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New Energy-Conserved Identities and Super-Convergence of the Symmetric EC-S-FDTD Scheme for Maxwell's Equations in 2D

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Abstract. The symmetric energy-conserved splitting FDTD scheme developed in [1] is a very new and efficient scheme for computing the Maxwell's equations. It is based on splitting the whole Maxwell's equations and matching the x-direction and y-direction electric fields associated to the magnetic field symmetrically. In this paper, we make further study on the scheme for the 2D Maxwell's equations with the PEC boundary condition. Two new energy-conserved identities of the symmetric EC-S-FDTD scheme in the discrete H^1-norm are derived. It is then proved that the scheme is unconditionally stable in the discrete H^1-norm. By the new energy-conserved identities, the super-convergence of the symmetric EC-S-FDTD scheme is further proved that it is of second order convergence in both time and space steps in the discrete H^1-norm. Numerical experiments are carried out and confirm our theoretical results.

AMS subject classifications: 65N06, 65N12, 65N15

Key words: Symmetric EC-S-FDTD, energy-conserved, unconditional stability, super convergence, Maxwell's equations, splitting.

1 Introduction

The finite-difference time-domain (FDTD) method, which was firstly introduced by Yee [18] in 1966, is a very popular and efficient numerical method in computational electromagnetics and is applicable to a broad range of problems (see, for example [16]). FDTD uses the central difference on the staggered grid points, second order accurate and easy to

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be implemented. However, FDTD is (conditionally) stable when the Courant-Friedrichs-Lewy (CFL) condition in the 2D case

$$\Delta t < \frac{1}{c}\Big(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\Big)^{-1/2},$$

is satisfied, where

$$c = 1/\sqrt{\varepsilon\mu}$$

is the wave velocity. In 1999, the alternating direction implicit FDTD (ADI-FDTD) methods were proposed in [13, 20] by Zheng, Chen, Zhang for the 3D case and Namiki for the 2D case and were proved to be unconditionally stable. Different from the ADI-FDTD schemes, the splitting FDTD schemes (S-FDTD) were proposed by Gao, Zhang and Liang in [6, 7], which firstly use the techniques of splitting the whole Maxwell's equations and reducing the perturbation errors by additional terms. It was proved that the splitting schemes are unconditionally stable, and the S-FDTDII scheme is of second order accuracy and has good merit in simulating a kind of scattering problems [6]. On the other hand, the electromagnetic energy of the wave keeps constant at different time in a lossless medium without sources. It is important to study the energy conservations of numerical schemes for the Maxwell's equations. Recently, in 2009, Chen, Li and Liang [1] proposed a new splitting finite difference method, called the symmetric energy-conserved splitting FDTD scheme (i.e. symmetric EC-S-FDTD), which reduces the perturbation errors due to the splitting of equations by the symmetry in the combination of the x-direction and y-direction electric fields and the magnetic field. It was proved in [1] that this method is unconditionally stable and of second order convergence in both time and space steps in the discrete L^2-norm, and specially, it is energy-conserved in the two energy identities in the discrete L^2-norm.

In this paper, we make further study on the energy conservation, stability and error estimates of the symmetric EC-S-FDTD scheme. We firstly give two new energy-conserved identities (in the H^1 norm) of the 2D Maxwell's equations with the PEC boundary conditions. These energy identities physically explain energy conservations of the variation of the electric and magnetic fields in a lossless medium without sources under the H^1 norm. Then, we strictly prove that the symmetric EC-S-FDTD scheme satisfies these two new energy-conserved identities in the discrete forms. By these new identities, it is proved that the symmetric EC-S-FDTD scheme is unconditionally stable in the discrete H^1-norm. Moreover, we prove the super-convergence of the symmetric EC-S-FDTD scheme that the scheme is of second order accuracy in the discrete H^1-norm. With the help of the super convergence result, it is easily proved that the divergence of the electric field of the symmetric ES-S-FDTD scheme is second order accurate. Numerical experiments are presented and numerical results confirm the theoretical results.

The remaining of this paper is organized as follows. In Section 2, new energy-conserved identities of the 2D Maxwell's equations in the H^1 norm are derived. In Section 3, we prove the symmetric EC-S-FDTD scheme to satisfy new energy-conserved identities and to be stable in the discrete H^1 norm. Section 4 gives the proof of the superconvergence of the scheme in the discrete H^1-norm and the divergence error of the electric field of the scheme. Numerical experiments and conclusions are given in Section 5 and Section 6.

2 Energy-conserved identities of Maxwell's equations

Consider the Maxwell's equations in two dimensions:

$$\frac{\partial E_x}{\partial t}=\frac{1}{\varepsilon}\frac{\partial H_z}{\partial y}, \tag{2.1}$$

$$\frac{\partial E_y}{\partial t}=-\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x}, \tag{2.2}$$

$$\frac{\partial H_z}{\partial t}=\frac{1}{\mu}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right), \tag{2.3}$$

in a lossless and isotropic medium with electric permittivity ε and magnetic permeability μ, where $\mathbf{E}=(E_x(t,x,y),E_y(t,x,y))$ and $H_z=H_z(t,x,y)$ denote the electric and magnetic fields, $(x,y)\in\Omega$, $t\in(0,T]$.

We assume that the domain Ω is a rectangle, $\Omega=[0,a]\times[0,b]$, where a and b are positive numbers, and the medium is covered by perfectly electric conductors. So, the perfectly electric conducting (PEC) boundary condition is satisfied:

$$(\vec{n},0)\times(\mathbf{E},0)=\vec{0}, \quad \text{on } (0,T]\times\partial\Omega, \tag{2.4}$$

where $\partial\Omega$ denotes the boundary of Ω, $\vec{n}$ is the outward norm to $\partial\Omega$. By the cross product of vectors, this PEC boundary condition can be written as

$$E_x(t,x,0)=E_x(t,x,b)=E_y(t,0,y)=E_y(t,a,y)=0, \quad \forall x\in[0,a],\ \forall y\in[0,b],\ \forall t\in(0,T]. \tag{2.5}$$

The initial conditions are given as

$$\mathbf{E}_0=\mathbf{E}(0,x,y)=(E_{x0}(x,y),E_{y0}(x,y)) \quad \text{and} \quad H_{z0}=H_{z0}(x,y). \tag{2.6}$$

We first give two new energy-conserved identities for the Maxwell's equations in two dimensions in the following Theorems 2.1 and 2.2.

Theorem 2.1. *Let* $\mathbf{E}=(E_x(t,x,y),E_y(t,x,y))$ *and* $H_z=H_z(t,x,y)$ *be the solution of the Maxwell's equations* (2.1)-(2.3) *with* (2.4) *and* (2.6). *Suppose that* $\mathbf{E}$ *and* H_z *are smooth enough, e.g.* $\mathbf{E}\in C([0,T];[C^2(\bar{\Omega})]^2)\cap C^1([0,T];[C^1(\bar{\Omega})]^2)$, $H_z\in C([0,T];C^2(\bar{\Omega}))\cap C^1([0,T];C^1(\bar{\Omega}))$. *Then, for* $w=x$ *or* y*, the following energy identities hold*

$$\begin{aligned}&\int_\Omega\left\{\varepsilon\Big(\frac{\partial E_x}{\partial w}\Big)^2+\varepsilon\Big(\frac{\partial E_y}{\partial w}\Big)^2+\mu\Big(\frac{\partial H_z}{\partial w}\Big)^2\right\}dxdy\\ =&\int_\Omega\left\{\varepsilon\Big(\frac{\partial E_{x0}}{\partial w}\Big)^2+\varepsilon\Big(\frac{\partial E_{y0}}{\partial w}\Big)^2+\mu\Big(\frac{\partial H_{z0}}{\partial w}\Big)^2\right\}dxdy\equiv Constant.\end{aligned} \tag{2.7}$$

Proof. The proof of (2.7) with $w=x$ is the same as that of (2.7) with $w=y$. Thus, we only derive (2.7) with $w=x$. Differentiating each of the equations from (2.1) to (2.3) with respect to x leads to

$$\frac{\partial^2 E_x}{\partial x\partial t}=\frac{1}{\varepsilon}\frac{\partial^2 H_z}{\partial x\partial y}, \tag{2.8}$$

$$\frac{\partial^2 E_y}{\partial x\partial t}=-\frac{1}{\varepsilon}\frac{\partial^2 H_z}{\partial x^2}, \tag{2.9}$$

$$\frac{\partial^2 H_z}{\partial x\partial t}=\frac{1}{\mu}\Big(\frac{\partial^2 E_x}{\partial x\partial y}-\frac{\partial^2 E_y}{\partial x^2}\Big). \tag{2.10}$$

By using the integration by parts and the PEC boundary condition (2.5), we have

$$\int_\Omega \frac{\partial^2 E_x}{\partial x\partial y}\frac{\partial H_z}{\partial x}dxdy=-\int_\Omega \frac{\partial E_x}{\partial x}\frac{\partial^2 H_z}{\partial x\partial y}dxdy, \tag{2.11}$$

$$\int_\Omega \frac{\partial^2 E_y}{\partial x^2}\frac{\partial H_z}{\partial x}dxdy=-\int_\Omega \frac{\partial E_y}{\partial x}\cdot\frac{\partial^2 H_z}{\partial x^2}dxdy+T_1-T_2, \tag{2.12}$$

where

$$T_1=\int_0^b \frac{\partial E_y}{\partial x}(t,a,y)\frac{\partial H_z}{\partial x}(t,a,y)dy, \quad T_2=\int_\Omega \frac{\partial E_y}{\partial x}(t,0,y)\frac{\partial H_z}{\partial x}(t,0,y)dy.$$

From Eq. (2.2) and the boundary condition (2.5), it holds that

$$T_1=-\int_0^b \lim_{x\to a}\varepsilon\frac{\partial E_y}{\partial t}\frac{\partial E_y}{\partial x}dy=0, \quad T_2=-\int_0^b \lim_{x\to 0}\varepsilon\frac{\partial E_y}{\partial t}\frac{\partial E_y}{\partial x}dy=0. \tag{2.13}$$

Then, multiplying Eqs. (2.8)-(2.10) by $\varepsilon\frac{\partial E_x}{\partial x}$, $\varepsilon\frac{\partial E_y}{\partial x}$ and $\mu\frac{\partial H_z}{\partial x}$ respectively, integrating both sides of the equations over $\Omega=[0,a]\times[0,b]$, and using (2.11)-(2.13), we have that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\Big\{\varepsilon\Big[(\frac{\partial E_x}{\partial x})^2+(\frac{\partial E_y}{\partial x})^2\Big]+\mu(\frac{\partial H_z}{\partial x})^2\Big\}dxdy=0. \tag{2.14}$$

So, the energy identity (2.7) with $w=x$ is obtained by integrating (2.14) over $[0,t]$. □

If we further differentiate Eqs. (2.1)-(2.5) with respect to t and repeat all the arguments above, we can obtain the following theorem.

Theorem 2.2. *Assume that* $\mathbf{E}=(E_x,E_y)$ *and* H_z *are the solution of the Maxwell's equations* (2.1)-(2.3) *with* (2.4) *and* (2.6) *and smooth enough, e.g.* $\mathbf{E}\in C^1([0,T];[C^2(\bar\Omega)]^2)\cap C^2([0,T];[C^1(\bar\Omega)]^2)$, $H_z\in C^1([0,T];C^2(\bar\Omega))\cap C^2([0,T];C^1(\bar\Omega))$. *Then, for* $w=x$ *or* y, *the following energy identities hold*

$$\int_\Omega\Big\{\varepsilon\Big(\frac{\partial^2 E_x}{\partial t\partial w}\Big)^2+\varepsilon\Big(\frac{\partial^2 E_y}{\partial t\partial w}\Big)^2+\mu\Big(\frac{\partial^2 H_z}{\partial t\partial w}\Big)^2\Big\}dxdy\equiv Constant. \tag{2.15}$$

Reduction of these two theorems to the case of $\frac{\partial}{\partial w}=I$ (the identity operator) leads to the following two important energy identities (the detailed proof see [2]).

