Commun. Comput. Phys.
doi: 10.4208/cicp.111209.160610s

Vol. **11**, No. 2, pp. 367-382
February 2012

# Antiplane Wave Scattering from a Cylindrical Void in a Pre-Stressed Incompressible Neo-Hookean Material

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Received 11 December 2009; Accepted (in revised version) 16 June 2010

Available online 24 October 2011

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**Abstract.** An isolated cylindrical void is located inside an incompressible nonlinear-elastic medium whose constitutive behaviour is governed by a neo-Hookean strain energy function. In-plane hydrostatic pressure is applied in the far-field so that the void changes its radius and an inhomogeneous region of deformation arises in the vicinity of the void. We consider scattering from the void in the deformed configuration due to an incident field (of small amplitude) generated by a horizontally polarized shear ($SH$) line source, a distance $r_0$ ($R_0$) away from the centre of the void in the deformed (undeformed) configuration. We show that the scattering coefficients of this scattered field are unaffected by the pre-stress (initial deformation). In particular, they depend not on the deformed void radius $a$ or distance $r_0$, but instead on the *original* void size $A$ and *original* distance $R_0$.

**AMS subject classifications**: 74B15, 74B20, 74J20, 74Q15

**Key words**: Pre-stress, incompressible, neo-Hookean, rubber, incremental $SH$ waves, scattering, line source.

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## 1 Introduction

The influence of nonlinear pre-stress on subsequent incremental linear wave propagation in elastic media has been studied in detail over the past few decades using the so-called theory of *small-on-large* [9, 15] where a linearization is performed about the nonlinear equilibrium state in order to determine the wave propagation characteristics of the pre-stressed medium. To the authors' knowledge, in the literature interest has centered exclusively on the influence of *homogeneous* stretch distributions (and hence induced anisotropy) on subsequent wave propagation, see e.g., [5, 10]. When the medium

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in question is *inhomogeneous* (for example a fibre-reinforced, or particulate composite material, where the host phase is nonlinear-elastic) then pre-stress will almost always lead to non-homogeneous stretch distributions, except in very special cases (see e.g., [17]). Degtyar et al. [6] analysed the case of stressed composites where residual stresses occur in the vicinity of inhomogeneities, in the context of linear elasticity. The interest was to determine how the presence of linear elastic residual stress affects the subsequent wave propagation through the inhomogeneous material and hence derive the associated effective wavenumber in the pre-stressed state. This work utilized the self consistent scattering formulation of the effective wavenumber derived for a single inclusion by Yang and Mal [19] which itself is a modification of the Waterman and Truell theory [18]. This modified formulation of multiple scattering theory was used since the stress distribution in the neighbourhood of an inhomogeneity was approximated as piecewise constant (i.e., a multi-layered cylindrical inclusion). We emphasize that this analysis took place in the linear elastic (small displacement gradient) regime. Of great interest however is how an initial *nonlinear* pre-stress affects subsequent multiple scattered wave fields which propagate through inhomogeneous media. In the low frequency regime this information would allow us to determine the incremental behaviour of the material and hence the effective homogenized properties. Recently, the so-called second-order homogenization method was applied to determine such homogenized behaviour in the static regime for nonlinear fibre reinforced composites [3]. The composite cylinder assemblage model for nonlinear composites has also been employed in this context [4]. Nonlinear pre-stress can be extremely useful in practice, allowing us to *tune* materials in order to permit or restrict the propagation of waves in specific frequency ranges. This property was described by Parnell [17] and discussed further in subsequent articles in different contexts [2,7].

Multiple scattering in the (unstressed) linear elastic quasi-static regime is relatively well understood [12] and much work has centered on the derivation of the effective wave number (and resulting effective elastic properties) given a random distribution of inhomogeneities. However, if the host medium is nonlinear-elastic (e.g., rubber) and we are interested in the multiple scattering problem in the pre-stressed configuration, a canonical single scattering problem of central importance, which must be studied before we can solve the multiple scattering problem, is the following: *How does an isolated inclusion embedded in a nonlinear-elastic pre-stressed host medium (where inhomogeneous deformation or stress is present), scatter incoming elastic waves?* To the authors' knowledge, no such problems of this type (i.e., incorporating inhomogeneous fields) have been solved before in the literature. In this article we shall consider the problem of antiplane elastic or horizontally-polarized shear $(SH)$ wave scattering from a cylindrical void embedded in an incompressible host medium which is capable of finite deformation and is neo-Hookean in its constitutive behaviour. This problem in the context of no pre-stress is discussed on pp. 123 of [12] and pp. 208 of [16]. A related pre-stress problem is studied in [11]. However, in [11] the pre-stress was assumed to be uniform, i.e., *all* stretch distributions in the host domain are homogeneous. This is a simpler problem than that to be discussed in the present article since it changes only the induced anisotropy of the host medium. Furthermore such a homogeneous deformation regime would *not*, in general, occur in reality, since some traction distribution would usually be applied in the far-field and the resulting deformation in the neighbourhood of the inhomogeneity would accommodate this imposed stress field, thereby setting up an inhomogeneous deformation in the material, as we show here.

