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Analysis of Convolution Quadrature Applied to the Time-Domain Electric Field Integral Equation

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Abstract. We show how to apply convolution quadrature (CQ) to approximate the time domain electric field integral equation (EFIE) for electromagnetic scattering. By a suitable choice of CQ, we prove that the method is unconditionally stable and has the optimal order of convergence. Surprisingly, the resulting semi discrete EFIE is dispersive and dissipative, and we analyze this phenomena. Finally, we present numerical results supporting and extending our convergence analysis.

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1 Introduction

The exterior electromagnetic scattering problem is often solved in the frequency domain, either by integral equations or by a finite element method. However, if the incoming wave is broad band, it may be attractive to solve the problem in the time domain. In this case we can again choose between volume based methods including the finite difference or discontinuous Galerkin methods and time domain integral equations. It is the latter technique that is the subject of this paper.

Historically the main difficulty with the time domain integral equation (TDIE) approach is stability. In recent years this problem has been overcome by using a time domain Petrov-Galerkin method [22] and this method is now the method of choice [1, 17]. However, in order to maintain stability, it is necessary to perform accurate integrations on

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complex domains obtained by intersecting regions between light cones with triangles in the spatial mesh [21]. This rules out simple quadrature on the spatial mesh, and implies that curvilinear patches (necessary for high order boundary representation) are difficult to implement because of the potentially much more complex domain of integration. In addition it is not easy to account for dispersive and dissipative media in such schemes because it is necessary to have an expression for the fundamental solution (although several engineering approaches have been suggested for specific media [14]). Convolution Quadrature (CQ) offers a potential alternative that is the subject of this paper.

We now detail the problem to be solved. Suppose a perfect conductor occupies a bounded Lipschitz polyhedron $\Omega\subset\mathbb{R}^3$ with boundary $\Gamma:=\partial\Omega$. Let $\Omega^e:=\mathbb{R}^3\backslash\overline{\Omega}$. In addition suppose Γ is connected and simply connected. The time domain electromagnetic scattering problem is then to find $\mathscr{E}:=\mathscr{E}(\boldsymbol{x},t)\in\mathbf{H}(\mathrm{curl},\Omega^e)$ such that

$$\begin{aligned}\mathscr{E}_{tt}+\mathrm{curl}\,\mathrm{curl}\,\mathscr{E}=0, &\quad \text{in}\,\Omega^e\times[0,T],\\ \mathscr{E}\times\boldsymbol{n}=\boldsymbol{g}, &\quad \text{on}\,\Gamma\times[0,T],\\ \mathscr{E}(\cdot,0)=\mathscr{E}_t(\cdot,0)=0, &\quad \text{on}\,\Omega^e.\end{aligned}\tag{1.1}$$

Here, $\boldsymbol{n}$ is the unit outward normal to Γ and $\boldsymbol{g}$ is a given tangential vector field on Γ, vanishing for $t\le 0$, usually obtained as a suitable trace of the incident electromagnetic field (that is $\boldsymbol{g}=-\mathscr{E}^i\times\boldsymbol{n}$ where $\mathscr{E}^i$ is the incident field taken to be a regular solution of Maxwell's equations). For convenience and without loss of generality, we have set the speed of light $c=1$.

To formulate an integral equation for this problem we can use the ansatz that there is a surface tangential field $\boldsymbol{J}:=\boldsymbol{J}(\boldsymbol{x},t)$ such that, for $\boldsymbol{x}\in\Omega^e$,

$$\begin{aligned}\mathscr{E}_t(\mathbf{x},t)=&M(\partial_t)\mathbf{J}(\mathbf{x},t)\\ :=&\int_0^t\int_\Gamma k(\mathbf{x}-\mathbf{y},t-\tau)\mathbf{J}_{tt}(\mathbf{y},\tau)d\sigma_y d\tau-\mathrm{grad}\int_0^t\int_\Gamma k(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{div}_\Gamma\mathbf{J}(\mathbf{y},\tau)d\sigma_y d\tau,\end{aligned}\tag{1.2}$$

where the time domain fundamental solution in three dimensions is $k(\mathbf{x},t):=\frac{\delta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}$ (this is just the inverse Fourier transform of the usual frequency domain fundamental solution [8]). In addition div_Γ is the surface divergence.