Corollary 2.1. *Let* $\{\mathbf{E},H_z\}$ *be the solution of the Maxwell's equations* (2.1)-(2.4) *and* (2.6). *If* $\mathbf{E}$ *and* H_z *are smooth enough, e.g.* $\mathbf{E}\in C([0,T];[C^1(\bar{\Omega})]^2)\cap C^1([0,T];[C(\bar{\Omega})]^2)$, $H_z\in C([0,T];C^1(\bar{\Omega}))\cap C^1([0,T];[C(\bar{\Omega})]^2)$, *then*

$$\int_\Omega\Big\{\varepsilon(E_x)^2+\varepsilon(E_y)^2+\mu(H_z)^2\Big\}dxdy\equiv Constant; \tag{2.16}$$

if $\mathbf{E}\in C^1([0,T];[C^1(\bar{\Omega})]^2)\cap C^2([0,T];[C(\bar{\Omega})]^2)$, $H_z\in C^1([0,T];C^1(\bar{\Omega}))\cap C^2([0,T];[C(\bar{\Omega})]^2)$, *then*

$$\int_\Omega\Big\{\varepsilon\Big(\frac{\partial E_x}{\partial t}\Big)^2+\varepsilon\Big(\frac{\partial E_y}{\partial t}\Big)^2+\mu\Big(\frac{\partial H_z}{\partial t}\Big)^2\Big\}dxdy\equiv Constant. \tag{2.17}$$

Now, we give some notations and the Yee's staggered meshes in two dimensions. Let $\Delta x=a/I$ and $\Delta y=b/J$ be the mesh sizes along the x and y directions respectively, and let $\Delta t=T/(2N)$ be the time step size, I, J and N are positive integers. Define

$$\begin{aligned}
&x_{i+\frac12}=x_i+\frac12\Delta x,\quad y_{j+\frac12}=y_j+\frac12\Delta y,\quad t^{n+\frac12}=t^n+\frac12\Delta t,\\
&U^m_{\alpha,\beta}=U(t^m,x_\alpha,y_\beta),\quad m=n,n+\frac12,\quad \alpha=i,i+\frac12,\quad \beta=j,j+\frac12,\\
&\delta_xU^m_{\alpha,\beta}=(U^m_{\alpha+\frac12,\beta}-U^m_{\alpha-\frac12,\beta})/\Delta x,\quad \delta_yU^m_{\alpha,\beta}=(U^m_{\alpha,\beta+\frac12}-U^m_{\alpha,\beta-\frac12})/\Delta y,\\
&\delta_tU^m_{\alpha,\beta}=(U^{m+1}_{\alpha,\beta}-U^{m-1}_{\alpha,\beta})/2\Delta t,\quad \delta_u\delta_vU^m_{\alpha,\beta}=\delta_u(\delta_vU^m_{\alpha,\beta}),\quad u,v=x,y,
\end{aligned}$$

where $i=0,1,\cdots,I-1$; $j=0,1,\cdots,J-1$; $n=0,1,\cdots,2N$. The Yee's staggered meshes associated with the fields $\mathbf{E}=(E_x,E_y)$ and H_z can be written as

$$\Omega_{E_x}=\{(x_{i+\frac12},y_j)|^{I-1}_{i=0}{}^{J-1}_{,j=1}\}, \tag{2.18a}$$

$$\Omega_{E_y}=\{(x_i,y_{j+\frac12})|^{I-1}_{i=1}{}^{J-1}_{,j=0}\}, \tag{2.18b}$$

$$\Omega_{H_z}=\{(x_{i+\frac12},y_{j+\frac12})|^{I-1}_{i=0}{}^{J-1}_{,j=0}\}. \tag{2.18c}$$

Similarly, the meshes corresponding to the differences of the electric and magnetic fields, δ_wE_x, δ_wE_y and $\delta_wH_z(w=x,y)$, and the boundary meshes to the boundary values $E_{x_{i+\frac12,j'}}$ and $E_{y_{i',j+\frac12}}$, where $i'=1,I-1$, $j'=1,J-1$, are denoted by

$$\begin{aligned}
&\Omega_{\delta_xE_x}=\Omega_{\delta_yE_y}=\{(x_i,y_j)|^{I-1}_{i=1}{}^{J-1}_{,j=1}\},\qquad \Omega_{\delta_yE_x}=\{(x_{i+\frac12},y_{j+\frac12})|^{I-1}_{i=0}{}^{J-2}_{,j=1}\},\\
&\Omega_{\delta_xE_y}=\{(x_{i+\frac12},y_{j+\frac12})|^{I-2}_{i=1}{}^{J-1}_{,j=0}\},\qquad \Omega_{\delta_xH_z}=\{(x_i,y_{j+\frac12})|^{I-1}_{i=1}{}^{J-1}_{,j=0}\},\\
&\Omega_{E_{xj'}}=\{(x_{i+\frac12},y_{j'})|^{I-1}_{i=0},j'=1,J-1\},\quad \Omega_{E_{yi'}}=\{(x_{i'},y_{j+\frac12})|^{J-1}_{j=0},i'=1,I-1\};
\end{aligned}$$

the meshes $\Omega_{\delta_y E_y}$ and $\Omega_{\delta_y H_z}$ can be similarly defined. For a grid function $U_{\alpha,\beta}$, where $\alpha=i,i+\frac{1}{2}$, $\beta=j,j+\frac{1}{2}$, define the following norms:

$$\|U\|_{E_x}^2=\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\varepsilon(U_{i+\frac{1}{2},j})^2\Delta x\Delta y, \qquad \|U\|_{E_y}^2=\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\varepsilon(U_{i,j+\frac{1}{2}})^2\Delta x\Delta y,$$

$$\|U\|_{H_z}^2=\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\mu(U_{i+\frac{1}{2},j+\frac{1}{2}})^2\Delta x\Delta y, \quad \|U\|_{\delta_x E_x}^2=\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\varepsilon(U_{i,j})^2\Delta x\Delta y,$$

$$|U|_{E_{xj'}}^2=\frac{1}{\Delta y}\sum_{i=0}^{I-1}\sum_{j'}\varepsilon(U_{i+\frac{1}{2},j'})^2\Delta x=\frac{1}{\Delta y}\sum_{i=0}^{I-1}\varepsilon\{(U_{i+\frac{1}{2},1})^2+(U_{i+\frac{1}{2},J-1})^2\}\Delta x.$$

In the norms above, the subscripts imply that the summations are done over the meshes on which the subscript fields are defined. Other norms, such as $\|U\|_{\delta_y E_x}^2$, $\|U\|_{\delta_x E_y}^2$, $\|U\|_{\delta_y E_y}^2$, $\|U\|_{\delta_x H_z}$, $\|U\|_{\delta_y H_z}$ and $|U|_{E_{yi'}}$ can be similarly defined.

3 Symmetric EC-S-FDTD and new discrete energy conservation

In this section, we firstly introduce the symmetric EC-S-FDTD scheme. Then, we prove that the scheme satisfies four new discrete energy-conserved identities corresponding to those of the Maxwell's equations.

The symmetric EC-S-FDTD scheme [1] for the 2D Maxwell's equations (2.1)-(2.3) is

$$\text{Stage 1:}\quad \frac{{E_y}_{i,j+\frac{1}{2}}^{2k+1}-{E_y}_{i,j+\frac{1}{2}}^{2k}}{\Delta t}=-\frac{1}{2\varepsilon}\delta_x\{{H_z}_{i,j+\frac{1}{2}}^{*}+{H_z}_{i,j+\frac{1}{2}}^{2k}\}, \tag{3.1}$$

$$\frac{{H_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{*}-{H_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{{E_y}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}+{E_y}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}\}; \tag{3.2}$$

$$\text{Stage 2:}\quad \frac{{E_x}_{i+\frac{1}{2},j}^{2k+1}-{E_x}_{i+\frac{1}{2},j}^{2k}}{\Delta t}=\frac{1}{2\varepsilon}\delta_y\{{H_z}_{i+\frac{1}{2},j}^{2k+1}+{H_z}_{i+\frac{1}{2},j}^{*}\}, \tag{3.3}$$

$$\frac{{H_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}-{H_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{*}}{\Delta t}=\frac{1}{2\mu}\delta_y\{{E_x}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}+{E_x}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}\}; \tag{3.4}$$

$$\text{Stage 3:}\quad \frac{{E_x}_{i+\frac{1}{2},j}^{2k+2}-{E_x}_{i+\frac{1}{2},j}^{2k+1}}{\Delta t}=\frac{1}{2\varepsilon}\delta_y\{{H_z}_{i+\frac{1}{2},j}^{**}+{H_z}_{i+\frac{1}{2},j}^{2k+1}\}, \tag{3.5}$$

$$\frac{{H_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{**}-{H_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}}{\Delta t}=\frac{1}{2\mu}\delta_y\{{E_x}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}+{E_x}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}\}; \tag{3.6}$$

$$\text{Stage 4:}\quad \frac{E_{y}{}_{i,j+\frac{1}{2}}^{2k+2}-E_{y}{}_{i,j+\frac{1}{2}}^{2k+1}}{\Delta t}=-\frac{1}{2\varepsilon}\delta_x\{H_{z}{}_{i,j+\frac{1}{2}}^{2k+2}+H_{z}{}_{i,j+\frac{1}{2}}^{**}\}, \tag{3.7}$$

$$\frac{H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}-H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{**}}{\Delta t}=-\frac{1}{2\mu}\delta_x\{E_{y}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+2}+E_{y}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}\}; \tag{3.8}$$

where the boundary conditions are

$$E_{x}{}_{i+\frac{1}{2},0}^{m}=E_{x}{}_{i+\frac{1}{2},J}^{m}=E_{y}{}_{0,j+\frac{1}{2}}^{m}=E_{y}{}_{I,j+\frac{1}{2}}^{m}=0,\quad m=2k,2k+1;\quad k=0,1,\cdots,N-1, \tag{3.9}$$

and the initial conditions are

$$\mathbf{E}^0_{\alpha,\beta}=\mathbf{E}_0(\alpha\Delta x,\beta\Delta y),\quad H_{z}{}_{\alpha,\beta}^{0}=H_{z0}(\alpha\Delta x,\beta\Delta y). \tag{3.10}$$

In order to derive the discrete energy-conserved identities, we will use the following lemma.

Lemma 3.1. *Let* $\{E_{x}{}_{i+\frac{1}{2}}^{n}\}$, $\{E_{y}{}_{i,j+\frac{1}{2}}^{n}\}$ *and* $\{H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{m}\}$, $n=2k,2k\pm1$; $m=2k,2k\pm1,*,**$; $k>0$, *be the grid function values in the symmetric EC-S-FDTD scheme and satisfy the boundary conditions* (3.9). *Then, it holds that*

$$\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\delta_x\delta_yE_x{}^n\cdot\delta_xH_z{}^m)_{i,j+\frac{1}{2}}\Delta x\Delta y=-\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\delta_xE_x{}^n\cdot\delta_x\delta_yH_z{}^m)_{i,j}\Delta x\Delta y \tag{3.11}$$

$$\begin{aligned}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}(\delta_x\delta_xE_y{}^n\cdot\delta_xH_z{}^m)_{i,j+\frac{1}{2}}\Delta x\Delta y=&-\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}(\delta_xE_y{}^n\cdot\delta_x\delta_xH_z{}^m)_{i+\frac{1}{2},j+\frac{1}{2}}\Delta x\Delta y\\&-\frac{\Delta t}{\Delta x}\sum_{i'}\sum_{j=0}^{J-1}(E_y{}^n\cdot\delta_xH_z{}^m)_{i',j+\frac{1}{2}}\Delta y,\end{aligned} \tag{3.12}$$

where the summation $\sum\limits_{i'}\sum\limits_{j=0}^{J-1}$ *means*

$$\sum_{i'}\sum_{j=0}^{J-1}(E_y{}^n\cdot\delta_xH_z{}^m)_{i',j+\frac{1}{2}}\Delta y=\sum_{j=0}^{J-1}\{E_{y}{}_{1,j+\frac{1}{2}}^{n}\cdot\delta_xH_{z}{}_{1,j+\frac{1}{2}}^{m}+E_{y}{}_{I-1,j+\frac{1}{2}}^{n}\cdot\delta_xH_{z}{}_{I-1,j+\frac{1}{2}}^{m}\}\Delta y.$$

This lemma is easy to be proved by using summation by parts and the boundary conditions (3.9). Now, we can derive two new discrete energy-conserved identities in the following theorems.