In Section 2 we summarize the results for scattering of a horizontally polarized shear line source from a cylindrical void in an otherwise uniform and unstressed host phase. In particular we obtain expressions for the scattering coefficients associated with each mode of scattering from the void. In Section 3 we begin the study of the influence of pre-stress on scattering. We derive the governing equilibrium equations for the initial nonlinear static deformation of a *neo-Hookean* host phase, given a uniform longitudinal stretch and hydrostatic pressure at infinity. In Section 4 we derive the incremental equation (Eq. (4.3)) for small-amplitude $SH$ waves superposed on top of the initial nonlinear static deformation derived in Section 3. We then show in Section 5 that we can solve the problem for the scattered field in the deformed configuration explicitly given an incident field due to a line source in the host domain. It transpires that if we maintain the magnitude of the *force* of the line source (and thus the coefficient of the line source *is* modified since this is a force per unit length in the longitudinal direction) then the resulting scattering coefficients in the deformed configuration are *identical* with those for scattering in the initial *undeformed* configuration. In particular, they depend not on the deformed void radius $a$ or distance $r_0$ from the center of the void to the source location in the deformed configuration. Instead the scattering coefficients depend on the *original* void size $A$ and *original* distance $R_0$ from source point to void location. This has implications from a non-destructive testing point of view since if these scattering coefficients were measured, they would not provide any information regarding the *deformed* void radius.

## 2 Scattering of an antiplane line source by a cylindrical void

In this section, in order to define notation and for later reference, we summarize results regarding the scattering of antiplane or horizontally polarized shear $(SH)$ waves from a cylindrical void due to an incident field generated by a line source (cf. the case of acoustic scattering). This problem is discussed in greater detail in numerous good textbooks on wave scattering theory (see for example [16]). Let us consider an isolated cylindrical void of radius $A$, inside an isotropic homogeneous elastic host region of infinite extent in all directions. With reference to Fig. 1, we specify a Cartesian coordinate system $(X,Y,Z)$, with origin at the centre of the void and whose $Z$ axis runs parallel with the axis of the cylindrical void. Consider the problem of scattering by a time-harmonic line source (of small-amplitude) which is polarized in the $Z$ direction and located at $X = X_0$, $Y = Y_0$, which is located at a distance $R_0$ away from the centre of the cylinder and at an angle $\Theta_0 \in [0,2\pi)$ subtended from the $X$-axis (see Fig. 1). This forcing therefore gives rise to linear elastic waves polarized in the $Z$ direction propagating in the $XY$ plane, i.e., $SH$

waves. The displacement field is therefore $\mathbf{U}'=\Re[(0,0,W(X,Y))\exp(-i\omega t)]$, where $\omega$ is the circular frequency and where $W(X,Y)$ is governed by [8]

$$(\mu\nabla^2+\rho\omega^2)W=C\delta(X-X_0)\delta(Y-Y_0)=\frac{C}{R_0}\delta(R-R_0)\delta(\Theta-\Theta_0). \tag{2.1}$$

Here we have defined the cylindrical polar coordinates $R$, $\Theta$ via the standard relations $X=R\cos\Theta$, $Y=R\cos\Theta$ and we note that $R_0$ and $\Theta_0$ are thus defined by $X_0=R_0\cos\Theta_0$, $Y_0=R_0\cos\Theta_0$. We also point out that $C$ is the force per unit length (in the $Z$ direction) of the imposed line source and $\rho$ and $\mu$ are the mass density and shear modulus of the host medium. We re-write (2.1) in the form

$$(\nabla^2+K^2)W=\frac{C}{\mu R_0}\delta(R-R_0)\delta(\Theta-\Theta_0), \tag{2.2}$$

where the wavenumber $K$ is defined by $K^2=\rho\omega^2/\mu$. Using the polar coordinate description, the traction-free boundary condition on the surface of the void at $R=A$ can then be specified as

$$\mu\frac{\partial W}{\partial R}=0. \tag{2.3}$$

We can write the solution of this problem in the form

$$W=W_i+W_s, \tag{2.4}$$

where $W_i$ and $W_s$ represent incident (incoming) and scattered (outgoing) fields from the cylindrical void. In particular, the incident field represents the solution of the inhomogeneous problem (2.2) which is outgoing from the source location. This is

$$W_i=\frac{C}{4i\mu}\mathrm{H}_0(KS), \tag{2.5}$$

where $S=\sqrt{(X-X_0)^2+(Y-Y_0)^2}$ and where we have defined $\mathrm{H}_0(KS)=\mathrm{H}_0^{(1)}(KS)=\mathrm{J}_0(KS)+i\mathrm{Y}_0(KS)$, the Hankel function of the first kind and $\mathrm{J}_0$ and $\mathrm{Y}_0$ are Bessel's functions of the first and second kind respectively, of order zero. Together with the time dependence in the problem, this ensures an outgoing field (from the source). Graf's addition theorem allows us to write this field relative to the coordinate system $(R,\Theta)$ center at the origin of the void in the form [12]

$$W_i=\frac{C}{4i\mu}\mathrm{H}_0(KS)=\frac{C}{4i\mu}\times\begin{cases}\displaystyle\sum_{n=-\infty}^{\infty}\mathrm{H}_n(KR_0)\mathrm{J}_n(KR)e^{in(\Theta-\Theta_0)}, & R<R_0,\\ \displaystyle\sum_{n=-\infty}^{\infty}\mathrm{H}_n(KR)\mathrm{J}_n(KR_0)e^{in(\Theta-\Theta_0)}, & R>R_0,\end{cases} \tag{2.6}$$

where $\mathrm{H}_n$ and $\mathrm{J}_n$ are respectively Hankel and Bessel functions of the first kind, and of order $n$.