If we now define the surface tangential projection $\Pi_T\mathbf{u}:=\mathbf{n}\times(\mathbf{u}\times\mathbf{n})|_\Gamma$, let $\boldsymbol{x}$ approach Γ in (1.2), and use the boundary data from (1.1), we obtain the Electric Field Integral Equation (EFIE). In particular, we need to find $\boldsymbol{J}$ such that, for all $\boldsymbol{x}\in\Gamma$ and $0\le t\le T$,

$$\begin{aligned}&V(\partial_t)\mathbf{J}(\mathbf{x},t)\\ :=&\Pi_T\int_0^t\int_\Gamma k(\mathbf{x}-\mathbf{y},t-\tau)\mathbf{J}_{tt}(\mathbf{y},\tau)d\sigma_y d\tau-\mathrm{grad}_\Gamma\int_0^t\int_\Gamma k(\mathbf{x}-\mathbf{y},t-\tau)\mathrm{div}_\Gamma\mathbf{J}(\mathbf{y},\tau)d\sigma_y d\tau\\ =&\mathbf{n}\times\mathbf{g}_t(\mathbf{x},t),\end{aligned}\tag{1.3}$$

where grad_Γ is the surface gradient. Once $\boldsymbol{J}$ is computed, we can compute the electric field $\mathscr{E}$ for $\mathbf{x}\notin\Gamma$ by integrating (1.2). Clearly these integral equations need to be carefully formulated in appropriate function spaces and we do this in the next section.

In [18], Lubich introduced a new technique, *convolution quadrature* (CQ) (or operational quadrature), for discretizing convolutions particularly those arising from initial value problems for partial differential equations of hyperbolic and parabolic type. In [19], he used this technique to discretize a time domain integral equation for the Helmholtz equation similar to the EFIE. In particular, he considered multistep methods applied to non homogeneous linear initial boundary value problems for the temporal discretization and a boundary integral Galerkin formulation for the spatial discretization. Under suitable constraints on the multistep method, this technique gives an unconditionally stable and provably convergent time discretization.

Motivated by the work of Hackbusch, Kress and Sauter [10], we have already shown numerical results from applying CQ to Maxwell's equations in [24]. The excellent stability and convergence properties of the method motivate us, in this paper, to analyze convergence. The error analysis we present in Section 3 rests on extending the analysis of [22] to Lipschitz domains (using results from Hiptmair and Schwab [13]), and then uses these results to obtain error estimates using the general theory from [19]. In particular, by modifying Lubich's theory slightly, we obtain a fully discrete error analysis (using Raviart-Thomas finite elements to discretize in space), and relate the method to a particular time semi-discretization of (1.1). This then shows, surprisingly, that the semi-discrete time domain integral equation is dissipative and dispersive, and we analyze this phenomenon for some standard time integration techniques in Section 3.3.

One advantage of using time domain boundary integral equations is that the problem is solved only on the boundary Γ, resulting in dimensional reduction. However a dense matrix problem needs to be solved at each time step, and the solution at previous time steps needs to be stored (at all previous time steps for CQ). However in [10], Hackbusch, Kress and Sauter give a way to save storage by using a cutoff to obtain sparse matrices. In addition H-matrix techniques can be used to further decrease computational cost [4] and the method does tolerate small errors due to numerical quadrature [15].