Theorem 3.1. *For* $k\geq0$, *let* $\mathbf{E}^{2k}=\{E_{x}{}_{i+\frac{1}{2},j}^{2k},E_{y}{}_{i,j+\frac{1}{2}}^{2k}\}$, $\{H_{z}{}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k}\}$ *be the solution of the symmetric EC-S-FDTD scheme* (3.1)-(3.10). *Then, the following energy-conserved identities hold*

$$\begin{aligned}&\|\delta_xE_x{}^{2k}\|^2_{\delta_xE_x}+\|\delta_xE_y{}^{2k}\|^2_{\delta_xE_y}+\|\delta_xH_z{}^{2k}\|^2_{\delta_xH_z}+|E_y{}^{2k}|^2_{E_{yi'}}\\=&\|\delta_xE_x{}^{0}\|^2_{\delta_xE_x}+\|\delta_xE_y{}^{0}\|^2_{\delta_xE_y}+\|\delta_xH_z{}^{0}\|^2_{\delta_xH_z}+|E_y{}^{0}|^2_{E_{yi'}},\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&\|\delta_y {E_x}^{2k}\|^2_{\delta_y E_x}+\|\delta_y {E_y}^{2k}\|^2_{\delta_y E_y}+\|\delta_y {H_z}^{2k}\|^2_{\delta_y H_z}+|{E_x}^{2k}|^2_{E_{xj'}}\\
=&\|\delta_y {E_x}^{0}\|^2_{\delta_y E_x}+\|\delta_y {E_y}^{0}\|^2_{\delta_y E_y}+\|\delta_y {H_z}^{0}\|^2_{\delta_y H_z}+|{E_x}^{0}|^2_{E_{xj'}}.
\end{aligned}\tag{3.14}
$$

Proof. Taking the difference δ_x to the equations in the symmetric EC-S-FDTD scheme leads to the corresponding scheme of δ_x-EC-S-FDTD as follows

δ_x-Stage 1:

$$\frac{\delta_x {E_y}^{2k+1}_{i+\frac12,j+\frac12}-\delta_x {E_y}^{2k}_{i+\frac12,j+\frac12}}{\Delta t}=-\frac{1}{2\varepsilon}\delta_x\delta_x\{{H_z}^{*}_{i+\frac12,j+\frac12}+{H_z}^{2k}_{i+\frac12,j+\frac12}\},\tag{3.15}$$

$$\frac{\delta_x {H_z}^{*}_{i,j+\frac12}-\delta_x {H_z}^{2k}_{i,j+\frac12}}{\Delta t}=-\frac{1}{2\mu}\delta_x\delta_x\{{E_y}^{2k+1}_{i,j+\frac12}+{E_y}^{2k}_{i,j+\frac12}\};\tag{3.16}$$

δ_x-Stage 2:

$$\frac{\delta_x {E_x}^{2k+1}_{i,j}-\delta_x {E_x}^{2k}_{i,j}}{\Delta t}=\frac{1}{2\varepsilon}\delta_y\delta_x\{{H_z}^{2k+1}_{i,j}+{H_z}^{*}_{i,j}\},\tag{3.17}$$

$$\frac{\delta_x {H_z}^{2k+1}_{i,j+\frac12}-\delta_x {H_z}^{*}_{i,j+\frac12}}{\Delta t}=\frac{1}{2\mu}\delta_y\delta_x\{{E_x}^{2k+1}_{i,j+\frac12}+{E_x}^{2k}_{i,j+\frac12}\};\tag{3.18}$$

δ_x-Stage 3:

$$\frac{\delta_x {E_x}^{2k+2}_{i,j}-\delta_x {E_x}^{2k+1}_{i,j}}{\Delta t}=\frac{1}{2\varepsilon}\delta_y\delta_x\{{H_z}^{**}_{i,j}+{H_z}^{2k+1}_{i,j}\},\tag{3.19}$$

$$\frac{\delta_x {H_z}^{**}_{i,j+\frac12}-\delta_x {H_z}^{2k+1}_{i,j+\frac12}}{\Delta t}=\frac{1}{2\mu}\delta_y\delta_x\{{E_x}^{2k+2}_{i,j+\frac12}+{E_x}^{2k+1}_{i,j+\frac12}\};\tag{3.20}$$

δ_x-Stage 4:

$$\frac{\delta_x {E_y}^{2k+2}_{i+\frac12,j+\frac12}-\delta_x {E_y}^{2k+1}_{i+\frac12,j+\frac12}}{\Delta t}=-\frac{1}{2\varepsilon}\delta_x\delta_x\{{H_z}^{2k+2}_{i+\frac12,j+\frac12}+{H_z}^{**}_{i+\frac12,j+\frac12}\},\tag{3.21}$$

$$\frac{\delta_x {H_z}^{2k+2}_{i,j+\frac12}-\delta_x {H_z}^{**}_{i,j+\frac12}}{\Delta t}=-\frac{1}{2\mu}\delta_x\delta_x\{{E_y}^{2k+2}_{i,j+\frac12}+{E_y}^{2k+1}_{i,j+\frac12}\}.\tag{3.22}$$

Here, the forms and the ranges of mesh points for the new fields $\delta_x {E_x}^m$, $\delta_x {E_y}^m$ $(m=2k,2k+1)$ and $\delta_x {H_z}^l$ $(l=2k,2k+1,*,**)$ in the δ_x-EC-S-FDTD scheme are changed. For example, $\{(x_{i+\frac12},y_j)|i=0,1,\cdots,I-1;j=1,2,\cdots,J-1\}$ for ${E_x}^{2k}$ is changed into $\{(x_i,y_j)|i=1,2,\cdots,I-1;j=1,2,\cdots,J-1\}$ for $\delta_x {E_x}^{2k}$.

Multiplying both sides of (3.15) and (3.16) by $\varepsilon\delta_x({E_y}^{2k+1}+{E_y}^{2k})\Delta x\Delta y$, $\mu\delta_x({H_z}^*+{H_z}^{2k})\Delta x\Delta y$, summing them over i,j in their valid ranges and using Lemma 3.1, we have

$$
\begin{aligned}
&\sum_{i=1}^{I-2}\sum_{j=0}^{J-1}\varepsilon\Big[(\delta_x {E_y}^{2k+1}_{i+\frac12,j+\frac12})^2-(\delta_x {E_x}^{2k}_{i+\frac12,j+\frac12})^2\Big]\Delta x\Delta y\\
&\quad+\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\mu\Big[(\delta_x {H_z}^{*}_{i,j+\frac12})^2-(\delta_x {H_z}^{2k}_{i,j+\frac12})^2\Big]\Delta x\Delta y\\
=&-\frac{\Delta t}{2\Delta x}\sum_{i'}\sum_{j=0}^{J-1}({E_y}^{2k+1}_{i',j+\frac12}+{E_y}^{2k}_{i',j+\frac12})(\delta_x {H_z}^{*}_{i',j+\frac12}+\delta_x {H_z}^{2k}_{i',j+\frac12})\Delta y.
\end{aligned}\tag{3.23}
$$

Noting that from (3.1) with $i=i'=1$, or $I-1$,

$$E_{y}{}^{2k+1}_{i',j+\frac{1}{2}}-E_{y}{}^{2k}_{i',j+\frac{1}{2}}=\frac{\Delta t}{2\varepsilon}\delta_x(H_{z}{}^{*}_{i',j+\frac{1}{2}}+H_{z}{}^{2k}_{i',j+\frac{1}{2}}), \tag{3.24}$$

multiplying both sides of this equation by $\varepsilon(E_{y}{}^{2k+1}_{i',j+\frac{1}{2}}+E_{y}{}^{2k}_{i',j+\frac{1}{2}})\frac{\Delta y}{\Delta x}$ and taking summation over $\Omega_{E_{yi'}}=\{(x_{i'},y_{j+\frac{1}{2}})|i'=1,I-1;j=0,1,\cdots,J-1\}$, we have

$$\begin{aligned}&-\Delta t\sum_{i'}\sum_{j=0}^{J-1}(E_{y}{}^{2k+1}_{i',j+\frac{1}{2}}+E_{y}{}^{2k}_{i',j+\frac{1}{2}})(\delta_x H_{z}{}^{*}_{i',j+\frac{1}{2}}+\delta_x H_{z}{}^{2k}_{i',j+\frac{1}{2}})\frac{\Delta y}{\Delta x}\\ =&2\sum_{i'}\sum_{j=0}^{J-1}\varepsilon\Big[(E_{y}{}^{2k}_{i',j+\frac{1}{2}})^2-(E_{y}{}^{2k+1}_{i',j+\frac{1}{2}})^2\Big]\frac{\Delta y}{\Delta x}.\end{aligned} \tag{3.25}$$

By the definitions of $|\cdot|_{E_{yi'}}$, $\|\cdot\|_{\delta_x E_y}$ and $\|\cdot\|_{\delta_x H_z}$ in Section 2 and the equation (3.25), the equation (3.23) becomes

$$\begin{aligned}&\|\delta_x E_y{}^{2k+1}\|^2_{\delta_x E_y}+\|\delta_x H_z{}^{*}\|^2_{\delta_x H_z}+|E_y{}^{2k+1}|^2_{E_{yi'}}\\ =&\|\delta_x E_y{}^{2k}\|^2_{\delta_x E_y}+\|\delta_x H_z{}^{2k}\|^2_{\delta_x H_z}+|E_y{}^{2k}|^2_{E_{yi'}}.\end{aligned} \tag{3.26}$$

Similarly, from the δ_x-Stage 2, δ_x-Stage 3 and δ_x-Stage 4, we get that

$$\|\delta_x E_x{}^{2k+1}\|^2_{\delta_x E_x}+\|\delta_x H_z{}^{2k+1}\|^2_{\delta_x H_z}=\|\delta_x E_x{}^{2k}\|^2_{\delta_x E_x}+\|\delta_x H_z{}^{*}\|^2_{\delta_x H_z}, \tag{3.27}$$

$$\|\delta_x E_x{}^{2k+2}\|^2_{\delta_x E_x}+\|\delta_x H_z{}^{**}\|^2_{\delta_x H_z}=\|\delta_x E_x{}^{2k+1}\|^2_{\delta_x E_x}+\|\delta_x H_z{}^{2k+1}\|^2_{\delta_x H_z}, \tag{3.28}$$

$$\begin{aligned}&\|\delta_x E_y{}^{2k+2}\|^2_{\delta_x E_y}+\|\delta_x H_z{}^{2k+2}\|^2_{\delta_x H_z}+|E_y{}^{2k+2}|^2_{E_{yi'}}\\ &\qquad=\|\delta_x E_y{}^{2k+1}\|^2_{\delta_x E_y}+\|\delta_x H_z{}^{**}\|^2_{\delta_x H_z}+|E_y{}^{2k+1}|^2_{E_{yi'}}.\end{aligned} \tag{3.29}$$

Combining the four relations (3.26)-(3.29), we have

$$\begin{aligned}&\|\delta_x E_x{}^{2k+2}\|^2_{\delta_x E_x}+\|\delta_x E_y{}^{2k+2}\|^2_{\delta_x E_y}+\|\delta_x H_z{}^{2k+2}\|^2_{\delta_x H_z}+|E_y{}^{2k+2}|^2_{E_{yi'}}\\ =&\|\delta_x E_x{}^{2k}\|^2_{\delta_x E_x}+\|\delta_x E_y{}^{2k}\|^2_{\delta_x E_y}+\|\delta_x H_z{}^{2k}\|^2_{\delta_x H_z}+|E_y{}^{2k}|^2_{E_{yi'}}.\end{aligned} \tag{3.30}$$

Summing up both sides of (3.30) over the time levels yields the discrete energy-conserved identity (3.13).

Similarly, taking the difference δ_y to the four stages of equations (3.1)-(3.8) and repeating all the processes above, we can obtain the identity (3.14). □

In the proof of Theorem 3.1, we can see that the summation by parts is independent of the time levels. Considering the operated scheme (3.1)-(3.8) by operators $\delta_w\delta_t$ ($w=x$ or y), we further obtain the following theorem

Theorem 3.2. *Let* $\mathbf{E}^{2k}=\{Ex^{2k}_{i+\frac{1}{2},j}, Ey^{2k}_{i,j+\frac{1}{2}}\}$ *and* $\{Hz^{2k}_{i+\frac{1}{2},j+\frac{1}{2}}\}$, $k\geq 0$, *be the solution of the symmetric EC-S-FDTD scheme* (3.1)-(3.8). *Then, the following energy-conserved identities hold*

$$\begin{aligned}
&\|\delta_t\delta_x E_x{}^{2k+1}\|^2_{\delta_x E_x}+\|\delta_t\delta_x E_y{}^{2k+1}\|^2_{\delta_x E_y}+\|\delta_t\delta_x H_z{}^{2k+1}\|^2_{\delta_x H_z}+|\delta_t E_y{}^{2k+1}|^2_{E_{yi'}}\\
=&\|\delta_t\delta_x E_x{}^{1}\|^2_{\delta_x E_x}+\|\delta_t\delta_x E_y{}^{1}\|^2_{\delta_x E_y}+\|\delta_t\delta_x H_z{}^{1}\|^2_{\delta_x H_z}+|\delta_t E_y{}^{1}|^2_{E_{yi'}}; \qquad (3.31)\\
&\|\delta_t\delta_y E_x{}^{2k+1}\|^2_{\delta_y E_x}+\|\delta_t\delta_y E_y{}^{2k+1}\|^2_{\delta_y E_y}+\|\delta_t\delta_y H_z{}^{2k+1}\|^2_{\delta_y H_z}+|\delta_t E_x{}^{2k+1}|^2_{E_{xj'}}\\
=&\|\delta_t\delta_y E_x{}^{1}\|^2_{\delta_y E_x}+\|\delta_t\delta_y E_y{}^{1}\|^2_{\delta_y E_y}+\|\delta_t\delta_y H_z{}^{1}\|^2_{\delta_y H_z}+|\delta_t E_x{}^{1}|^2_{E_{xj'}}. \qquad (3.32)
\end{aligned}$$

If regarding δ_x and $\delta_t\delta_x$ as I (the identity operator) and δ_t in Theorems 3.1 and 3.2, the following two discrete energy-conserved identities are obtained (which were proved by Chen, Li and Liang in [1]).

