-\frac{1}{2},k+\frac{1}{2}]}c_{per}}\right)^{-1} \\ &= N\left(N(a_{per}^*)^{-1}-\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{sc_{per}}{a_{per}(a_{per}+sc_{per})}-\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{tc_{per}}{a_{per}(a_{per}+tc_{per})}\right)^{-1} \\ &= N\left(N(a_{per}^*)^{-1}-f(s)-f(t)\right)^{-1}. \end{aligned}$$

Then

$$a_2^{s,t,0,k}\left(\frac{d}{dx}w^{2,s,t,0,k,N}+1\right) = a_{per}^* + \frac{(a_{per}^*)^2}{N}(f(s)+f(t)) + \frac{(a_{per}^*)^3}{N^2}(f(s)+f(t))^2 + o(N^{-2}).$$

Notice that this expression is independent of k (and so of the distance between the two defects), so that $a_2^{*,N}$ defined by (3.27) here reads

$$a_2^{*,N} = \frac{N(N-1)}{2}\left\langle d\bar{P}_1(s)d\bar{P}_1(t), a_{per}^* + \frac{(a_{per}^*)^2}{N}(f(s)+f(t)) + \frac{(a_{per}^*)^3}{N^2}(f(s)+f(t))^2\right\rangle$$
$$+ N\left\langle d\bar{P}_2(s), a_{per}^* + \frac{(a_{per}^*)^2}{N}f(s)\right\rangle + o(1). \tag{A.17}$$

Since we know from Lemma 3.2 that $\langle d\bar{P}_1(s),1\rangle = 0$ and $\langle d\bar{P}_2(s),1\rangle = 0$, (A.17) reduces to

$$a_2^{*,N} = (a_{per}^*)^3\langle d\bar{P}_1(s),f(s)\rangle^2 + (a_{per}^*)^2\langle d\bar{P}_2(s),f(s)\rangle + o(1).$$

Thus

$$a_2^{*,N}\underset{N\to\infty}{\longrightarrow}\bar{a}_2^* = (a_{per}^*)^3\langle d\bar{P}_1(s),f(s)\rangle^2 + (a_{per}^*)^2\langle d\bar{P}_2(s),f(s)\rangle. \tag{A.18}$$

Finally, since

$$f(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{a_{per}} - \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{a_{per}+sc_{per}}\quad\text{and}\quad\langle d\bar{P}_1(s),1\rangle = \langle d\bar{P}_2(s),1\rangle = 0,$$

we have

$$\left\langle d\bar{P}_1(s),f(s)\right\rangle = -\left\langle d\bar{P}_1(s),\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{a_{per}+sc_{per}}\right\rangle, \tag{A.19}$$

and

$$\left\langle d\bar{P}_2(s),f(s)\right\rangle = -\left\langle d\bar{P}_2(s),\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{a_{per}+sc_{per}}\right\rangle. \tag{A.20}$$

In view of (A.14), (A.16), (A.18), (A.19) and (A.20), we have proved

$$a_\eta^* = a_{per}^* + \eta\bar{a}_1^* + \eta^2\bar{a}_2^* + o(\eta^2).$$

So, the proof is completed.   □

A.3 A proof of the approach of Section 3 in a specific setting

The purpose of this final section is to prove that the formal approach of Section 3 is rigorous in a setting related to that of Corollary 2.2.

More precisely, we assume that the random field b_η satisfies the assumptions of Corollary 2.2. These assumptions do not imply that the image measure dP_η satisfies assumption (3.3) which is at the heart of the approach of Section 3, so that we have to impose that dP_η additionally satisfies (3.3). The following preliminary result then gives the necessary form of the expansion of the image measure dP_η.

Lemma A.6. *Assume that b_η satisfies*

$$b_\eta(x,\omega) = \sum_{k \in \mathbb{Z}^d} \mathbb{1}_{Q+k}(x) B_\eta^k(\omega), \tag{A.21}$$

where the B_η^k are i.i.d random variables, the distribution of which is given by a "mother variable" B_η satisfying

$$\forall \eta > 0, \ \|B_\eta\|_{L^\infty(\Omega)} \leq M, \tag{A.22a}$$

$$B_\eta = \eta \bar{B}_0 + \eta^2 \bar{R}_0 + o(\eta^2) \ \text{weakly in } L^2(\Omega). \tag{A.22b}$$

Assume further that the image measure dP_η of B_η satisfies (3.3). Then

$$dP_\eta = \delta_0 - \eta \mathbb{E}(\bar{B}_0)\delta_0' + \frac{\eta^2}{2}\mathbb{E}(\bar{B}_0^2)\delta_0'' - \eta^2 \mathbb{E}(\bar{R}_0)\delta_0' + o(\eta^2), \quad \text{in } \mathcal{E}'(\mathbb{R}). \tag{A.23}$$