The structure of the rest of the paper is the following. We introduce the variational formulation in Section 2, which will give the properties of the boundary integral operators. Based on these properties and Lubich's technique, we can analyze the time discretization error in Section 3. In addition, we investigate dispersion and dissipation. In Section 4, we give the full space-time discretization using Raviart-Thomas elements on the surface. Finally, some numerical experiments are performed in the Section 5, showing that the numerical results having the predicted time convergence order and stability property. Finally, we draw some conclusions.

2 Variational formulation

Before giving the precise statement of the EFIE, we give the notation used here (c.f. [6,13]). Let curl_Γ denote the surface curl on Γ, then let

$$V_\gamma := \gamma_T(\mathbf{H}^1(\Omega)), \quad \text{where} \quad \gamma_T : \mathbf{v} \mapsto \mathbf{v} \times \mathbf{n}|_\Gamma,$$

$$V_\Pi := \Pi_T(\mathbf{H}^1(\Omega)), \quad \text{where} \quad \Pi_T : \mathbf{v} \mapsto \mathbf{n}\times(\mathbf{v}\times\mathbf{n})|_\Gamma,$$
$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \left\{ \mathbf{v}\in V'_\Pi \mid \mathrm{div}_\Gamma \mathbf{v} \in H^{-1/2}(\Gamma) \right\},$$
$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma) := \left\{ \mathbf{v}\in V'_\gamma \mid \mathrm{curl}_\Gamma \mathbf{v} \in H^{-1/2}(\Gamma) \right\},$$
$$\mathbf{H}(\mathrm{curl},\Omega^e) := \left\{ \mathbf{v}\in \mathbf{L}^2(\Omega^e), \mathrm{curl}\,\mathbf{u}\in \mathbf{L}^2(\Omega^e) \right\}.$$

The norm for $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ is

$$\|\mathbf{v}\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} := \|\mathbf{v}\|^2_{V'_\Pi} + \|\mathrm{div}_\Gamma \mathbf{v}\|^2_{H^{-1/2}(\Gamma)},$$

where V'_Π is the dual space of V_Π. Similar spaces and norms can be defined for $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$.

The analysis of CQ applied to the EFIE is based on understanding Maxwell's equation in the Laplace domain using the Laplace transform in time. Recall that the Laplace transform of a function f is given by

$$\mathscr{L}f(s) := \int_0^\infty f(t)\exp(-st)dt. \tag{2.1}$$

The corresponding Laplace domain problem, for $s=\sigma+i\eta$ with $\sigma,\eta\in\mathbb{R}$, $\sigma\geq\sigma_0>0$, is to find $\mathbf{E}:=\mathscr{L}(\mathscr{E})\in\mathbf{H}(\mathrm{curl},\Omega^e)$ such that

$$\begin{aligned} \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}+s^2\mathbf{E}=0, &\quad \text{on}\,\Omega^e,\\ \mathbf{E}\times\mathbf{n}=\hat{\mathbf{g}}, &\quad \text{in}\,\Gamma, \end{aligned} \tag{2.2}$$

where $\hat{\mathbf{g}}=\mathscr{L}(\mathbf{g})\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$. This motivates the use of a special norm on $\mathbf{H}(\mathrm{curl},\Omega^e)$ given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^{1,s}(\mathrm{curl},\Omega^e)} = |s|^2\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^e)} + \|\mathrm{curl}\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega^e)}.$$

As for the time domain problem, we have the ansatz representation in the Laplace domain

$$s\mathbf{E}(\mathbf{x}) = M(s)\hat{\mathbf{J}}(\mathbf{x}) := s^2\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\hat{\mathbf{J}}(\mathbf{y})d\sigma_y - \mathrm{grad}\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathrm{div}_\Gamma\hat{\mathbf{J}}(\mathbf{y})d\sigma_y, \quad \mathbf{x}\notin\Gamma, \tag{2.3}$$