Remark 3.1. Let $\mathbf{E}^{2k}=\{Ex^{2k}_{i+\frac{1}{2},j}, Ey^{2k}_{i,j+\frac{1}{2}}\}$ and $\{Hz^{2k}_{i+\frac{1}{2},j+\frac{1}{2}}\}$ be the same as in Theorems 3.1 and 3.2. Then, for $k\geq 0$, the symmetric EC-S-FDTD scheme satisfies

$$\|E_x{}^{2k}\|^2_{E_x}+\|E_y{}^{2k}\|^2_{E_y}+\|H_z{}^{2k}\|^2_{H_z}=\|E_x{}^{0}\|^2_{E_x}+\|E_y{}^{0}\|^2_{E_y}+\|H_z{}^{0}\|^2_{H_z}, \qquad (3.33)$$

$$\|\delta_t E_x{}^{2k-1}\|^2_{E_x}+\|\delta_t E_y{}^{2k-1}\|^2_{E_y}+\|\delta_t H_z{}^{2k-1}\|^2_{H_z}=\|\delta_t E_x{}^{1}\|^2_{E_x}+\|\delta_t E_y{}^{1}\|^2_{E_y}+\|\delta_t H_z{}^{1}\|^2_{H_z}. \qquad (3.34)$$

Finally, let further define the discrete H^1-norms as follows

$$\begin{aligned}
\|E_x{}^m\|_1^2&=\|E_x{}^m\|^2_{E_x}+\|\delta_x E_x{}^m\|^2_{\delta_x E_x}+\|\delta_y E_x{}^m\|^2_{\delta_y E_x}+|E_y{}^m|^2_{E_{yi'}},\\
\|E_y{}^m\|_1^2&=\|E_y{}^m\|^2_{E_y}+\|\delta_x E_y{}^m\|^2_{\delta_x E_y}+\|\delta_y E_y{}^m\|^2_{\delta_y E_y}+|E_x{}^m|^2_{E_{xj'}},\\
\|H_z{}^m\|_1^2&=\|H_z{}^m\|^2_{H_z}+\|\delta_x H_z{}^m\|^2_{\delta_x H_z}+\|\delta_y H_z{}^m\|^2_{\delta_y H_z},
\end{aligned}$$

for $m=2k,2k+1$, or 0. Combining the identities in Theorems 3.1 and 3.2 and Remark 3.1, we obtain the forms of the energy-conserved identities in the discrete H^1-norm, as shown in the following theorem.

Theorem 3.3. *The solution of the symmetric EC-S-FDTD scheme* (3.1)-(3.10), $\mathbf{E}^m=\{Ex^{m}_{i+\frac{1}{2},j}, Ey^{m}_{i,j+\frac{1}{2}}\}$ *and* $\{Hz^{m}_{i+\frac{1}{2},j+\frac{1}{2}}\}$ *with* $m=2k,2k+1,k\geq 0$, *satisfies*

$$\|E_x{}^{2k}\|_1^2+\|E_y{}^{2k}\|_1^2+\|H_z{}^{2k}\|_1^2=\|E_x{}^{0}\|_1^2+\|E_y{}^{0}\|_1^2+\|H_z{}^{0}\|_1^2, \qquad (3.35)$$

$$\|\delta_t E_x{}^{2k+1}\|_1^2+\|\delta_t E_y{}^{2k+1}\|_1^2+\|\delta_t H_z{}^{2k+1}\|_1^2=\|\delta_t E_x{}^{1}\|_1^2+\|\delta_t E_y{}^{1}\|_1^2+\|\delta_t H_z{}^{1}\|_1^2. \qquad (3.36)$$

Thus, the solution of the symmetric EC-S-FDTD scheme and its δ_t*-difference are energy conserved and unconditionally stable in the discrete* H^1*-norm.*

4 Super-convergence Analysis in the discrete H^1-norm

In this section, we consider the error estimates and super-convergence analysis by using the new energy-conserved identities. Then we give a simple estimate of the divergence of the electric field of the symmetric EC-S-FDTD scheme. First, we derive the error equations and truncation errors of the operated symmetric EC-S-FDTD schemes by δ_x by the method used in [1].

From the equations (3.15)-(3.22) in the δ_x-EC-S-FDTD, we have the following expressions

$$\begin{aligned}\delta_x {H_z}^{2k+1}|_{i,j+\frac{1}{2}} =& \frac{1}{2}\delta_x({H_z}^{2k+2}+{H_z}^{2k})|_{i,j+\frac{1}{2}} \\ &-\frac{\Delta t}{4\mu}\delta_x\{\delta_y({E_x}^{2k+2}-{E_x}^{2k})-\delta_x({E_y}^{2k+2}-{E_y}^{2k})\}|_{i,j+\frac{1}{2}},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\delta_x {H_z}^{*}|_{i,j+\frac{1}{2}} =& \frac{1}{2}\delta_x({H_z}^{2k+1}+{H_z}^{2k})|_{i,j+\frac{1}{2}} \\ &-\frac{\Delta t}{4\mu}\delta_x\{\delta_y({E_x}^{2k+1}+{E_x}^{2k})+\delta_x({E_y}^{2k+1}+{E_y}^{2k})\}|_{i,j+\frac{1}{2}},\end{aligned} \tag{4.2}$$

$$\begin{aligned}\delta_x {H_z}^{**}|_{i,j+\frac{1}{2}} =& \frac{1}{2}\delta_x({H_z}^{2k+2}+{H_z}^{2k+1})|_{i,j+\frac{1}{2}} \\ &+\frac{\Delta t}{4\mu}\delta_x\{\delta_y({E_x}^{2k+2}+{E_x}^{2k+1})+\delta_x({E_y}^{2k+2}+{E_y}^{2k+1})\}|_{i,j+\frac{1}{2}}.\end{aligned} \tag{4.3}$$

Combining (3.15)-(3.22) and using (4.1)-(4.3), we derive the equivalent form of the δ_x-EC-S-FDTD which is

$$\begin{aligned}\frac{\delta_x {E_x}^{2k+2}-\delta_x {E_x}^{2k}}{\Delta t} =& \frac{1}{\varepsilon}\delta_y\delta_x\{{H_z}^{2k+2}+{H_z}^{2k}\}+\frac{\Delta t}{2\mu\varepsilon}\delta_x\delta_y\delta_x({E_y}^{2k+2}-{E_y}^{2k}) \\ &+\frac{\Delta t}{4\mu\varepsilon}\delta_x\delta_y\delta_y({E_x}^{2k+2}-{E_x}^{2k}),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\frac{\delta_x {E_y}^{2k+2}-\delta_x {E_y}^{2k}}{\Delta t} =& -\frac{1}{\varepsilon}\delta_x\delta_x\{{H_z}^{2k+2}+{H_z}^{2k}\} \\ &-\frac{\Delta t}{4\mu\varepsilon}\delta_x\delta_x\delta_x({E_y}^{2k+2}-{E_y}^{2k}),\end{aligned} \tag{4.5}$$

$$\begin{aligned}\frac{\delta_x {H_z}^{2k+2}-\delta_x {H_z}^{2k}}{\Delta t} =& \frac{1}{2\mu}\delta_x\{\delta_y({E_x}^{2k+2}+2{E_x}^{2k+1}+{E_x}^{2k}) \\ &-\delta_x({E_y}^{2k+2}+2{E_y}^{2k+1}+{E_y}^{2k})\}.\end{aligned} \tag{4.6}$$

The corresponding truncation errors of the three equations in (4.4)-(4.6) are:

$$\begin{aligned}\delta_x \xi_{x\,i,j}^{2k+1} =& \frac{\delta_x {e_x}^{2k+2}-\delta_x {e_x}^{2k}}{\Delta t}|_{i,j}-\frac{1}{\varepsilon}\delta_y\delta_x\{{h_z}^{2k+2}+{h_z}^{2k}\}|_{i,j} \\ &-\frac{\Delta t}{2\mu\varepsilon}\delta_x\delta_y\{\delta_x({e_y}^{2k+2}-{e_y}^{2k})+\frac{1}{2}\delta_y({e_x}^{2k+2}-{e_x}^{2k})\}|_{i,j},\end{aligned} \tag{4.7}$$

$$\delta_x \xi_{y}{}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{\delta_x({e_y}^{2k+1}-{e_y}^{2k})}{\Delta t}\Big|_{i+\frac{1}{2},j+\frac{1}{2}} + \frac{1}{\varepsilon}\delta_x\delta_x\{{h_z}^{2k+2}+{h_z}^{2k}\}\Big|_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$+\frac{\Delta t}{4\mu\varepsilon}\delta_x\delta_x\delta_x({e_y}^{2k+2}-{e_y}^{2k})\Big|_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{4.8}$$

$$\delta_x \eta_{z}{}^{2k+1}_{i,j+\frac{1}{2}} = \frac{\delta_x({h_z}^{2k+2}-{h_z}^{2k})}{\Delta t}\Big|_{i,j+\frac{1}{2}} - \frac{1}{2\mu}\delta_x\{\delta_y\tilde{\delta}_t {e_x}^{2k+1} - \delta_x\tilde{\delta}_t {e_y}^{2k+1}\}\Big|_{i,j+\frac{1}{2}}, \tag{4.9}$$

where e_x,e_y and h_z denote the exact solution components of the Maxwell's equations, and $\tilde{\delta}_t f^{2k+1} = f^{2k+2}+2f^{2k+1}+f^{2k}$ for $f=e_x,e_y$. If e_x,e_y and h_z are smooth enough, then, by the Taylor's theorem, the truncation errors are bounded by

$$\|\delta_x {\xi_x}^{2k+1}\|_{\delta_x E_x} + \|\delta_x {\xi_y}^{2k+1}\|_{\delta_x E_y} + \|\delta_x {\eta_z}^{2k+1}\|_{\delta_x H_z} \le C\{\Delta t^2+\Delta x^2+\Delta y^2\}. \tag{4.10}$$

In order to get the system of the error equations, we define some intermediate variables as follows.

$$\delta_x {e_x}^{2k+1}|_{i,j} = \frac{1}{2}\delta_x\Big({e_x}^{2k+2}+{e_x}^{2k} - \frac{\Delta t}{4\varepsilon}\delta_y({h_z}^{2k+2}-{h_z}^{2k})\Big)\Big|_{i,j}$$
$$-\frac{\Delta t^2}{16\mu\varepsilon}\delta_y\delta_x\{\delta_x\tilde{\delta}_t {e_y}^{2k+1}+\delta_y\tilde{\delta}_t {e_x}^{2k+1}\}\Big|_{i,j}, \tag{4.11}$$

$$\delta_x {e_y}^{2k+1}|_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{2}\delta_x\Big({e_y}^{2k+2}+{e_y}^{2k} + \frac{3\Delta t}{4\varepsilon}\delta_x({h_z}^{2k+2}-{h_z}^{2k})\Big)\Big|_{i+\frac{1}{2},j+\frac{1}{2}}$$
$$+\frac{\Delta t^2}{16\mu\varepsilon}\delta_x\delta_x\{\delta_x\tilde{\delta}_t {e_y}^{2k+1}+\delta_y\tilde{\delta}_t {e_x}^{2k+1}\}\Big|_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{4.12}$$

$$\delta_x {h_z}^{2k+1}|_{i,j+\frac{1}{2}} = -\frac{\Delta t}{4\mu}\delta_x\{\delta_y({e_x}^{2k+2}-{e_x}^{2k}) - \delta_x({e_y}^{2k+2}-{e_y}^{2k})\}\Big|_{i,j+\frac{1}{2}}$$
$$+\frac{1}{2}\delta_x({h_z}^{2k+2}+{h_z}^{2k})\Big|_{i,j+\frac{1}{2}}, \tag{4.13}$$

$$\delta_x {h_z}^{*}|_{i,j+\frac{1}{2}} = -\frac{\Delta t}{4\mu}\delta_x\{\delta_y({e_x}^{2k+1}+{e_x}^{2k}) + \delta_x({e_y}^{2k+1}+{e_y}^{2k})\}\Big|_{i,j+\frac{1}{2}}$$
$$+\frac{1}{2}\delta_x({h_z}^{2k+1}+{h_z}^{2k})\Big|_{i,j+\frac{1}{2}}, \tag{4.14}$$

$$\delta_x {h_z}^{**}|_{i,j+\frac{1}{2}} = \frac{\Delta t}{4\mu}\delta_x\{\delta_y({e_x}^{2k+2}+{e_x}^{2k+1}) + \delta_x({e_y}^{2k+2}+{e_y}^{2k+1})\}\Big|_{i,j+\frac{1}{2}}$$
$$+\frac{1}{2}\delta_x({h_z}^{2k+2}+{h_z}^{2k+1})\Big|_{i,j+\frac{1}{2}}, \tag{4.15}$$

where $\tilde{\delta}_t f^{2k+1}$, with $f=e_x,e_y$, are the same as those in (4.9).