Proof. Firstly, notice that B_η/η converges strongly to $\bar{B}_0$ in $L^2(\Omega)$ because of (A.22). Now consider $\varphi \in \mathcal{D}(\mathbb{R})$. We have on the one hand

$$\mathbb{E}\left(\frac{B_\eta^2}{\eta^2}\varphi(B_\eta)\right) \rightarrow \mathbb{E}(\bar{B}_0^2)\varphi(0),$$

and on the other hand

$$\mathbb{E}\left(B_\eta^2\varphi(B_\eta)\right) = \eta\langle s^2 d\bar{P}_1, \varphi\rangle + \eta^2\langle s^2 d\bar{P}_2, \varphi\rangle + o(\eta^2).$$

Thus $s^2 d\bar{P}_1 = 0$ and $s^2 d\bar{P}_2 = \mathbb{E}(\bar{B}_0^2)\delta_0$ in $\mathcal{D}'(\mathbb{R})$. It is then well known that there exist γ_1, κ_1, γ_2, κ_2 in $\mathbb{R}$ such that

$$d\bar{P}_1 = \gamma_1\delta_0 + \kappa_1\delta_0' \quad \text{and} \quad d\bar{P}_2 = \gamma_2\delta_0 + \kappa_2\delta_0' + \frac{\mathbb{E}(\bar{B}_0^2)}{2}\delta_0''.$$

Lemma 3.2 implies $\gamma_1 = \gamma_2 = 0$. Then, we have

$$\mathbb{E}(B_\eta) = \eta\mathbb{E}(\bar{B}_0) + \eta^2\mathbb{E}(\bar{R}_0) + o(\eta^2)$$

and also

$$\mathbb{E}(B_\eta) = \eta\langle sd\bar{P}_1, 1\rangle + \eta^2\langle sd\bar{P}_2, 1\rangle + o(\eta^2).$$

Thus $\langle sd\bar{P}_1, 1\rangle = \mathbb{E}(\bar{B}_0)$ and $\langle sd\bar{P}_2, 1\rangle = \mathbb{E}(\bar{R}_0)$, from which we deduce $\kappa_1 = -\mathbb{E}(\bar{B}_0)$ and $\kappa_2 = -\mathbb{E}(\bar{R}_0)$. $\qquad\square$

Theorem 2.1 and Corollary 2.2 rigorously yield the second-order expansion

$$A_\eta^* = A_{per}^* + \eta\widetilde{A_1^*} + \eta^2\widetilde{A_2^*} + o(\eta^2)$$

with $\widetilde{A_1^*}$ and $\widetilde{A_2^*}$ respectively defined by (2.13) and (2.22).

On the other hand, using (A.23), Section 3 yields the formal expansion

$$A_\eta^* = A_{per}^* + \eta \bar{A}_1^* + \eta^2 \bar{A}_2^* + o(\eta^2).$$

where $\bar{A}_1^*$ is the limit of the sequence $A_1^{*,N}$ defined by (3.23) or equivalently by (3.28), and $\bar{A}_2^*$ the limit of the sequence $A_2^{*,N}$ defined by (3.27).

The rest of this section is devoted to verifying that $\bar{A}_1^*$ coincides with $\widetilde{A_1^*}$ and $\bar{A}_2^*$ coincides with $\widetilde{A_2^*}$ in the specific setting of Lemma A.6.

A.3.1   First-order term

Using (A.23), (3.28) reads

$$A_1^{*,N} e_i \cdot e_j = -\mathbb{E}(\bar{B}_0)\left\langle \delta_0'(s), \int_Q sC_{per}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) \right\rangle,$$

and we compute

$$A_1^{*,N} = \mathbb{E}(\bar{B}_0)\int_Q C_{per}(\nabla w_i^{1,0,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0).$$

Setting $s = 0$ in (3.22), it is clear that $w_i^{1,0,0,N}$ is equal to the periodic corrector w_i^0. Then

$$A_1^{*,N} = \mathbb{E}(\bar{B}_0)\int_Q C_{per}(\nabla w_i^0 + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0). \tag{A.24}$$

Clearly $A_1^{*,N}$ does not depend on N and its limit is then

$$\bar{A}_1^* = \mathbb{E}(\bar{B}_0)\int_Q C_{per}(\nabla w_i^0 + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0). \tag{A.25}$$

We recognize in the right-hand side of (A.25) the first-order coefficient in (2.17), which we know from Remark 2.2 is equivalent to (2.13). Theorem 2.1 therefore shows that the first-order expansion

$$A_\eta^* = A_{per}^* + \eta \bar{A}_1^* + o(\eta)$$

is correct with the values of the coefficients given by our formal approach of Section 3.