and $\hat{\mathbf{J}}$ satisfies the Laplace domain EFIE

$$\begin{aligned} V(s)\hat{\mathbf{J}}(\mathbf{x}) :&= s^2\Pi_T\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\hat{\mathbf{J}}(\mathbf{y})d\sigma_y - \mathrm{grad}_\Gamma\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathrm{div}_\Gamma\hat{\mathbf{J}}(\mathbf{y})d\sigma_y\\ &= s\mathbf{n}\times\hat{\mathbf{g}}(\mathbf{x}), \quad \mathbf{x}\in\Gamma, \end{aligned} \tag{2.4}$$

where

$$\Phi(\mathbf{x},\mathbf{y}) := \frac{e^{-s|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}, \quad \mathbf{x}\neq\mathbf{y},$$

is the fundamental solution of the Helmholtz equation in the Laplace domain. Note that the operator $V(s)$ (or $M(s)$) is the Laplace transform of $V(\partial_t)$ (or$M(\partial_t)$), and $V(s)$: $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\to\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ [12].

Lemma 2.1. *Given* $\hat{\mathbf{g}}\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$*, problem* (2.2) *admits a unique solution* $\mathbf{E}\in\mathbf{H}(\mathrm{curl},\Omega^e)$*, and*

$$\|\mathbf{E}\|_{\mathbf{H}^{1,s}(\mathrm{curl},\Omega^e)}\le C(\Gamma,\sigma_0)|s|^2\|\hat{\mathbf{g}}\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}.$$

Proof. For a smooth domain, the proof follows [22]. For a Lipschitz domain, again motivated by [22], the above inequality can be proved by using techniques from [26, Chapter I] and also trace and lifting theorems in [7,9]. We do not give the detailed proof here. □

In the same way, we also have

Lemma 2.2. *Given* $\mathbf{E}\in\mathbf{H}(\mathrm{curl},\Omega\cup\Omega^e)$ *satisfying* $\mathrm{curl}\,\mathrm{curl}\mathbf{E}+s^2\mathbf{E}=0$*, let* $\varphi:=[\mathrm{curl}\mathbf{E}\times\mathbf{n}]|_\Gamma/s$*, where* $[\cdot]$ *denotes the jump across* Γ*, we have*

$$\|\varphi\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\le C(\Gamma,\sigma_0)|s|\|\mathbf{E}\|_{\mathbf{H}^{1,s}(\mathrm{curl},\Omega\cup\Omega^e)}.$$

To analyse the properties of $V(s)$ (and $V(\partial_t)$), we define the following sesquilinear form b_s on $H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ by

$$b_s(\varphi,\xi):=\int_\Gamma\int_\Gamma s^2\Phi(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})\cdot\overline{\xi(\mathbf{x})}d\sigma_y d\sigma_x+\int_\Gamma\int_\Gamma\Phi(\mathbf{x},\mathbf{y})\mathrm{div}_\Gamma\varphi(\mathbf{y})\overline{\mathrm{div}_\Gamma\xi(\mathbf{x})}d\sigma_y d\sigma_x. \tag{2.5}$$

The following theorem implies the invertibility of $V(s)$ and the existence and uniqueness of the solution to the time domain EFIE (2.4).

Theorem 2.1. *For all* $\hat{\mathbf{g}}\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$*, the integral equation* $V(s)\hat{\mathbf{J}}=s\mathbf{n}\times\hat{\mathbf{g}}$ *has the following equivalent variational formulation: Find* $\hat{\mathbf{J}}\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$*, such that* $\forall\xi\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$*,*

$$b_s(\hat{\mathbf{J}},\xi)=s\int_\Gamma\mathbf{n}(\mathbf{x})\times\hat{\mathbf{g}}(\mathbf{x})\cdot\overline{\xi(\mathbf{x})}d\sigma_x. \tag{2.6}$$

Then b_s *is continuous and coercive on* $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *and*

$$|b_s(\varphi,s\varphi)|\ge C(\Gamma,\sigma_0)\frac{\sigma}{|s|^2}\|\varphi\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},$$

where $C(\Gamma,\sigma_0)$ *is a constant depending on* Γ *and* σ_0*.*

Remark 2.1. The coercivity of $|b_s(\varphi,s\varphi)|$ implies the invertibility of $V(s)$ and the boundedness $\|V(s)^{-1}\|\le C(\Gamma,\sigma_0)|s|^3/\sigma\le C(\Gamma,\sigma_0)|s|^3$. Furthermore, the time domain EFIE $V(\partial_t)$ $\mathbf{J}=\mathbf{n}\times\mathbf{g}_t$ has a unique solution in a suitable time function space, i.e. $V(\partial_t)$ is invertible.