By using these five intermediate variables in (4.11)-(4.15) and careful calculation we derive that

δ_x-Stage 1 for exact solution:

$$\frac{\delta_x({e_y}^{2k+1}-{e_y}^{2k})}{\Delta t}|_{i+\frac{1}{2},j+\frac{1}{2}}=-\frac{1}{2\varepsilon}\delta_x\delta_x\{{h_z}^{*}+{h_z}^{2k}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}\delta_x{\xi_y}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{4.16}$$

$$\frac{\delta_x({h_z}^{*}-{h_z}^{2k})}{\Delta t}|_{i,j+\frac{1}{2}}=-\frac{1}{2\mu}\delta_x\delta_x\{{e_y}^{2k+1}+{e_y}^{2k}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}^{2k+1}_{i,j+\frac{1}{2}}; \tag{4.17}$$

δ_x-Stage 2 for exact solution:

$$\frac{\delta_x({e_x}^{2k+1}-{e_x}^{2k})}{\Delta t}|_{i,j}=\frac{1}{2\varepsilon}\delta_y\delta_x\{{h_z}^{2k+1}+{h_z}^{*}\}_{i+\frac{1}{2},j}+\frac{1}{2}\delta_x{\xi_x}^{2k+1}_{i,j}, \tag{4.18}$$

$$\frac{\delta_x({h_z}^{2k+1}-{h_z}^{*})}{\Delta t}|_{i,j+\frac{1}{2}}=\frac{1}{2\mu}\delta_y\delta_x\{{e_x}^{2k+1}+{e_x}^{2k}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}^{2k+1}_{i,j+\frac{1}{2}}; \tag{4.19}$$

δ_x-Stage 3 for exact solution:

$$\frac{\delta_x({e_x}^{2k+2}-{e_x}^{2k+1})}{\Delta t}|_{i,j}=\frac{1}{2\varepsilon}\delta_y\delta_x\{{h_z}^{**}+{h_z}^{2k+1}\}_{i,j}+\frac{1}{2}\delta_x{\xi_x}^{2k+1}_{i,j}, \tag{4.20}$$

$$\frac{\delta_x({h_z}^{**}-{h_z}^{2k+1})}{\Delta t}|_{i,j+\frac{1}{2}}=\frac{1}{2\mu}\delta_y\delta_x\{{e_x}^{2k+2}+{e_x}^{2k+1}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}^{2k+1}_{i,j+\frac{1}{2}}; \tag{4.21}$$

δ_x-Stage 4 for exact solution:

$$\frac{\delta_x({e_y}^{2k+2}-{e_y}^{2k+1})}{\Delta t}|_{i+\frac{1}{2},j+\frac{1}{2}}=-\frac{1}{2\varepsilon}\delta_x\delta_x\{{h_z}^{2k+2}+{h_z}^{**}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}\delta_x{\xi_y}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{4.22}$$

$$\frac{\delta_x({h_z}^{2k+2}-{h_z}^{**})}{\Delta t}|_{i,j+\frac{1}{2}}=-\frac{1}{2\mu}\delta_x\delta_x\{{e_y}^{2k+2}+{e_y}^{2k+1}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}^{2k+1}_{i,j+\frac{1}{2}}. \tag{4.23}$$

Define the errors of the fields appeared in the operated scheme δ_x-EC-S-FDTD as

$$\delta_x{\mathcal{E}_x}^m=\delta_x{e_x}^m-\delta_x{E_x}^m,\quad \delta_x{\mathcal{E}_y}^m=\delta_x{e_y}^m-\delta_x{E_y}^m,\quad \delta_x{\mathcal{H}_z}^m=\delta_x{h_z}^m-\delta_x{H_z}^m, \tag{4.24a}$$

$$\delta_x{\mathcal{H}_z}^{**}=\delta_x{h_z}^{**}-\delta_x{H_z}^{**},\quad \delta_x{\mathcal{H}_z}^{*}=\delta_x{h_z}^{*}-\delta_x{H_z}^{*},\quad m=2k,2k+1. \tag{4.24b}$$

Then, subtracting the equations in the scheme (3.15)-(3.22) from those in (4.16)-(4.23), we have the system of the error equations for δ_x-EC-S-FDTD:

δ_x-Stage 1 of error equations:

$$\frac{\delta_x({\mathcal{E}_y}^{2k+1}-{\mathcal{E}_y}^{2k})}{\Delta t}|_{i+\frac{1}{2},j+\frac{1}{2}}=-\frac{1}{2\varepsilon}\delta_x\delta_x\{{\mathcal{H}_z}^{*}+{\mathcal{H}_z}^{2k}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}\delta_x{\xi_y}^{2k+1}_{i+\frac{1}{2},j+\frac{1}{2}}, \tag{4.25}$$

$$\frac{\delta_x({\mathcal{H}_z}^{*}-{\mathcal{H}_z}^{2k})}{\Delta t}|_{i,j+\frac{1}{2}}=-\frac{1}{2\mu}\delta_x\delta_x\{{\mathcal{E}_y}^{2k+1}+{\mathcal{E}_y}^{2k}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}^{2k+1}_{i,j+\frac{1}{2}}; \tag{4.26}$$

δ_x-Stage 2 of error equations:

$$\frac{\delta_x({\mathcal{E}_x}^{2k+1}-{\mathcal{E}_x}^{2k})}{\Delta t}|_{i,j}=\frac{1}{2\varepsilon}\delta_y\delta_x\{{\mathcal{H}_z}^{2k+1}+{\mathcal{H}_z}^{*}\}_{i+\frac{1}{2},j}+\frac{1}{2}\delta_x{\xi_x}_{i,j}^{2k+1}, \tag{4.27}$$

$$\frac{\delta_x({\mathcal{H}_z}^{2k+1}-{\mathcal{H}_z}^{*})}{\Delta t}|_{i,j+\frac{1}{2}}=\frac{1}{2\mu}\delta_y\delta_x\{{\mathcal{E}_x}^{2k+1}+{\mathcal{E}_x}^{2k}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}_{i,j+\frac{1}{2}}^{2k+1}; \tag{4.28}$$

δ_x-Stage 3 of error equations:

$$\frac{\delta_x({\mathcal{E}_x}^{2k+2}-{\mathcal{E}_x}^{2k+1})}{\Delta t}|_{i,j}=\frac{1}{2\varepsilon}\delta_y\delta_x\{{\mathcal{H}_z}^{**}+{\mathcal{H}_z}^{2k+1}\}_{i,j}+\frac{1}{2}\delta_x{\xi_x}_{i,j}^{2k+1}, \tag{4.29}$$

$$\frac{\delta_x({\mathcal{H}_z}^{**}-{\mathcal{H}_z}^{2k+1})}{\Delta t}|_{i,j+\frac{1}{2}}=\frac{1}{2\mu}\delta_y\delta_x\{{\mathcal{E}_x}^{2k+2}+{\mathcal{E}_x}^{2k+1}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}_{i,j+\frac{1}{2}}^{2k+1}; \tag{4.30}$$

δ_x-Stage 4 of error equations:

$$\frac{\delta_x({\mathcal{E}_y}^{2k+2}-{\mathcal{E}_y}^{2k+1})}{\Delta t}|_{i+\frac{1}{2},j+\frac{1}{2}}=-\frac{1}{2\varepsilon}\delta_x\delta_x\{{\mathcal{H}_z}^{2k+2}+{\mathcal{H}_z}^{**}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}\delta_x{\xi_y}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}, \tag{4.31}$$

$$\frac{\delta_x({\mathcal{H}_z}^{2k+2}-{\mathcal{H}_z}^{**})}{\Delta t}|_{i,j+\frac{1}{2}}=-\frac{1}{2\mu}\delta_x\delta_x\{{\mathcal{E}_y}^{2k+2}+{\mathcal{E}_y}^{2k+1}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\delta_x{\eta_z}_{i,j+\frac{1}{2}}^{2k+1}. \tag{4.32}$$

Besides the error equations above, we also need the error equations of the original scheme, symmetric EC-S-FDTD (3.1)-(3.8). Using the same technique of deriving (4.25)-(4.32) (or see [1] for the details), we can obtain the error equations of the symmetric EC-S-FDTD scheme:

Stage 1 of error equations of Symmetric EC-S-FDTD:

$$\frac{{\mathcal{E}_y}^{2k+1}-{\mathcal{E}_y}^{2k}}{\Delta t}|_{i,j+\frac{1}{2}}=-\frac{1}{2\varepsilon}\delta_x\{{\mathcal{H}_z}^{*}+{\mathcal{H}_z}^{2k}\}_{i,j+\frac{1}{2}}+\frac{1}{2}{\xi_y}_{i,j+\frac{1}{2}}^{2k+1}, \tag{4.33}$$

$$\frac{{\mathcal{H}_z}^{*}-{\mathcal{H}_z}^{2k}}{\Delta t}|_{i+\frac{1}{2},j+\frac{1}{2}}=-\frac{1}{2\mu}\delta_x\{{\mathcal{E}_y}^{2k+1}+{\mathcal{E}_y}^{2k}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}{\eta_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}; \tag{4.34}$$

Stage 2 of error equations of Symmetric EC-S-FDTD:

$$\frac{{\mathcal{E}_x}^{2k+1}-{\mathcal{E}_x}^{2k}}{\Delta t}|_{i+\frac{1}{2},j}=\frac{1}{2\varepsilon}\delta_y\{{\mathcal{H}_z}^{2k+1}+{\mathcal{H}_z}^{*}\}_{i+\frac{1}{2},j}+\frac{1}{2}{\xi_x}_{i+\frac{1}{2},j}^{2k+1}, \tag{4.35}$$

$$\frac{{\mathcal{H}_z}^{2k+1}-{\mathcal{H}_z}^{*}}{\Delta t}|_{i+\frac{1}{2},j+\frac{1}{2}}=\frac{1}{2\mu}\delta_y\{{\mathcal{E}_x}^{2k+1}+{\mathcal{E}_x}^{2k}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}{\eta_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}; \tag{4.36}$$

Stage 3 of error equations of Symmetric EC-S-FDTD:

$$\frac{{\mathcal{E}_x}^{2k+2}-{\mathcal{E}_x}^{2k+1}}{\Delta t}|_{i+\frac{1}{2},j}=\frac{1}{2\varepsilon}\delta_y\{{\mathcal{H}_z}^{**}+{\mathcal{H}_z}^{2k+1}\}_{i+\frac{1}{2},j}+\frac{1}{2}{\xi_x}_{i+\frac{1}{2},j}^{2k+1}, \tag{4.37}$$

$$\frac{{\mathcal{H}_z}^{**}-{\mathcal{H}_z}^{2k+1}}{\Delta t}|_{i+\frac{1}{2},j+\frac{1}{2}}=\frac{1}{2\mu}\delta_y\{{\mathcal{E}_x}^{2k+2}+{\mathcal{E}_x}^{2k+1}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}{\eta_z}_{i+\frac{1}{2},j+\frac{1}{2}}^{2k+1}; \tag{4.38}$$