Upon writing the scattered field in the form

$$W_s = \sum_{n=-\infty}^{\infty} (-i)^n A_n \mathrm{H}_n(KR) e^{in(\Theta-\Theta_0)} \tag{2.7}$$

and using this together with the first of (2.6) and (2.4) in (2.3) we find that the scattering coefficients due to the incident wave from the line source are

$$A_n = \frac{C(-1)^n}{4\mu i^{n-1}} \frac{\mathrm{J}_n'(KA)}{\mathrm{H}_n'(KA)} \mathrm{H}_n(KR_0). \tag{2.8}$$

Let us now consider the limit as $R_0 \to \infty$ in (2.5) and the first of (2.6). Using the fact that (pp. 364 of [1])

$$H_n(KR_0) \sim (-i)^n \sqrt{\frac{2}{\pi K R_0}} e^{i(KR_0 - \frac{\pi}{4})} \tag{2.9}$$

as $R_0 \to \infty$ and upon setting

$$C = 2i\mu\sqrt{2\pi K R_0} e^{i(\frac{\pi}{4} - KR_0)}, \tag{2.10}$$

we find that in this limit,

$$W_i \sim \sum_{n=-\infty}^{\infty} i^n \mathrm{J}_n(KR) e^{in(\Theta-\theta_{inc})} = e^{iK(X\cos\theta_{inc} + Y\sin\theta_{inc})}, \tag{2.11}$$

which is an incident plane wave, propagating at an angle of incidence $\theta_{inc} = \Theta_0 - \pi \in [-\pi, \pi)$ to the $X$ axis (since $\Theta_0 \in [0, 2\pi)$). The scattering coefficients appearing in (2.7) for this forcing are thus

$$A_n^{(pw)} = -\frac{\mathrm{J}_n'(KA)}{\mathrm{H}_n'(KA)}, \tag{2.12}$$

where the superscript $(pw)$ indicates a plane-wave forcing. These are the standard scattering coefficients associated with plane wave scattering from a circular cylindrical void (see e.g., [12], pp. 123, Eq. (4.5)), noting the form of the scattered field in (2.7) with $\Theta_0 = \theta_{inc} + \pi$.

## 3 Initial finite static deformation (pre-stress)

Consider an initial pre-stress of the host phase in question. We now assume that this host phase is isotropic and nonlinear-elastic (e.g., rubber) so that it is capable of finite deformation. Its constitutive behaviour may be described by a strain energy function $\mathcal{W}_{SEF} = \mathcal{W}_{SEF}(I_1, I_2, I_3)$, where $I_j$ are the principal strain invariants of the deformation [9, 15]. Since the host material is envisaged to be of a rubber type, we assume *incompressibility* as a first approximation. This means that the strain invariant $I_3 = 1$ and

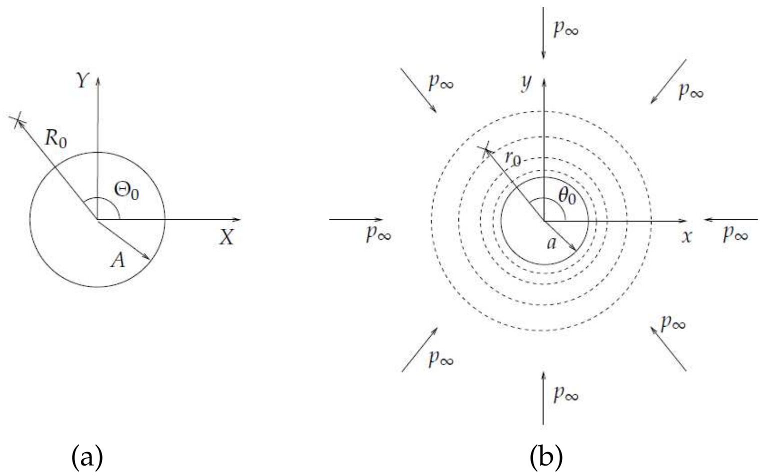


Figure 1: (a) shows the undeformed cavity, with a line source indicated by crossed lines at the position $(R_0,\Theta_0)$. In (b) we show how the cavity deforms due to the hydrostatic pressure $p_\infty$, successive dotted circles indicating decreasing strain as we move away from the cavity. Under this deformation the line source moves to the location $(r_0,\theta_0)$, noting that $\theta_0=\Theta_0$ due to symmetry.

thus $\mathcal{W}_{SEF}=\mathcal{W}_{SEF}(I_1,I_2)$. The incompressible limit is discussed on p. 64 of Green and Zerna [9]. In particular because there is an additional constraint in this case (Eq. (3.4) here) an additional unknown (the Lagrange multiplier $p$ here) is required in the analysis. This hydrostatic pressure term appears in the Eqs. (3.13a)-(3.13c) below and takes the form $p=2\sqrt{I_3}\partial W/\partial I_3$ in the *compressible* case. In this incompressible scenario this expression is, of course, undefined and therefore $p$ becomes an unknown which is determined via the additional incompressibility constraint.