We now proceed similarly with the second-order coefficient.

A.3.2   Second-order term

Using the adjoint cell problems (2.16) in (3.27) as in the proof of Proposition 3.1, let us first rewrite

$$A_2^{*,N} e_i \cdot e_j = \sum_{k \in \mathcal{T}_N, k \neq 0} \left\langle d\bar{P}_1(s) d\bar{P}_1(t), \int_Q sC_{per} \nabla w_i^{2,s,t,0,k,N} \cdot (e_j + \nabla \widetilde{w}_j^0) \right\rangle$$

$$+ \left\langle d\bar{P}_2(s), \int_Q sC_{per}(\nabla w_i^{1,s,0,N} + e_i) \cdot (e_j + \nabla \widetilde{w}_j^0) \right\rangle. \tag{A.26}$$

Inserting (A.23) in (A.26), we start by focusing on

$$\left\langle d\bar{P}_2(s), \int_Q sC_{per}(\nabla w_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle$$

$$=\frac{1}{2}(\mathbb{E}(\bar{B}_0))^2\left\langle \delta_0''(s), \int_Q sC_{per}(\nabla w_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle$$

$$-\mathbb{E}(\bar{R}_0)\left\langle \delta_0'(s), \int_Q sC_{per}(\nabla w_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle.$$

Denoting by $\partial_s w_i^{1,0,0,N}$, the first derivative of $w_i^{1,s,0,N}$ evaluated at $s=0$, we compute

$$\left\langle d\bar{P}_2(s), \int_Q sC_{per}(\nabla w_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle$$

$$=\mathbb{E}(\bar{B}_0^2)\int_Q C_{per}\nabla\partial_s w_i^{1,0,0,N}\cdot(\nabla\widetilde{w}_j^0+e_j)+\mathbb{E}(\bar{R}_0)\int_Q C_{per}(\nabla w_i^{1,0,0,N}+e_i)\cdot(\nabla\widetilde{w}_j^0+e_j)$$

$$=\mathbb{E}(\bar{B}_0^2)\int_Q C_{per}\nabla\partial_s w_i^{1,0,0,N}\cdot(\nabla\widetilde{w}_j^0+e_j)+\mathbb{E}(\bar{R}_0)\int_Q C_{per}(\nabla w_i^0+e_i)\cdot(\nabla\widetilde{w}_j^0+e_j).$$

It follows from (3.22) that $\partial_s w_i^{1,0,0,N}$ solves

$$\begin{cases} -\mathrm{div}(A_{per}\nabla\partial_s w_i^{1,0,0,N})=\mathrm{div}\big(\mathbb{1}_Q C_{per}(\nabla w_i^0+e_i)\big), & \text{in } I_N, \\ \partial_s w_i^{1,0,0,N} \ (N\mathbb{Z})^d\text{-periodic.} \end{cases} \qquad (A.27)$$

Applying Lemma A.2 to (A.27), we deduce that $\nabla\partial_s w_i^{1,0,0,N}$ converges in $L^2(Q)$, when $N\rightarrow\infty$, to ∇t_i defined by (2.28) in Corollary 2.2. Consequently,

$$\left\langle d\bar{P}_2(s), \int_Q sC_{per}(\nabla w_i^{1,s,0,N}+e_i)\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle \underset{N\rightarrow\infty}{\longrightarrow} \mathbb{E}(\bar{B}_0^2)\int_Q C_{per}\nabla t_i\cdot(\nabla\widetilde{w}_j^0+e_j)$$

$$+\mathbb{E}(\bar{R}_0)\int_Q C_{per}(\nabla w_i^0+e_i)\cdot(\nabla\widetilde{w}_j^0+e_j). \quad (A.28)$$

Next, we address

$$\sum_{k\in\mathcal{T}_N,k\neq 0}\left\langle d\bar{P}_1(s)d\bar{P}_1(t), \int_Q sC_{per}\nabla w_i^{2,s,t,0,k,N}\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle$$

$$=\mathbb{E}(\bar{B}_0)^2\sum_{k\in\mathcal{T}_N,k\neq 0}\left\langle \delta_0'(s)\delta_0'(t), \int_Q sC_{per}\nabla w_i^{2,s,t,0,k,N}\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle.$$