Proof. We sketch the proof. Full detail in the case of a smooth domain can be found in [22] with different s-dependent norms. Our case follows closely Terrasse [22] but generalized to Lipschitz domains. The proof consists of the following steps:

Step 1. The continuity of b_s follows from Bamberger and HaDuong's [3] results for a single layer potential,

$$|b_s(\varphi,\xi)|\le C(\Gamma,\sigma_0)|s|^3\|\varphi\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\|\xi\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},\quad\forall\varphi,\xi\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma). \tag{2.7}$$

Step 2. For $\varphi\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$, define $\mathbf{u}:=M(s)\varphi/s$ in $\Omega\cup\Omega^e$, which satisfies

$$\begin{aligned}\mathrm{curl}\;\mathrm{curl}\mathbf{u}+s^2\mathbf{u}=0, &\quad \text{in } \Omega\cup\Omega^e,\\ [\mathrm{curl}\mathbf{u}\times\boldsymbol{n}]/s=\varphi, &\quad \text{on } \Gamma.\end{aligned}$$

Here $[\cdot]$ denotes the jump across Γ. For any $\mathbf{v}\in\mathbf{H}(\mathrm{curl},\mathbb{R}^3)$, integration by parts and use of the boundary condition shows that

$$a(\mathbf{u},\mathbf{v}):=\int_{\Omega\cup\Omega^e}\mathrm{curl}\mathbf{u}\cdot\mathrm{curl}\overline{\mathbf{v}}+s^2\mathbf{u}\cdot\overline{\mathbf{v}}=\langle s\varphi,\mathbf{v}_T\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$. Then $|a(\mathbf{u},s\mathbf{u})|\geq\sigma\|\mathbf{u}\|^2_{\mathbf{H}^{1,s}(\mathrm{curl},\Omega\cup\Omega^e)}$. Now, define $\mathbf{f}:=\mathbf{u}\times\mathbf{n}|_\Gamma\in\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$, then by the definition of $\mathbf{u}$ and the property of $M(s)$, $V_s\varphi=s\mathbf{n}\times\mathbf{f}$. So

$$a(\mathbf{u},\mathbf{u})=\langle s\varphi,\mathbf{n}\times\mathbf{f}\rangle.$$

It is not hard to conclude the following equality

$$a(\mathbf{u},\mathbf{u})=\frac{s}{\bar{s}}\overline{b_s(\varphi,\varphi)}.$$

Step 3. Using the results of step 2 and Lemma 2.2, we obtain

$$|b_s(\varphi,s\varphi)|=|a(\mathbf{u},s\mathbf{u})|\geq\sigma\|\mathbf{u}\|^2_{\mathbf{H}^{1,s}(\mathrm{curl},\Omega\cup\Omega^e)}\geq C(\Gamma,\sigma_0)\frac{\sigma}{|s|^2}\|\varphi\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},$$

which completes the proof of the theorem. □

3 Time discretization

Having proved the properties of $V(s)$, we can discretize in time by using CQ [19] which we now describe. Using the definition of Laplace transform (2.1), for $s=\sigma+i\eta$ with $\sigma\geq\sigma_0>0$ we have $\mathscr{L}(\delta(\cdot-R))(s)=\exp(-sR)$, and the corresponding inverse Laplace transform gives