In this article we shall restrict attention to an incompressible material whose strain energy function is of neo-Hookean type, i.e., $\mathcal{W}_{SEF}=\mu(I_1-3)/2$, where $\mu$ is the linear-elastic shear modulus of the material [15]. We shall impose an initial finite deformation and consider how this pre-stress affects the subsequent incremental scattering of small-amplitude $SH$ waves from the void. In particular we wish to focus on the influence of the pre-stress on the scattering coefficients associated with the scattered field. Since we are working in cylindrical polar coordinates it will be convenient to employ the tensorial notation of Green and Zerna [9], in order to firstly determine the equation of static equilibrium of the finite deformation and then to derive the equation governing the small-amplitude incremental motions. For ease of reading, some of this analysis will be located in the Appendix.

As in the previous section we define the cylindrical polar coordinate system $(R,\Theta,Z)$ with origin at the centre of the cylindrical void in the undeformed configuration. Additionally we define the cylindrical polar coordinate system $(r,\theta,z)$ associated with the *deformed* configuration. In the far-field we impose an *in-plane* hydrostatic pressure $\sigma^{rr}=-p_\infty$ (per unit area in the deformed configuration, corresponding to a Cauchy stress) as $r,R\to\infty$ (uniform in the longitudinal direction), as shown in Fig. 1. The material is also held at a fixed stretch $L$ in the longitudinal $Z$ direction. The ensuing deformation is therefore described by (see pp. 87 of [9])

$$R = rQ(r), \qquad \Theta = \theta, \qquad Z = \frac{z}{L}. \tag{3.1}$$

Note the convention introduced in (3.1) above, i.e., that upper case variables correspond to the *undeformed* configuration whilst lower case corresponds to the *deformed* configuration. The function $Q(r)$ is to be determined from the incompressibility condition and equilibrium equations. Note that it will be convenient for us to derive equations in terms of coordinates in the *deformed* configuration since we linearize about this state when we consider incremental motions. Cartesian position vectors of a material point in the undeformed (upper case) and deformed (lower case) configurations are respectively

$$\mathbf{X} = \begin{pmatrix} R\cos\Theta \\ R\sin\Theta \\ Z \end{pmatrix} = \begin{pmatrix} rQ(r)\cos\theta \\ rQ(r)\sin\theta \\ \frac{z}{L} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}. \tag{3.2}$$

Under the deformation described above, the principal stretches in the radial, azimuthal and longitudinal directions are therefore given by

$$\lambda_r = \frac{dr}{dR} = \Big(Q + r\frac{dQ}{dr}\Big)^{-1}, \qquad \lambda_\theta = \frac{r}{R} = \frac{1}{Q}, \qquad \lambda_z = L. \tag{3.3}$$

The condition of incompressibility is (see pp. 64 of [9])

$$I_3 = \lambda_r^2 \lambda_\theta^2 \lambda_z^2 = 1, \tag{3.4}$$

where $I_3$ is the third principal strain invariant. This means that $Q(r)$ must satisfy the relation

$$Q\Big(Q + r\frac{dQ}{dr}\Big) = L, \tag{3.5}$$

which can be integrated directly in order to find that

$$Q^2(r) = \frac{L(r^2 + M)}{r^2}, \tag{3.6}$$

where $M$ is a constant to be determined. Since $R = rQ(r)$, we can determine a relationship between $M$ and the undeformed and deformed cylinder radii $A$ and $a$ respectively, i.e.,

$$M = \frac{A^2}{L} - a^2, \tag{3.7}$$

where $A$ is the (specified) initial radius of the cavity and $a$ will be determined from the resulting equation of equilibrium. Note also that (3.5) means that the principal stretch $\lambda_r$

in (3.3) can be written simply as $\lambda_r = Q/L$. Since we know the principal stretches, the two strain invariants $I_1$ and $I_2$ are thus given by (see pp. 57 of [9])

$$I_1 = \lambda_r^2 + \lambda_\theta^2 + \lambda_z^2 = \frac{Q^2}{L^2} + \frac{1}{Q^2} + L^2, \qquad I_2 = \lambda_r^2\lambda_z^2 + \lambda_\theta^2\lambda_z^2 + \lambda_r^2\lambda_\theta^2 = Q^2 + \frac{L^2}{Q^2} + \frac{1}{L^2}. \tag{3.8}$$

On introducing the notation (Green and Zerna [9]) $(\theta_1,\theta_2,\theta_3) = (r,\theta,z)$ for the curvilinear (cylindrical polar) coordinate system in the deformed configuration, we can define the following covariant basis vectors in the undeformed and deformed configurations respectively:

$$\mathbf{G}_r = \frac{\partial X_s}{\partial \theta_r}\mathbf{i}_s, \qquad \mathbf{g}_r = \frac{\partial x_s}{\partial \theta_r}\mathbf{i}_s, \tag{3.9}$$

where $\mathbf{i}_s$ are the unit basis vectors in a Cartesian coordinate system, e.g., $\mathbf{i}_1 = (1,0,0)$. Note here that we have used slightly different notation to that employed by Green and Zerna: we have employed upper case script for the undeformed configuration and lower case for the deformed configuration. We feel that this will be notation which the reader is more familiar with. Also, note the convention introduced here that superscripts denote contravariant tensors, whilst subscripts denote covariant tensors. We can therefore derive the covariant metric tensors $G_{ij} = \mathbf{G}_i.\mathbf{G}_j$ and $g_{ij} = \mathbf{g}_i.\mathbf{g}_j$ associated with the undeformed and deformed configurations:

$$G_{ij} = \begin{pmatrix} L^2/Q^2 & 0 & 0 \\ 0 & r^2Q^2 & 0 \\ 0 & 0 & 1/L^2 \end{pmatrix}, \qquad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

where we have used the incompressibility constraint (3.5), to simplify $G_{ij}$. Finally, since these metric tensors are diagonal, we can immediately obtain the *contravariant* metric tensors $G^{ij}$ and $g^{ij}$ in the form

$$G^{ij} = \begin{pmatrix} Q^2/L^2 & 0 & 0 \\ 0 & 1/r^2Q^2 & 0 \\ 0 & 0 & L^2 \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.11}$$