Denoting by $\partial_t w_i^{2,0,0,0,k,N}$ the first derivative of $w_i^{2,s,t,0,k,N}$ with respect to t evaluated at $s=t=0$, we have

$$\sum_{k\in\mathcal{T}_N,k\neq 0}\left\langle d\bar{P}_1(s)d\bar{P}_1(t), \int_Q sC_{per}\nabla w_i^{2,s,t,0,k,N}\cdot(e_j+\nabla\widetilde{w}_j^0)\right\rangle$$

$$=\mathbb{E}(\bar{B}_0)^2\sum_{k\in\mathcal{T}_N,k\neq 0}\int_Q C_{per}\nabla\partial_t w_i^{2,0,0,0,k,N}\cdot(e_j+\nabla\widetilde{w}_j^0). \qquad (A.29)$$

It follows from (3.26) that $\partial_t w_i^{2,0,0,0,k,N}$ solves

$$\begin{cases} -\mathrm{div}\big(A_{per}\nabla\partial_t w_i^{2,0,0,0,k,N}\big) = \mathrm{div}\big(\mathbb{1}_{Q+k}C_{per}(\nabla w_i^0+e_i)\big), & \text{in } I_N, \\ \partial_t w_i^{2,0,0,0,k,N} \ (N\mathbb{Z})^d\text{-periodic.} \end{cases} \qquad (A.30)$$

Defining $d_i^N = \sum_{k\in\mathcal{T}_N}\partial_t w_i^{2,0,0,0,k,N}$, it is easy to see that d_i^N is a $\mathbb{Z}^d$-periodic function that solves

$$\begin{cases} -\mathrm{div}(A_{per}\nabla d_i^N) = \mathrm{div}\big(C_{per}(\nabla w_i^0+e_i)\big), & \text{in } Q, \\ d_i^N \ \mathbb{Z}^d\text{-periodic.} \end{cases} \qquad (A.31)$$

Since problem (A.31) has a unique solution up to an additive constant, $\nabla d_i^N = \nabla s_i$ where s_i is defined by (2.29) in Corollary 2.2.

Finally, comparing (A.27) to (A.30) for $k=0$, we find that $\nabla\partial_t w_i^{2,0,0,0,0,N}$ is equal to $\nabla\partial_s w_i^{1,0,0,N}$ and then also converges in $L^2(Q)$ to ∇t_i when $N\to\infty$.

Then, starting from (A.29),

$$\sum_{k\in\mathcal{T}_N, k\neq 0}\Big\langle d\bar{P}_1(s)d\bar{P}_1(t), \int_Q sC_{per}\nabla w_i^{2,s,t,0,k,N}\cdot(e_j+\nabla\widetilde{w}_j^0)\Big\rangle$$

$$=(\mathbb{E}(\bar{B}_0))^2\int_Q C_{per}\sum_{k\in\mathcal{T}_N}\nabla\partial_t w_i^{2,0,0,0,k,N}\cdot(e_j+\nabla\widetilde{w}_j^0) - (\mathbb{E}(\bar{B}_0))^2\int_Q C_{per}\nabla\partial_t w_i^{2,0,0,0,0,N}\cdot(e_j+\nabla\widetilde{w}_j^0)$$

$$=(\mathbb{E}(\bar{B}_0))^2\int_Q C_{per}\nabla s_i\cdot(e_j+\nabla\widetilde{w}_j^0) - (\mathbb{E}(\bar{B}_0))^2\int_Q C_{per}\nabla\partial_t w_i^{2,0,0,0,0,N}\cdot(e_j+\nabla\widetilde{w}_j^0)$$

$$\xrightarrow[N\to\infty]{} (\mathbb{E}(\bar{B}_0))^2\int_Q C_{per}\nabla s_i\cdot(e_j+\nabla\widetilde{w}_j^0) - (\mathbb{E}(\bar{B}_0))^2\int_Q C_{per}\nabla t_i\cdot(e_j+\nabla\widetilde{w}_j^0). \qquad (A.32)$$

It entails from (A.26), (A.28) and (A.32) that $A_2^{*,N}$ converges to a limit $\bar{A}_2^*$ defined by

$$\bar{A}_2^*e_i\cdot e_j = \mathbb{E}(\bar{R}_0)\int_Q C_{per}(\nabla w_i^0+e_i)\cdot(\nabla\widetilde{w}_j^0+e_j) + Var(\bar{B}_0)\int_Q C_{per}\nabla t_i\cdot(\nabla\widetilde{w}_j^0+e_j)$$

$$+ (\mathbb{E}(\bar{B}_0))^2\int_Q\nabla s_i\cdot(e_j+\nabla\widetilde{w}_j^0).$$

$\bar{A}_2^*$ is equal to the second-order term given by (2.27) in Corollary 2.2 since we deal with independent random variables in each cell of $\mathbb{Z}^d$. Thus the second-order expansion

$$A_\eta^* = A_{per}^* + \eta\bar{A}_1^* + \eta^2\bar{A}_2^* + o(\eta^2)$$

derived from the formal approach of Section 3 is correct in this specific setting.

Acknowledgments

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