$$\delta(t-R)=\frac{1}{2\pi}\int_{\sigma+i\mathbb{R}}\exp(st)\exp(-sR)ds.$$

Using this expression and interchanging the order of integration

$$\int_0^t\delta(t-R-\tau)f(\tau)d\tau=\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}}\exp(-sR)F(s,t)ds,$$

where $F(s,t)=\int_0^t\exp(s(t-\tau))f(\tau)d\tau$. For fixed s, F satisfies the ordinary differential equation (ODE)

$$F_t=sF+f,\quad t>0,\quad \text{with } F(s,0)=0.$$

To discretize the convolution, we proceed to approximate F by a suitable ODE method. Define the step size $\Delta t > 0$ and let $t_n = n\Delta t$, we shall use a multistep method with k-steps as follows:

$$\sum_{j=0}^{k} \alpha_j F_{n-j} = \Delta t \sum_{j=0}^{k} \beta_j (sF_{n-j} + f_{n-j}),$$

where $f_n = f(t_n)$, and α_j and β_j are coefficients of the multistep method. Then the discrete convolution can be defined by

$$\int_0^{t_n} \delta(t_n - R - \tau) f(\tau) \approx \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} \exp(-sR) F_n(s) ds = \sum_{j=0}^{\infty} w_{n-j}^{(0)}(R) f_j,$$

where $w_j^{(0)}(R)$ are quadrature weights obtained as follows. We define

$$D(z) := \frac{\alpha_0 + \alpha_1 z + \cdots + \alpha_k z^k}{\beta_0 + \beta_1 z + \cdots + \beta_k z^k} = \sum_{n=0}^{\infty} d_n z^n,$$

where α_j and β_j are the multistep coefficients mentioned before. For the multistep methods we will use, $\alpha_0/\beta_0 > 0$ [19].

Using generating functions, we can compute the quadrature weights by

$$\sum_{j=0}^{\infty} w_j^{(0)} z^j = \exp\left(-R \frac{D(z)}{\Delta t}\right).$$

To apply Lubich's theory to the EFIE, it is necessary to choose the multistep method to be A-stable, which implies that $\Re(D(z)) > 0$ for $|z| < 1$. Note that the multistep method is of order p if $D(e^{-\Delta t})/\Delta t = 1 + \mathcal{O}(\Delta t^p)$ as $\Delta t \to 0$, but the order p for an A-stable method is at most 2. One example of second order A-stable methods is the Backward Differentiation Formulas 2 (BDF2): $D(z) = (3 - 4z + z^2)/2$, another is the Trapezoidal rule: $D(z) = (2 - 2z)/(1 + z)$. The standard first order example is the Backward Euler (BE) method where $D(z) = 1 - z$. For BE we have

$$\exp\left(-R \frac{1-z}{\Delta t}\right) = \exp\left(-\frac{R}{\Delta t}\right) \sum_{j=0}^{\infty} \frac{1}{n!} \left(\frac{R}{\Delta t}\right)^n z^n,$$

and so

$$w_j^{(0)}(R) = \frac{\exp(-R/\Delta t)}{n!} \left(\frac{R}{\Delta t}\right)^n.$$

In Eq. (1.3), time derivatives are also involved. By the same arguments, using the fact that s corresponds to a time derivative, we can obtain, for all z,

$$\int_0^{t_n} \delta(t_n - R - \tau) f_{tt}(\tau) d\tau \approx \sum_{j=0}^{n} w_{n-j}^{(2)}(R) f_j,$$

with

$$\sum_{j=0}^{\infty} w_j^{(2)}(R) z^j = \left(\frac{D(z)}{\Delta t}\right)^2 \exp\left(-R\frac{D(z)}{\Delta t}\right).$$

More generally, we can use the FFT to compute the coefficients numerically [19]. Convolution Quadrature can be interpreted via the discrete Laplace transform (the z-transform) and is then seen to correspond to techniques in control engineering (see [24] for more details).