Stage 4 of error equations of Symmetric EC-S-FDTD:

$$\frac{\mathcal{E}_y^{2k+2}-\mathcal{E}_y^{2k+1}}{\Delta t}\Big|_{i,j+\frac{1}{2}}=-\frac{1}{2\varepsilon}\delta_x\{\mathcal{H}_z^{2k+2}+\mathcal{H}_z^{**}\}_{i,j+\frac{1}{2}}+\frac{1}{2}\xi_{y\,i,j+\frac{1}{2}}^{\,2k+1}, \tag{4.39}$$

$$\frac{\mathcal{H}_z^{2k+2}-\mathcal{H}_z^{**}}{\Delta t}\Big|_{i+\frac{1}{2},j+\frac{1}{2}}=-\frac{1}{2\mu}\delta_x\{\mathcal{E}_y^{2k+2}+\mathcal{E}_y^{2k+1}\}_{i+\frac{1}{2},j+\frac{1}{2}}+\frac{1}{2}\eta_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{\,2k+1}; \tag{4.40}$$

where $\xi_x^{2k+\frac{1}{2}}$, $\xi_y^{2k+\frac{1}{2}}$ and $\eta_z^{2k+\frac{1}{2}}$ are defined as

$$\begin{aligned}\xi_{x\,i+\frac{1}{2},j}^{\,2k+1}=&\frac{e_x^{2k+2}-e_x^{2k}}{\Delta t}\Big|_{i+\frac{1}{2},j}-\frac{1}{\varepsilon}\delta_y\{h_z^{2k+2}+h_z^{2k}\}\big|_{i+\frac{1}{2},j}\\&-\frac{\Delta t}{2\mu\varepsilon}\delta_y\{\delta_x(e_y^{2k+2}-e_y^{2k})+\frac{1}{2}\delta_y(e_x^{2k+2}-e_x^{2k})\}\big|_{i+\frac{1}{2},j},\end{aligned} \tag{4.41}$$

$$\begin{aligned}\xi_{y\,i,j+\frac{1}{2}}^{\,2k+1}=&\frac{e_y^{2k+1}-e_y^{2k}}{\Delta t}\Big|_{i,j+\frac{1}{2}}+\frac{1}{\varepsilon}\delta_x\{h_z^{2k+2}+h_z^{2k}\}\big|_{i,j+\frac{1}{2}}\\&+\frac{\Delta t}{4\mu\varepsilon}\delta_x\delta_x(e_y^{2k+2}-e_y^{2k})\big|_{i,j+\frac{1}{2}},\end{aligned} \tag{4.42}$$

$$\eta_{z\,i+\frac{1}{2},j+\frac{1}{2}}^{\,2k+1}=\frac{h_z^{2k+2}-h_z^{2k}}{\Delta t}\Big|_{i+\frac{1}{2},j+\frac{1}{2}}-\frac{1}{2\mu}\{\delta_y\tilde{\delta}_t e_x^{2k+1}-\delta_x\tilde{\delta}_t e_y^{2k+1}\}\big|_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{4.43}$$

The terms of ξ_x^{2k+1}, ξ_y^{2k+1} and η_z^{2k+1} are the truncation errors of the following equivalent form of the symmetric EC-S-FDTD scheme (which can be obtained by the same technique of deriving the equivalent form of the δ_x-EC-S-FDTD, or see [1] for the derivation):

$$\begin{aligned}\frac{E_x^{2k+2}-E_x^{2k}}{\Delta t}\Big|_{i+\frac{1}{2},j}=&\frac{1}{\varepsilon}\delta_y\{(H_z^{2k+2}+H_z^{2k})+\frac{\Delta t}{2\mu}\delta_x(E_y^{2k+2}-E_y^{2k})\}\big|_{i+\frac{1}{2},j}\\&+\frac{\Delta t}{4\mu\varepsilon}\delta_y\delta_y(E_x^{2k+2}-E_x^{2k})\big|_{i+\frac{1}{2},j},\end{aligned} \tag{4.44}$$

$$\frac{E_y^{2k+2}-E_y^{2k}}{\Delta t}\Big|_{i,j+\frac{1}{2}}=-\frac{1}{\varepsilon}\delta_x\{(H_z^{2k+2}+H_z^{2k})+\frac{\Delta t}{4\mu}\delta_x(E_y^{2k+2}-E_y^{2k})\})\big|_{i,j+\frac{1}{2}}, \tag{4.45}$$

$$\frac{H_z^{2k+2}-H_z^{2k}}{\Delta t}\Big|_{i+\frac{1}{2},j+\frac{1}{2}}=\frac{1}{2\mu}\{\delta_y\tilde{\delta}_t E_x^{2k+1}-\delta_x\tilde{\delta}_t E_y^{2k+1}\}\big|_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{4.46}$$

From the expressions of ξ_x^{2k+1}, ξ_y^{2k+1} and η_z^{2k+1}, we can obtain that

$$\|\xi_x^{2k+1}\|_{E_x}+\|\xi_y^{2k+1}\|_{E_y}+\|\eta_z^{2k+1}\|_{H_z}\le C\{\Delta t^2+\Delta x^2+\Delta y^2\}. \tag{4.47}$$

Similar to the definitions of the error in (4.24), the error functions in (4.33)-(4.40) are defined as

$$\mathcal{E}_x^m=e_x^m-E_x^m,\quad \mathcal{E}_y^m=e_y^m-E_y^m,\quad \mathcal{H}_z^m=h_z^m-H_z^m, \tag{4.48a}$$

$$\mathcal{H}_z^*=h_z^*-H_z^*,\quad \mathcal{H}_z^{**}=h_z^{**}-H_z^{**},\quad m=2k,2k+1, \tag{4.48b}$$

where the intermediate variables ${h_z}^*$ and ${h_z}^{**}$ are defined as

$$\begin{aligned} {h_z}^*|_{i+\frac{1}{2},j+\frac{1}{2}} =& -\frac{\Delta t}{4\mu}\{\delta_y({e_x}^{2k+1}+{e_x}^{2k})+\delta_x({e_y}^{2k+1}+{e_y}^{2k})\}|_{i+\frac{1}{2},j+\frac{1}{2}} \\ &+\frac{1}{2}({h_z}^{2k+1}+{h_z}^{2k})|_{i+\frac{1}{2},j+\frac{1}{2}}, \end{aligned} \tag{4.49}$$

$$\begin{aligned} {h_z}^{**}|_{i+\frac{1}{2},j+\frac{1}{2}} =& \frac{\Delta t}{4\mu}\{\delta_y({e_x}^{2k+2}+{e_x}^{2k+1})+\delta_x({e_y}^{2k+2}+{e_y}^{2k+1})\}|_{i+\frac{1}{2},j+\frac{1}{2}} \\ &+\frac{1}{2}({h_z}^{2k+2}+{h_z}^{2k+1})|_{i+\frac{1}{2},j+\frac{1}{2}}. \end{aligned} \tag{4.50}$$

Using these error equations, we can now prove the following super-convergence in Theorem 4.1.

Theorem 4.1. *Assume that $\mathbf{e}=(e_x,e_y)$ and h_z, the solution components of the Maxwell's equations, are smooth enough such as $\mathbf{e}\in C^4([0,T];[C^4(\bar{\Omega})]^2)$ and $h_z\in C^4([0,T];C^4(\bar{\Omega}))$. Let ${E_x}^{2k}_{i+\frac{1}{2},j}$, ${E_y}^{2k}_{i,j+\frac{1}{2}}$ and ${H_z}^{2k}_{i+\frac{1}{2},j+\frac{1}{2}}$ be the solution components of the symmetric EC-S-FDTD scheme, and let*

$$\mathcal{E}_x{}^{2k}={e_x}^{2k}-{E_x}^{2k},\quad \mathcal{E}_y{}^{2k}={e_y}^{2k}-{E_y}^{2k},\quad \mathcal{H}_z{}^{2k}={h_z}^{2k}-{H_z}^{2k}$$

be the errors. Then, for $k\ge 0$, we have the following error estimates:

$$\begin{aligned} &\|\delta_x\mathcal{E}_x{}^{2k}\|_{\delta_x E_x}+\|\delta_x\mathcal{E}_y{}^{2k}\|_{\delta_x E_y}+\|\delta_x\mathcal{H}_z{}^{2k}\|_{\delta_x H_z}+|\mathcal{E}_y{}^{2k}|_{E_{yi'}} \\ \le& C\{\|\delta_x\mathcal{E}_x{}^{0}\|_{\delta_x E_x}+\|\delta_x\mathcal{E}_y{}^{0}\|_{\delta_x E_y}+\|\delta_x\mathcal{H}_z{}^{0}\|_{\delta_x H_z}+|\mathcal{E}_y{}^{0}|_{E_{yi'}}+\Delta t^2+\Delta x^2+\Delta y^2\}, \end{aligned} \tag{4.51}$$

$$\begin{aligned} &\|\delta_y\mathcal{E}_x{}^{2k}\|_{\delta_y E_x}+\|\delta_y\mathcal{E}_y{}^{2k}\|_{\delta_y E_y}+\|\delta_y\mathcal{H}_z{}^{2k}\|_{\delta_y H_z}+|\mathcal{E}_x{}^{2k}|_{E_{xj'}} \\ \le& C\{\|\delta_y\mathcal{E}_x{}^{0}\|_{\delta_y E_x}+\|\delta_y\mathcal{E}_y{}^{0}\|_{\delta_y E_y}+\|\delta_y\mathcal{H}_z{}^{0}\|_{\delta_y H_z}+|\mathcal{E}_x{}^{0}|_{E_{xj'}}+\Delta t^2+\Delta x^2+\Delta y^2\}. \end{aligned} \tag{4.52}$$

Proof. First, we prove (4.51). Similarly to the derivation of the energy identities, from the error equations (4.25) and (4.26) and using Lemma 3.1, we have

$$\begin{aligned} &\|\delta_x\mathcal{E}_y{}^{2k+1}\|^2_{\delta_x E_y}-\|\delta_x\mathcal{E}_y{}^{2k}\|^2_{\delta_x E_y}+\|\delta_x\mathcal{H}_z{}^{*}\|^2_{\delta_x H_z}-\|\delta_x\mathcal{H}_z{}^{2k}\|^2_{\delta_x H_z} \\ =&\frac{\Delta t}{2\Delta x}\sum_{i'}\sum_{j=0}^{J-1}(\mathcal{E}_y{}^{2k+1}+\mathcal{E}_y{}^{2k})\delta_x(\mathcal{H}_z{}^*+\mathcal{H}_z{}^{2k})_{i',j+\frac{1}{2}}\Delta y \\ &+\sum_{i=1}^{I-2}\sum_{j=1}^{J-1}\frac{\varepsilon}{2}\delta_x\xi_y{}^{2k+1}\delta_x(\mathcal{E}_y{}^{2k+1}+\mathcal{E}_y{}^{2k})\Delta x\Delta y\Delta t \\ &+\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\frac{\mu}{2}\delta_x\eta_z{}^{2k+1}\delta_x(\mathcal{H}_z{}^*+\mathcal{H}_z{}^{2k})\Delta x\Delta y\Delta t. \end{aligned} \tag{4.53}$$

From (4.33) with $i=i'=1,I-1$ and $j=0,1,\cdots,J-1$, we get that

$$\begin{aligned}
&-\frac{\Delta t}{2}({\mathcal{E}_y}^{2k+1}+{\mathcal{E}_y}^{2k})\delta_x({\mathcal{H}_z}^{*}+{\mathcal{H}_z}^{2k})|_{i',j+\frac{1}{2}}\\
=&\varepsilon({\mathcal{E}_y}^{2k+1}_{i',j+\frac{1}{2}})^2-\varepsilon({\mathcal{E}_y}^{2k}_{i',j+\frac{1}{2}})^2-\varepsilon\frac{\Delta t}{2}{\xi_y}^{2k+1}_{i',j+\frac{1}{2}}({\mathcal{E}_y}^{2k+1}+{\mathcal{E}_y}^{2k})_{i',j+\frac{1}{2}}.
\end{aligned} \tag{4.54}$$

Combining (4.53) and (4.54) and using the estimates of the truncation errors in (4.10) and (4.47), we have

$$\begin{aligned}
&\Big(1-\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_y}^{2k+1}\|^2_{\delta_x E_y}+\|\delta_x{\mathcal{H}_z}^{*}\|^2_{\delta_x H_z}+|{\mathcal{E}_y}^{2k+1}|^2_{E_{yi'}}\}\\
\le&\Big(1+\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_y}^{2k}\|^2_{\delta_x E_y}+\|\delta_x{\mathcal{H}_z}^{2k}\|^2_{\delta_x H_z}+|{\mathcal{E}_y}^{2k}|^2_{E_{yi'}}\}+C\{\Delta t^4+\Delta x^4+\Delta y^4\}.
\end{aligned} \tag{4.55}$$

Similarly, from (4.27) and (4.28), we have

$$\begin{aligned}
&\Big(1-\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_x}^{2k+1}\|^2_{\delta_x E_x}+\|\delta_x{\mathcal{H}_z}^{2k+1}\|^2_{\delta_x H_z}\}\\
\le&\Big(1+\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_x}^{2k}\|^2_{\delta_x E_x}+\|\delta_x{\mathcal{H}_z}^{*}\|^2_{\delta_x H_z}\}+C\{\Delta t^4+\Delta x^4+\Delta y^4\}.
\end{aligned} \tag{4.56}$$

Combining (4.55) with (4.56), we get that

$$\begin{aligned}
&\Big(1-\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_x}^{2k+1}\|^2_{\delta_x E_x}+\|\delta_x{\mathcal{E}_y}^{2k+1}\|^2_{\delta_x E_y}+\|\delta_x{\mathcal{H}_z}^{2k+1}\|^2_{\delta_x H_z}+|{\mathcal{E}_y}^{2k+1}|^2_{E_{yi'}}\}\\
\le&\Big(1+\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_x}^{2k}\|^2_{\delta_x E_x}+\|\delta_x{\mathcal{E}_y}^{2k}\|^2_{\delta_x E_y}+\|\delta_x{\mathcal{H}_z}^{2k}\|^2_{\delta_x H_z}+|{\mathcal{E}_y}^{2k}|^2_{E_{yi'}}\}\\
&+\Delta t\|\delta_x{\mathcal{H}_z}^{*}\|^2_{\delta_x H_z}+C\{\Delta t^4+\Delta x^4+\Delta y^4\}.
\end{aligned} \tag{4.57}$$