Let us now assume that the constitutive response of the incompressible material is *neo-Hookean*, i.e.,

$$\mathcal{W}_{SEF} = \frac{\mu}{2}(I_1 - 3), \tag{3.12}$$

where $\mu$ is the linear-elastic shear modulus associated with the material. This form of strain energy function has been utilized with great success in applications of nonlinear elasticity, especially in the context of rubber at reasonable deformations [13, 14]. Following Green and Zerna [9] for a material governed by the strain energy function $\mathcal{W}_{SEF}$ in

(3.12), the non-zero contravariant Cauchy stress tensor components $\tau^{ij}$ are given by

$$\tau^{11}=G^{11}\mu+g^{11}p=\frac{Q^2}{L^2}\mu+p=\sigma^{rr}, \tag{3.13a}$$

$$r^2\tau^{22}=r^2G^{22}\mu+r^2g^{22}p=\frac{1}{Q^2}\mu+p=\sigma^{\theta\theta}, \tag{3.13b}$$

$$\tau^{33}=G^{33}\mu+g^{33}p=L^2\mu+p=\sigma^{zz}, \tag{3.13c}$$

where $\sigma^{ij}$ are the *physical* components of the Cauchy stress tensor and the function $p$ is the Lagrange multiplier introduced in order to accommodate the incompressibility constraint (3.5). Below we show that we can integrate the equations of equilibrium exactly and therefore determine $\sigma^{rr}$ directly. Therefore, the function $p$ is then determined from (3.13a).

The equations of equilibrium are given by

$$\tau^{ij}\|_i=\tau^{ij}_{,i}+\Gamma^i_{ir}\tau^{rj}+\Gamma^j_{ir}\tau^{ir}=0, \tag{3.14}$$

where $\|_i$ denotes the covariant derivative with respect to the deformed configuration, $f_{,i}$ denotes differentiation of the quantity $f$ with respect to $\theta_i$ and $\Gamma^i_{jk}$ is the Christoffel symbol, derived from the metric tensor of the deformed configuration, i.e.,

$$\Gamma^i_{jk}=\mathbf{g}^i.\mathbf{g}_{j,k}. \tag{3.15}$$

Note here that $\mathbf{g}^i$ is the $i$th *contravariant* basis vector defined by $\mathbf{g}_i.\mathbf{g}^j=\delta^j_i$, where $\delta^j_i$ is the mixed Kronecker delta tensor. It may be shown that the only non-zero components of $\Gamma^i_{jk}$ are

$$\Gamma^1_{22}=-r, \qquad \Gamma^2_{12}=\frac{1}{r}=\Gamma^2_{21}. \tag{3.16}$$

Therefore, the azimuthal and longitudinal equilibrium equations reduce to

$$\tau^{22}_{,2}=\frac{\partial p}{\partial\theta}=0, \qquad \tau^{33}_{,3}=\frac{\partial p}{\partial z}=0, \tag{3.17}$$

which means that $p=p(r)$. The radial equilibrium equation, in terms of physical stresses $\sigma^{ij}$ is

$$\frac{d\sigma^{rr}}{dr}+\frac{1}{r}\left(\sigma^{rr}-\sigma^{\theta\theta}\right)=0, \tag{3.18}$$

which, on using (3.13a) and (3.13b) can be directly integrated for $\sigma^{rr}$, i.e.,

$$\sigma^{rr}(r)=-\mu\int_a^r\left(\frac{Q^2(r')}{L^2}-\frac{1}{Q^2(r')}\right)\frac{dr'}{r'}, \tag{3.19}$$

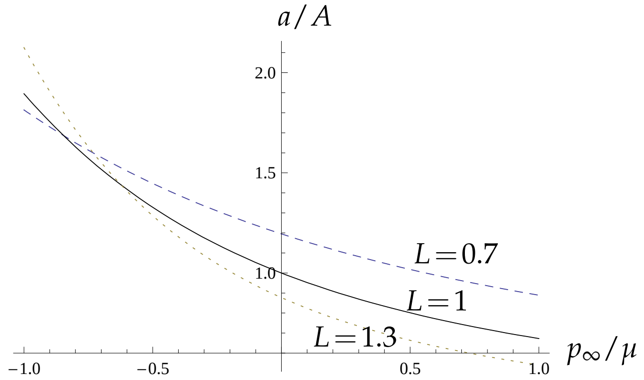


Figure 2: Plot of the deformed to undeformed radius ratio $a/A$ as a function of $p_\infty/\mu$ for the three prescribed values of axial stretch $L=0.7$, 1, 1.3.

where we have imposed $\sigma^{rr}(r=a)=0$. With $Q^2$ given by (3.6), (3.19) becomes

$$\frac{\sigma^{rr}(r)}{\mu}=\frac{1}{2L}\Big[M\Big(\frac{1}{r^2}-\frac{1}{a^2}\Big)-\log\Big(\frac{r^2}{a^2}\Big)+\log\Big(\frac{r^2+M}{a^2+M}\Big)\Big], \tag{3.20}$$

where $M$ was defined in (3.7). When $r\to\infty$ in (3.20), given that $\sigma_{rr}\to -p_\infty$, we find that

$$\frac{p_\infty}{\mu}=\frac{1}{2L}\Big[\frac{A^2}{La^2}-1+\log\Big(\frac{A^2}{La^2}\Big)\Big], \tag{3.21}$$

and we see therefore that this is a (nonlinear) equation for the determination of the deformed to undeformed radius ratio $a/A$ as a function of the ratio $p_\infty/\mu$ and the stretch $L$, both of which are assumed specified. Note that $M=0$ if and only if $p_\infty=0$. We plot $a/A$ as a function of $p_\infty/\mu$ for three different prescribed values of $L=0.7$, 1, 1.3 in Fig. 2. The exhibited behaviour is as one would expect. Note that we can also use (3.20) to determine the function $p(r)$ from (3.13a).

# 4 Incremental deformations

In order to model scattering from the void in the deformed configuration, we now consider the propagation of small-amplitude time-harmonic waves through the pre-stressed medium. We use the theory of *small-on-large*, i.e., linearization about a nonlinear deformation state [9]. The total displacement field may therefore be represented by

$$\hat{\mathbf{u}}=\mathbf{u}+\eta\mathbf{u}', \tag{4.1}$$

where $\mathbf{u}=u^i\mathbf{g}_i=u_i\mathbf{g}^i$ is the displacement field derived from the finite deformation (3.1) and $\eta\ll 1$ is a small parameter associated with the magnitude of the incremental displacement. Let us assume that the incremental displacement is of an antiplane nature, i.e., of the form

$$\mathbf{u}' = u_3'\mathbf{g}^3 = u^{3'}\mathbf{g}_3 = \Re\left[w(r,\theta)\exp(-i\omega t)\right]\mathbf{g}_3, \tag{4.2}$$

so that the time-harmonic wave is a $SH$ wave, polarized in the $z$ direction and propagating in the $r\theta$ plane. The initial finite deformation leads to a modified wave equation for $w$, as we now show.

In the Appendix, we show, using the theory of *small-on-large* (i.e., linearizing about the nonlinear deformation) that the incremental equation governing $w$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\left(r+\frac{M}{r}\right)\frac{\partial w}{\partial r}\right] + \frac{1}{(r^2+M)}\frac{\partial^2 w}{\partial \theta^2} + k^2 w = 0, \tag{4.3}$$

where we remind the reader that $M = A^2/L - a^2$. Furthermore we have defined the modified wavenumber $k$ at infinity by $k^2 = LK^2$, where $K = \omega\sqrt{\rho/\mu}$ as introduced in Section 2 is the wavenumber of the host material in the undeformed configuration. Note the special case when $L=1$, corresponding to no longitudinal stretch, when $k^2 = K^2$. Note also the special case $A^2 = La^2$ leading to $M=0$ (this only occurs when $p_\infty = 0$), when the modified wave equation (4.3) becomes the standard wave equation in cylindrical polar coordinates with the modified wavenumber $k=\sqrt{L}K$. As stated earlier, Eq. (3.21) reveals that $M \neq 0$ for any non-zero hydrostatic pressure $p_\infty$. Thus for any $L$, the pre-stress always has an influence on the wave field close to the void.

At this point we also note that Eq. (4.3) is equivalent to the following:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu_r(r)\frac{\partial w}{\partial r}\right] + \frac{\mu_\theta(r)}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \rho\omega^2 w = 0, \tag{4.4}$$

where we have defined the anisotropic (and spatially varying) shear moduli as

$$\mu_r(r) = \frac{\mu}{L}\left(\frac{r^2+M}{r^2}\right), \qquad \mu_\theta(r) = \frac{\mu}{L}\left(\frac{r^2}{r^2+M}\right). \tag{4.5}$$

This can be derived from the appropriate equation of equilibrium in cylindrical polar coordinates for an anisotropic material, i.e.,

$$\frac{\partial \sigma'_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma'_{\theta z}}{\partial \theta} + \frac{1}{r}\sigma'_{rz} + \rho\omega^2 w = 0 \tag{4.6}$$

with the incremental stresses $\sigma'_{rz}$ and $\sigma'_{\theta z}$ defined as

$$\sigma'_{rz} = \mu_r(r)\frac{\partial w}{\partial r}, \qquad \sigma'_{\theta z} = \mu_\theta(r)\frac{\partial w}{\partial \theta}. \tag{4.7}$$

We therefore recognize that the deformed medium is equivalent to a material possessing curvilinear (cylindrical) anisotropy and inhomogeneous elastic moduli. Also note that as $r \to \infty$, (4.3) reduces to

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + k^2 w = 0, \tag{4.8}$$

i.e., the standard wave equation for horizontally-polarized shear waves with a modified wavenumber due to the pre-stress, noting that when $L=1$, $k=K$, the shear wavenumber in the undeformed material.