Similarly, from the other stages (4.29)-(4.32), we can obtain

$$\begin{aligned}
&\Big(1-\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_x}^{2k+2}\|^2_{\delta_x E_x}+\|\delta_x{\mathcal{E}_y}^{2k+2}\|^2_{\delta_x E_y}+\|\delta_x{\mathcal{H}_z}^{2k+2}\|^2_{\delta_x H_z}+|{\mathcal{E}_y}^{2k+2}|^2_{E_{yi'}}\}\\
\le&\Big(1+\frac{\Delta t}{2}\Big)\{\|\delta_x{\mathcal{E}_x}^{2k+1}\|^2_{\delta_x E_x}+\|\delta_x{\mathcal{E}_y}^{2k+1}\|^2_{\delta_x E_y}+\|\delta_x{\mathcal{H}_z}^{2k+1}\|^2_{\delta_x H_z}+|{\mathcal{E}_y}^{2k+1}|^2_{E_{yi'}}\}\\
&+\Delta t\|\delta_x{\mathcal{H}_z}^{**}\|^2_{\delta_x H_z}+C\{\Delta t^4+\Delta x^4+\Delta y^4\}.
\end{aligned} \tag{4.58}$$

Solving $\delta_x{\mathcal{H}_z}^{*}$ from (4.26) and (4.28), $\delta_x{\mathcal{H}_z}^{**}$ from (4.30) and (4.32), respectively, we have

$$\delta_x{\mathcal{H}_z}^{*}=\frac{1}{2}\delta_x({\mathcal{H}_z}^{2k+1}+{\mathcal{H}_z}^{2k})-\frac{\Delta t}{4\mu}\delta_x\{\delta_y({\mathcal{E}_x}^{2k+1}+{\mathcal{E}_x}^{2k})+\delta_x({\mathcal{E}_y}^{2k+1}+{\mathcal{E}_y}^{2k})\}, \tag{4.59}$$

$$\delta_x{\mathcal{H}_z}^{**}=\frac{1}{2}\delta_x({\mathcal{H}_z}^{2k+2}+{\mathcal{H}_z}^{2k+1})+\frac{\Delta t}{4\mu}\delta_x\{\delta_y({\mathcal{E}_x}^{2k+2}+{\mathcal{E}_x}^{2k+1})+\delta_x({\mathcal{E}_y}^{2k+2}+{\mathcal{E}_y}^{2k+1})\}. \tag{4.60}$$

Thus, we have

$$
\begin{aligned}
\|\delta_x \mathcal{H}_z{}^*\|^2_{\delta_x H_z} \le & \|\delta_x \mathcal{H}_z{}^{2k+1}\|^2_{\delta_x H_z} + \|\delta_x \mathcal{H}_z{}^{2k}\|^2_{\delta_x H_z} + \frac{2\Delta t^2}{\mu\varepsilon}\Big\{ \frac{1}{\Delta y^2}(\|\delta_x \mathcal{E}_x{}^{2k+1}\|^2_{\delta_x E_x} \\
& + \|\delta_x \mathcal{E}_x{}^{2k}\|^2_{\delta_x E_x}) + \frac{1}{\Delta x^2}(\|\delta_x \mathcal{E}_y{}^{2k+1}\|^2_{\delta_x E_y} + \|\mathcal{E}_y{}^{2k}\|^2_{\delta_x E_y})\Big\}, \qquad (4.61) \\
\|\delta_x \mathcal{H}_z{}^{**}\|^2_{\delta_x H_z} \le & \|\delta_x \mathcal{H}_z{}^{2k+2}\|^2_{\delta_x H_z} + \|\delta_x \mathcal{H}_z{}^{2k+1}\|^2_{\delta_x H_z} + \frac{2\Delta t^2}{\mu\varepsilon}\Big\{ \frac{1}{\Delta y^2}(\|\delta_x \mathcal{E}_x{}^{2k+2}\|^2_{\delta_x E_x} \\
& + \|\delta_x \mathcal{E}_x{}^{2k+1}\|^2_{\delta_x E_x}) + \frac{1}{\Delta x^2}(\|\delta_x \mathcal{E}_y{}^{2k+2}\|^2_{\delta_x E_y} + \|\mathcal{E}_y{}^{2k+1}\|^2_{\delta_x E_y})\Big\}. \qquad (4.62)
\end{aligned}
$$

Combining (4.57) and (4.58) with the above two estimates (4.61) and (4.62), we obtain

$$
\begin{aligned}
& (1-r\Delta t)^2\{\|\delta_x \mathcal{E}_x{}^{2k+2}\|^2_{\delta_x E_x} + \|\delta_x \mathcal{E}_y{}^{2k+2}\|^2_{\delta_x E_y} + \|\delta_x \mathcal{H}_z{}^{2k+2}\|^2_{\delta_x H_z} + |\mathcal{E}_y{}^{2k+2}|^2_{E_{yi'}}\} \\
\le & (1+r\Delta t)^2\{\|\delta_x \mathcal{E}_x{}^{2k}\|^2_{\delta_x E_x} + \|\delta_x \mathcal{E}_y{}^{2k}\|^2_{\delta_x E_y} + \|\delta_x \mathcal{H}_z{}^{2k}\|^2_{\delta_x H_z} + |\mathcal{E}_y{}^{2k}|^2_{E_{yi'}}\} \\
& + C\Delta t\{\Delta t^4 + \Delta x^4 + \Delta y^4\}, \qquad (4.63)
\end{aligned}
$$

where

$$
r = \max\Big\{\frac{1}{2} + \frac{2\Delta t^2}{\mu\varepsilon\Delta x^2}, \frac{1}{2} + \frac{2\Delta t^2}{\mu\varepsilon\Delta y^2}, \frac{3}{2}\Big\}.
$$

So, using the transitivity of inequalities, we have

$$
\begin{aligned}
& \|\delta_x \mathcal{E}_x{}^{2k}\|^2_{\delta_x E_x} + \|\delta_x \mathcal{E}_y{}^{2k}\|^2_{\delta_x E_y} + \|\delta_x \mathcal{H}_z{}^{2k}\|^2_{\delta_x H_z} + |\mathcal{E}_y{}^{2k}|^2_{E_{yi'}} \\
\le & e^{r_1 T}\{\|\delta_x \mathcal{E}_x{}^{0}\|^2_{\delta_x E_x} + \|\delta_x \mathcal{E}_y{}^{0}\|^2_{\delta_x E_y} + \|\delta_x \mathcal{H}_z{}^{0}\|^2_{\delta_x H_z} + |\mathcal{E}_y{}^{0}|^2_{E_{yi'}}\} \\
& + C\frac{e^{r_1 T}-1}{2r_1 + r_1^2}\{\Delta t^4 + \Delta x^4 + \Delta y^4\}, \qquad (4.64)
\end{aligned}
$$

where e is the Euler number and r_1 is a constant. This completes the proof of (4.51). The other estimate (4.52) can be symmetrically proved by repeating the above argument with δ_x being changed into δ_y. □

If regarding δ_w as $\delta_w\delta_t$ $(w=x,y)$, we can further obtain the following theorem.

Theorem 4.2. *Let* $\mathbf{e}=(e_x,e_y)$, h_z; ${E_x}^{2k}_{i+\frac{1}{2},j}$, ${E_y}^{2k}_{i,j+\frac{1}{2}}$, ${H_z}^{2k}_{i+\frac{1}{2},j+\frac{1}{2}}$; *and* $\mathcal{E}_x^{2k}$, $\mathcal{E}_y^{2k}$ *and* $\mathcal{H}_z^{2k}$ *be the same as those in Theorem* 4.1. *If* $\mathbf{e}\in C^5([0,T];[C^4(\bar{\Omega})]^2)$ *and* $h_z\in C^5([0,T];C^4(\bar{\Omega}))$, *then for any* $k\ge 0$, *we have the following error estimates:*

$$
\begin{aligned}
& \|\delta_t\delta_x \mathcal{E}_x{}^{2k+1}\|_{\delta_x E_x} + \|\delta_t\delta_x \mathcal{E}_y{}^{2k+1}\|_{\delta_x E_y} + \|\delta_t\delta_x \mathcal{H}_z{}^{2k+1}\|_{\delta_x H_z} + |\delta_t \mathcal{E}_y{}^{2k+1}|_{E_{yi'}} \\
\le & C\{\|\delta_t\delta_x \mathcal{E}_x{}^{1}\|_{\delta_x E_x} + \|\delta_t\delta_x \mathcal{E}_y{}^{1}\|_{\delta_x E_y} + \|\delta_t\delta_x \mathcal{H}_z{}^{1}\|_{\delta_x H_z} \\
& + |\delta_t \mathcal{E}_y{}^{1}|_{E_{yi'}} + \Delta t^2 + \Delta x^2 + \Delta y^2\}, \qquad (4.65)
\end{aligned}
$$

$$\begin{aligned}&\|\delta_t\delta_y{\mathcal{E}_x}^{2k+1}\|_{\delta_y E_x}+\|\delta_t\delta_y{\mathcal{E}_y}^{2k+1}\|_{\delta_y E_y}+\|\delta_t\delta_y{\mathcal{H}_z}^{2k+1}\|_{\delta_y H_z}+|\delta_t{\mathcal{E}_x}^{2k+1}|_{E_{xj'}}\\ \le& C\{\|\delta_t\delta_y{\mathcal{E}_x}^1\|_{\delta_y E_x}+\|\delta_t\delta_y\mathcal{E}_y^1\|_{\delta_y E_y}+\|\delta_t\delta_y\mathcal{H}_z^1\|_{\delta_y H_z}\\ &+|\delta_t{\mathcal{E}_x}^1|_{E_{xj'}}+\Delta t^2+\Delta x^2+\Delta y^2\}. \end{aligned}\tag{4.66}$$

On the other hand, if regarding δ_w ($w=x,y$) as I (the identity operator), we obtain the following two estimates in Remark 4.1 (which were proved by Chen, Li and Liang in [1]).

Remark 4.1. Let $\mathbf{e}=(e_x,e_y)$, h_z; ${E_x}^{2k}_{i+\frac12,j}$, ${E_y}^{2k}_{i,j+\frac12}$, ${H_z}^{2k}_{i+\frac12,j+\frac12}$; and $\mathcal{E}_x^{2k}$, $\mathcal{E}_y^{2k}$ and $\mathcal{H}_z^{2k}$ be the same as those in Theorems 4.1 and 4.2. If $\mathbf{e}\in C^3([0,T];[C^3(\bar\Omega)]^2)$ and $h_z\in C^3([0,T];C^3(\bar\Omega))$, then

$$\begin{aligned}&\|{\mathcal{E}_x}^{2k}\|_{E_x}+\|{\mathcal{E}_y}^{2k}\|_{E_y}+\|{\mathcal{H}_z}^{2k}\|_{H_z}\\ \le& C\{\|{\mathcal{E}_x}^0\|_{E_x}+\|{\mathcal{E}_y}^0\|_{E_y}+\|{\mathcal{H}_z}^0\|_{H_z}+\Delta t^2+\Delta x^2+\Delta y^2\},\end{aligned}\tag{4.67}$$

and if $\mathbf{e}\in C^4([0,T];[C^3(\bar\Omega)]^2)$ and $h_z\in C^4([0,T];C^3(\bar\Omega))$, then

$$\begin{aligned}&\|\delta_t{\mathcal{E}_x}^{2k+1}\|_{E_x}+\|\delta_t{\mathcal{E}_y}^{2k+1}\|_{E_y}+\|\delta_t{\mathcal{H}_z}^{2k+1}\|_{H_z}\\ \le& C\{\|\delta_t{\mathcal{E}_x}^1\|_{E_x}+\|\delta_t{\mathcal{E}_y}^1\|_{E_y}+\|\delta_t{\mathcal{H}_z}^1\|_{H_z}+\Delta t^2+\Delta x^2+\Delta y^2\}.\end{aligned}\tag{4.68}$$

Finally, we combine all results in Theorems 4.1 and 4.2 and Remark 4.1, we have the following super-convergence estimates in Theorem 4.3.