## 5 The influence of pre-stress on scattering of a line source by a cylindrical void

We wish to assess how the scattering coefficients $A_n$ as derived in (2.8) (associated with an incoming wave from a line source) or (2.12) (associated with an incoming plane wave, derived from the limiting case of a line source as its distance from the void tends to infinity), become modified due to the pre-stress. We note that an incident plane-wave must be addressed in this manner (i.e., as the limit of a line source moving to infinity). If one attempts to impose a plane-wave forcing from the outset, this will not satisfy the governing incremental equation (4.3) in the pre-stressed state and hence gives rise to various difficulties, including additional forcing terms in this governing equation. All of these difficulties are avoided by considering a line source forcing and moreover this case is more physically meaningful. As such, let us consider a line source in the deformed configuration located at $(x_0,y_0)$ (equivalently at $(r_0,\theta_0)$), which in the undeformed configuration was in the position $(X_0,Y_0)$. The coefficient $C$ associated with this line source (i.e., the coefficient on the right-hand-side of (2.1) in the undeformed configuration) is a force per unit length, $C=F/L_0$ say where $F$ is the force and $L_0$ is a unit length in the undeformed configuration. Let us associate the coefficient $c$ with the line source in the *deformed* configuration. We wish to ensure that the magnitude of the force $F$ of the line source remains unchanged (so that we can assess the effects of pre-stress alone on scattering) and therefore $c=F/L_1$ where $L_1$ is a unit length in the deformed configuration. From the pre-stress it is evident that $L_1/L_0=L$, the longitudinal stretch, and therefore we must have $c=C/L$ for $F$ to remain unchanged.

We now consider the analogous problem to (2.1) in the *deformed* configuration. Hence take a line source on the right hand side of (4.4) with magnitude $c=C/L$, i.e.,

$$\frac{\mu}{L}\Big[\frac{1}{r}\frac{\partial}{\partial r}\Big(\Big(r+\frac{M}{r}\Big)\frac{\partial w}{\partial r}\Big)+\frac{1}{(r^2+M)}\frac{\partial^2 w}{\partial \theta^2}\Big]+\rho\omega^2 w=\frac{C}{Lr_0}\delta(r-r_0)\delta(\theta-\theta_0). \tag{5.1}$$

In order to proceed we introduce the mapping

$$R^2=L(r^2+M), \qquad \Theta=\theta, \tag{5.2}$$

which is, in fact, the mapping corresponding to the initial finite deformation (3.1) and (3.6). Defining $W(R,\Theta)=w(r(R),\theta(\Theta))$, we find that

$$\frac{\partial^2 W}{\partial R^2}+\frac{1}{R}\frac{\partial W}{\partial R}+\frac{1}{R^2}\frac{\partial^2 W}{\partial \Theta^2}+K^2 W=\frac{C}{L\mu}\frac{1}{r_0}\delta(r-r_0)\delta(\Theta-\Theta_0) \tag{5.3}$$

and $K=\rho\omega^2/\mu$ is the wavenumber associated with the undeformed material.

It is straightforward to show that

$$\frac{1}{r_0}\delta(r-r_0)=\frac{L}{R_0}\delta(R-R_0) \tag{5.4}$$

and hence (5.3) becomes

$$\nabla^2 W+K^2 W=\frac{C}{\mu}\frac{1}{R_0}\delta(R-R_0)\delta(\Theta-\Theta_0), \tag{5.5}$$

where we note that, importantly, the factors of $L$ have cancelled on the right-hand-side. Therefore, the solution of (5.5) is entirely equivalent to the solution of (2.2) obtained in Section 2. The scattered field associated with (5.5) is thus (cf. (2.7))

$$W_s(R)=\sum_{n=-\infty}^{\infty}(-i)^n a_n \mathrm{H}_n(KR)e^{in(\Theta-\Theta_0)}, \tag{5.6}$$

where

$$a_n=A_n=\frac{C(-1)^n}{4\mu i^{n-1}}\frac{\mathrm{J}_n'(KA)}{\mathrm{H}_n'(KA)}\mathrm{H}_n(KR_0). \tag{5.7}$$

Mapping back to the deformed configuration, using (5.2) in (5.6), the scattered field in the deformed configuration may therefore be written as

$$w_s(r)=\sum_{n=-\infty}^{\infty}(-i)^n a_n \mathrm{H}_n\big(k\sqrt{r^2+M}\big)e^{in(\theta-\theta_0)}, \tag{5.8}$$

where we note that the scattering coefficients $a_n$ depend on the *initial* distance $R_0$ between the centre of the void and the source location, and the *undeformed* void radius $A$. They are completely unchanged from those in the undeformed configuration (i.e., $a_n=A_n$). In particular the scattering coefficients $a_n$ are *completely unaffected by both the change in void radius and the change in the distance between the centre of the void and the source location*. The wave field in the vicinity of the void *is* significantly affected by the pre-stress (and change in radius of the void) due to the argument of the Hankel functions in (5.8) but this does not affect what is seen in the far-field, i.e., the scattering coefficients $a_n$.

Note that taking the limit $R_0\to\infty$ in (5.7) with $C$ as defined in (2.10) simulates plane wave incidence in this problem. As described above we do not assume an incident plane wave from the outset since this induces numerous unnecessary difficulties, and moreover the line source problem is more physically meaningful.

To conclude, we note that in general (i.e., for more complicated strain energy functions) mappings of the type specified in (5.2) do not reduce the problem to Helmholtz' equation in the mapped variables, although often the governing equation does have more simple form. This means that the scattering coefficients in these more general problems *will* be modified by the pre-stress.

# 6 Conclusions

In this article we have studied the problem of horizontally-polarized shear wave scattering from a cylindrical void in a pre-stressed incompressible neo-Hookean material. The pre-stress consists of a uniform longitudinal stretch and an in-plane hydrostatic pressure imposed in the far-field, thus altering the size of the void radius. Importantly, this pre-stress generates an inhomogeneous deformation in the host domain. Scattering is due to an incident field from a line source a distance $R_0$ $(r_0)$ away from the centre of the void in the undeformed (deformed) configuration respectively.

The theory of *small-on-large* was used to derive the incremental equation in the pre-stressed configuration. By mapping back to the undeformed configuration, we have shown that the scattering coefficients $a_n$ in the deformed configuration are completely unchanged if the magnitude of the force associated with the line source remains unchanged (this means that the coefficient of the line source is modified, since this is a force *per unit length*). In particular it is important to note that $a_n$ are independent of the deformed void radius $a$ and the distance $r_0$ between the centre of the void and the line source. It is important to stress that this is a rather special result, dependent on the neo-Hookean form of the strain energy function. Mappings such as (5.2) do not give the same result for more general strain energy functions. Indeed we shall show in a forthcoming article that the more general incompressible Mooney-Rivlin material leads to modified scattering coefficients $a_n$ which *do* depend on $a$ and $r_0$ in general.

The above information is important from a non-destructive testing viewpoint. It appears that for an incompressible neo-Hookean material no conclusions can be drawn regarding the size of the deformed radius of the void by analyzing scattering coefficients associated with incident $SH$ waves.

# Appendix

We shall use the theory of *small-on-large* as developed and presented for general curvilinear coordinates by Green and Zerna [9]. We shall continue to use the lower case notation for variables in the deformed configuration. This is at odds with the notation used by Green and Zerna but it makes more sense in our current setting. Note that Green and Zerna did not consider the specific small-on-large application considered here.

Given the perturbed displacement field (4.1), the modified covariant base vectors are written as

$$\hat{\mathbf{g}}_i = \mathbf{g}_i + \eta \mathbf{g}'_i, \tag{A.1}$$

where $\mathbf{g}'_i = \mathbf{u}'_{,i}$. The covariant derivative with respect to the deformed configuration is defined as

$$u'_i \|_j = u'_{i,j} - \Gamma^k_{ij} u'_k, \tag{A.2}$$

where $\Gamma^k_{ij}$ was introduced in (3.16). With the assumption of the perturbed field in the form in (4.2), the perturbed metric tensor becomes $g_{ij}+\eta g'_{ij}$, where (A.2) can be used to show that the only non-zero components of the incremental covariant metric tensor $g'_{ij}$ are

$$g'_{13}=g'_{31}=\frac{\partial u'_3}{\partial r}, \qquad g'_{23}=g'_{32}=\frac{\partial u'_3}{\partial \theta}. \tag{A.3}$$

Similarly it may be shown that the only non-zero components of the incremental *contravariant* metric tensor are

$$g^{13'}=g^{31'}=-\frac{\partial u'_3}{\partial r}, \qquad g^{23'}=g^{32'}=-\frac{1}{r^2}\frac{\partial u'_3}{\partial \theta}. \tag{A.4}$$

It transpires that the incremental stress tensor is simply

$$\tau^{ij'}=g^{ij'}p, \tag{A.5}$$

where $p$ is defined by (3.13a) with (3.20). Its only non-zero components are thus

$$\tau^{13'}=\tau^{31'}=-p\frac{\partial u'_3}{\partial r}, \qquad \tau^{23'}=\tau^{32'}=-\frac{p}{r^2}\frac{\partial u'_3}{\partial \theta}. \tag{A.6}$$

The incremental equation of motion is therefore given by

$$T^{i3}\|_i=\frac{\partial T^{13}}{\partial r}+\frac{\partial T^{23}}{\partial \theta}+\frac{1}{r}T^{13}=\rho\frac{\partial^2 u'_3}{\partial t^2}, \tag{A.7}$$

where

$$T^{i3}=\tau^{i3'}+\tau^{i1}\frac{\partial u'_3}{\partial r}+\tau^{i2}\frac{\partial u'_3}{\partial \theta}. \tag{A.8}$$

Therefore

$$\frac{\partial}{\partial r}\left[\tau^{13'}+\tau^{11}\frac{\partial u'_3}{\partial r}\right]+\frac{\partial}{\partial \theta}\left[\tau^{23'}+\tau^{22}\frac{\partial u'_3}{\partial \theta}\right]+\frac{1}{r}\left[\tau^{13'}+\tau^{11}\frac{\partial u'_3}{\partial r}\right]=\rho\frac{\partial^2 u'_3}{\partial t^2}, \tag{A.9}$$

which on using (A.6), (3.13a), (3.13b) and (4.2) becomes

$$\left(1+\frac{M}{r^2}\right)\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\left(1-\frac{M}{r^2}\right)\frac{\partial w}{\partial r}+\frac{1}{(r^2+M)}\frac{\partial^2 w}{\partial \theta^2}+k^2 w=0, \tag{A.10}$$

where $k^2=L\rho\omega^2/\mu=LK^2$. This can be written in the neater form

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\left(r+\frac{M}{r}\right)\frac{\partial w}{\partial r}\right]+\frac{1}{(r^2+M)}\frac{\partial^2 w}{\partial \theta^2}+k^2 w=0. \tag{A.11}$$

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