Theorem 4.3. *Suppose that the assumptions in Theorems* 4.1 *and* 4.2 *are satisfied. Then, we have following super-convergence estimates in the discrete H^1-norm.*

$$\begin{aligned}&\|{\mathcal{E}_x}^{2k}\|_1+\|{\mathcal{E}_y}^{2k}\|_1+\|{\mathcal{H}_z}^{2k}\|_1\\ \le& C\{\|{\mathcal{E}_x}^0\|_1+\|{\mathcal{E}_y}^0\|_1+\|{\mathcal{H}_z}^0\|_1+\Delta t^2+\Delta x^2+\Delta y^2\},\end{aligned}\tag{4.69}$$

$$\begin{aligned}&\|\delta_t{\mathcal{E}_x}^{2k+1}\|_1+\|\delta_t{\mathcal{E}_y}^{2k+1}\|_1+\|\delta_t{\mathcal{H}_z}^{2k+1}\|_1\\ \le& C\{\|\delta_t{\mathcal{E}_x}^1\|_1+\|\delta_t{\mathcal{E}_y}^1\|_1+\|\delta_t{\mathcal{H}_z}^1\|_1+\Delta t^2+\Delta x^2+\Delta y^2\}.\end{aligned}\tag{4.70}$$

The super-convergence of the symmetric EC-S-FDTD scheme provides us a simple method to estimate the error bound of the divergence of electric field, as shown in Theorem 4.4.

Theorem 4.4. *Let $\{{E_x}^{2k}_{i+\frac12,j}\}$, $\{{E_y}^{2k}_{i,j+\frac12}\}$ and $\{{H_z}^{2k}_{i+\frac12,j+\frac12}\}$ be the solution of the symmetric EC-S-FDTD scheme. If the exact solution components* **e** *and h_z of the Maxwell's equations is smooth enough, then for any $k\ge 1$, the divergence of the electric field is estimated by*

$$\begin{aligned}\|\sqrt{\varepsilon}(\delta_x{E_x}^{2k}+\delta_y{E_y}^{2k})\|^2:=&\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\varepsilon\Big(\delta_x{E_x}^{2k}_{i,j}+\delta_y{E_y}^{2k}_{i,j}\Big)^2\Delta x\Delta y\\ \le& C\{\Delta t^4+\Delta x^4+\Delta y^4\}.\end{aligned}\tag{4.71}$$

Proof. Let e_x and e_y be the exact solution of the Maxwell's equations, and $\mathcal{E}_x = e_x - E_x$, $\mathcal{E}_y = e_y - E_y$. Then

$$\begin{aligned} &\|\delta_x E_x^{2k} + \delta_y E_y^{2k}\|_{\delta_x E_x} \le \|\delta_x \mathcal{E}_x^{2k} + \delta_y \mathcal{E}_y^{2k}\|_{\delta_x E_x} + \|\delta_x e_x^{2k} + \delta_y e_y^{2k}\|_{\delta_x E_x} \\ \le &\|\delta_x \mathcal{E}_x^{2k}\|_{\delta_x E_x} + \|\delta_y \mathcal{E}_y^{2k}\|_{\delta_y E_y} + \|\delta_x e_x^{2k} + \delta_y e_y^{2k}\|_{\delta_x E_x}, \end{aligned} \tag{4.72}$$

where we have used the fact that $\|\cdot\|_{\delta_x E_x}$ is the same as $\|\cdot\|_{\delta_y E_y}$. By the property that the exact electric field (e_x, e_y) is divergence free and the error estimate of ${\mathcal{E}_x}^{2k}$ and ${\mathcal{E}_y}^{2k}$ in Theorem 4.1 we see that (4.71) holds. $\square$

5 Numerical experiments

In this section we do experiments to test the energy conservations and the super-convergence of the symmetric EC-S-FDTD scheme. The considered model is a wave guide problem in $\Omega = [0,1] \times [0,1]$ and with $\varepsilon = \mu = 1$ and a PEC boundary condition. The exact solution of the problem is

$$\begin{aligned} &e_x = e_x(t,x,y) = \cos(\sqrt{2}\pi t)\cos\pi(1-x)\sin\pi(1-y), \\ &e_y = e_y(t,x,y) = -\cos(\sqrt{2}\pi t)\sin\pi(1-x)\cos\pi(1-y), \\ &h_z = h_z(t,x,y) = -\sqrt{2}\sin(\sqrt{2}\pi t)\cos\pi(1-x)\cos\pi(1-y). \end{aligned}$$

5.1 Computation of energy in the different norms

For the fields of $(\mathbf{E}^m, H_z^m)$, we define the discrete energy functionals of I_x, I_y, I_0 and I_1 by using the energy norms defined in Section 2, i.e., for $m = 2k$ or $2k+1$, $k \ge 0$

$$I_x((\mathbf{E}^m, H_z^m)) = \Big(\|\delta_x E_x^m\|^2_{\delta_x E_x} + \|\delta_x E_y^m\|^2_{\delta_x E_y} + \|\delta_x H_z^m\|^2_{\delta_x H_z} + |E_y^m|^2_{E_{yi'}} \Big)^{\frac{1}{2}}, \tag{5.1a}$$

$$I_y((\mathbf{E}^m, H_z^m)) = \Big(\|\delta_y E_x^m\|^2_{\delta_y E_x} + \|\delta_y E_y^m\|^2_{\delta_y E_y} + \|\delta_y H_z^m\|^2_{\delta_y H_z} + |E_x^m|^2_{E_{xj'}} \Big)^{\frac{1}{2}}, \tag{5.1b}$$

$$I_0((\mathbf{E}^m, H_z^m)) = \Big(\|E_x^m\|^2_{E_x} + \|E_y^m\|^2_{E_y} + \|H_z^m\|^2_{H_z} \Big)^{\frac{1}{2}}, \tag{5.1c}$$

$$I_1((\mathbf{E}^m, H_z^m)) = \Big(|I_x((\mathbf{E}^m, H_z^m))|^2 + |I_y((\mathbf{E}^m, H_z^m))|^2 + |I_0((\mathbf{E}^m, H_z^m))|^2 \Big)^{\frac{1}{2}}. \tag{5.1d}$$

Tables 1 and 2 give the discrete energy functionals of the approximate fields $(\mathbf{E}^{2N}, H_z^{2N})$ and $\delta_t(\mathbf{E}^{2N-1}, H_z^{2N-1})$ and the absolute values of their differences with the initial energies as well as the energy functionals of the exact fields $(\mathbf{e}, h_z)$ at $t = T$. The spatial and time steps are $\Delta x = \Delta y = 0.02$ and $\Delta t = 0.01$, respectively. $N = 100$ and $T = 2N\Delta t = 2$. From results in Tables 1 and 2, we can see clearly that the solution of the symmetric EC-S-FDTD scheme and its δ_t-difference are energy-conserved in terms of the four discrete energy norms.

Table 1: Energy of $(\mathbf{E},H_z)$ in four energy norms at $t=0$ and $t=2$.

Energy Functionals	I_x	I_y	I_0	I_1
$(\mathbf{e},h_z)$	$\frac{\sqrt{2}}{2}\pi$	$\frac{\sqrt{2}}{2}\pi$	$\frac{\sqrt{2}}{2}$	$\sqrt{0.5+\pi^2}$
$(\mathbf{E}^0,H_z^0)$	2.2213	2.2213	0.7071	3.2201
$(\mathbf{E}^{2N},H_z^{2N})$	2.2213	2.2213	0.7071	3.2201
$(\mathbf{E}^{2N},H_z^{2N})$-$(\mathbf{E}^0,H_z^0)$	2.2204e-15	9.7699e-15	8.7708e-15	3.1086e-15

Table 2: Energy of $\delta_t(\mathbf{E},H_z)$ in four energy norms at $t=0$ and $t=2-\Delta t$.

Energy Functionals	I_x	I_y	I_0	I_1
$\partial_t(\mathbf{e},h_z)$	π^2	π^2	π	$\pi\sqrt{1+2\pi^2}$
$\delta_t(\mathbf{E}^1,H_z^1)$	9.8676	9.8676	3.1411	14.3040
$\delta_t(\mathbf{E}^{2N-1},H_z^{2N-1})$	9.8676	9.8676	3.1411	14.3040
$\delta_t(\mathbf{E}^{2N-1},H_z^{2N-1})$-$\delta_t(\mathbf{E}^1,H_z^1)$	3.3751e-14	3.9080e-14	2.7978e-14	5.6843e-14

Table 3: Energy of $(\mathbf{E},H_z)$ in four energy norms at $t=0$ and $t=8$.

Energy Functionals	I_x	I_y	I_0	I_1
$(\mathbf{e},h_z)$	$\frac{\sqrt{2}}{2}\pi$	$\frac{\sqrt{2}}{2}\pi$	$\frac{\sqrt{2}}{2}$	$\sqrt{0.5+\pi^2}$
$(\mathbf{E}^0,H_z^0)$	2.2214	2.2214	0.7071	3.2201
$(\mathbf{E}^{2N},H_z^{2N})$	2.2214	2.2214	0.7071	3.2201
$(\mathbf{E}^{2N},H_z^{2N})$-$(\mathbf{E}^0,H_z^0)$	5.5511e-14	5.5067e-14	1.6542e-14	7.9936e-14

Table 4: Energy of $\delta_t(\mathbf{E},H_z)$ in four energy norms at $t=\Delta t$ and $t=8-\Delta t$.

Energy Functionals	I_x	I_y	I_0	I_1
$\partial_t(\mathbf{e},h_z)$	π^2	π^2	π	$\pi\sqrt{1+2\pi^2}$
$\delta_t(\mathbf{E}^1,H_z^1)$	9.8676	9.8676	3.1411	14.3040
$\delta_t(\mathbf{E}^{2N-1},H_z^{2N-1})$	9.8676	9.8676	3.1411	14.3040
$\delta_t(\mathbf{E}^{2N-1},H_z^{2N-1})$-$\delta_t(\mathbf{E}^0,H_z^0)$	2.4336e-13	2.1494e-13	6.8834e-14	3.3218e-13

In order to observe the behavior in a long time, Tables 3 and 4 present numerical results of the discrete energy functionals at time $T=8$ or $N=800$ when $\Delta x=\Delta y=0.01$, and $\Delta t=0.005$ are used. It is clearly shown that for a long time, the numerical solution of the symmetric EC-S-FDTD scheme and its δ_t-difference keep energy-conserved in the discrete energy norms and are consistent with theoretical results obtained in Theorems 3.2-3.4 in Section 3.

Table 5: Errors and convergence rates of $(\mathbf{E}^{2N}, H_z^{2N})$ in four energy norms.

$\Delta t=\Delta x=\Delta y$	I_x	I_y	Rate	I_0	Rate	I_1	Rate
$h=0.02$	1.672e-2	1.672e-2		5.323e-3		2.424e-2	
$h=0.01$	4.184e-3	4.184e-3	1.9989	1.332e-3	1.9917	6.065e-3	1.9990
$h=0.005$	1.046e-3	1.046e-3	1.9997	3.329e-4	1.9998	1.516e-3	1.9997
$h=0.0025$	2.6153e-4	2.6153e-4	1.9999	8.3248e-5	1.9999	3.7922e-4	1.9999

Table 6: Errors and convergence rates of $\delta_t(\mathbf{E}^{2N-1}, H_z^{2N-1})$ in four energy norms.

$\Delta t=\Delta x=\Delta y$	I_x	I_y	Rate	I_0	Rate	I_1	Rate
$h=0.02$	7.453e-2	7.453e-2		2.373e-2		1.080e-1	
$h=0.01$	1.874e-2	1.874e-2	1.9917	5.966e-3	1.9919	2.717e-2	1.9918
$h=0.005$	4.697e-3	4.697e-3	1.9964	1.495e-3	1.9964	6.809e-3	1.9964
$h=0.0025$	1.176e-3	1.176e-3	1.9983	3.742e-4	1.9983	1.7041e-3	1.9983

5.2 Computation of errors and convergence rates

As given in Section 4, let $(\mathcal{E}^n, \mathcal{H}_z^n) = (e_x^n - E_x^n, e_y^n - E_y^n, h_z^n - H_z^n)$ denote the vector of the errors between the numerical solutions and the exact solution. I_x, I_y, I_0 and I_1 are the discrete energy norms or the discrete energy functionals defined in (5.1). Take $\Delta x = \Delta y = \Delta t = 0.02$, 0.01, 0.005 and 0.0025. Let the time length $T=2$ and $N = \frac{T}{2\Delta t}$. The errors and convergence rates of the numerical solutions computed by the symmetric SC-S-FDTD scheme are presented in four discrete energy norms in Tables 5 and 6.

Numerical results in Tables 5 and 6 show clearly that the errors of the numerical solution and its δ_t-difference of the symmetric EC-S-FDTD scheme in the discrete H^1-norm are of second order convergence in both time and space steps, which confirm the theoretical results of the super-convergence in the discrete H^1-norm.

6 Conclusions

We have derived new energy-conserved identities of the symmetric EC-S-FDTD scheme, which show that the scheme is energy-conserved and unconditionally stable in the discrete H^1 norm. By the new energy-conserved identities, it is proved that the symmetric EC-S-FDTD scheme is second order convergent in the discrete H^1 norm. This shows the scheme has the property of super convergence. By the super convergence, we strictly proved that the error of the divergence of the electric field of the symmetric EC-S-FDTD scheme is second order accurate.

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