3.1 Time discretization error analysis

Using convolution quadrature, we have the time discretized EFIE

$$\begin{aligned} V(\partial_t^{\Delta t})\mathbf{J}_n(\mathbf{x}) = \Pi_T \int_\Gamma \frac{\sum_{j=0}^n w_{n-j}^{(2)}(|\mathbf{x}-\mathbf{y}|)\mathbf{J}_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} d\sigma_y \\ -\mathrm{grad}_\Gamma \int_\Gamma \frac{\sum_{j=0}^n w_{n-j}^{(0)}(|\mathbf{x}-\mathbf{y}|)\mathrm{div}_\Gamma \mathbf{J}_j(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} d\sigma_y = \mathbf{n}(\mathbf{x}) \times \mathbf{g}_t(\mathbf{x}, t_n). \end{aligned} \tag{3.1}$$

Note that the quadrature coefficients $w_j^{(2)}(R)$ and $w_j^{(0)}(R)$ are zero for $j<0$, and this implies that the scheme can be marched on in time since $\mathbf{J}_n$ is independent of $\mathbf{J}_j$ for $j>n$.

Theorem 3.1. *Consider a convolution quadrature method based on an A-stable multistep method of order p and having no poles for $D(z)$ on the unit circle. Given boundary data $\mathbf{g}$ with sufficiently many time derivatives and vanishing at $t=0$, the error in the surface current is bounded by*

$$\left(\Delta t \sum_{n=0}^{N} \|\mathbf{J}_n - \mathbf{J}(\cdot, t_n)\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)}\right)^{1/2} \leq C \Delta t^p \|\mathbf{n} \times \mathbf{g}\|_{\mathbf{H}^{r,-1/2}([0,T]; \mathrm{curl}_\Gamma, \Gamma)}, \tag{3.2}$$

where r satisfies $r \geq p+4$. The coefficient C depends on Γ and $T = N\Delta t < \infty$, but is independent of $\Delta t < \Delta t_0$, for some $\Delta t_0 > 0$.

Remark 3.1. The space

$$\begin{aligned} \mathbf{H}^{\alpha,-1/2}([0,T]; \mathrm{curl}_\Gamma, \Gamma) := \Big\{ \mathbf{v}(\cdot, t) \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \forall t \in [0,T] \\ \text{and } \|\mathbf{v}(\cdot, t)\|_{\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)} \in H^\alpha([0,T]) \Big\}. \end{aligned}$$

Remark 3.2. The condition on the poles of $D(z)$ rules out the trapezoidal rule. Admissible multistep methods are BE and BDF2. BDF3 is ruled out since it is not A-stable (nevertheless, as we shall see, it can give conditionally convergent results).

Proof. The result follows from the remark after Theorem 2.1 and Lubich's Theorem 5.1 of [19]. □

Once we obtain the time discretized surface current $\mathbf{J}_n(\mathbf{x})$, we can compute

$$\mathbf{u}^{\Delta t}(\mathbf{x}) = M(\partial_t^{\Delta t})\mathbf{J}^{\Delta t}(\mathbf{x}), \tag{3.3}$$

where $\mathbf{J}^{\Delta t} = (\mathbf{J}_n, \mathbf{J}_{n-1}, \cdots, \mathbf{J}_0)^T$ and similarly for $\mathbf{u}^{\Delta t}$. Then we solve $\partial_t^{\Delta t}\mathscr{E}^{\Delta t} = \mathbf{u}^{\Delta t}$, where $\partial_t^{\Delta t}\mathscr{E}^{\Delta}$ denotes applying the multistep method to the time discrete electric field each time step. Now we have the following important result connecting the boundary integral problem to the electrical field:

Theorem 3.2. *Given smooth compatible data* $\mathbf{g}$ *and a multistep method as in the previous theorem,* $\mathbf{u}^{\Delta t}$ *computed by* (3.3) *satisfies the semi-discrete Maxwell system obtained by applying the multistep method at each timestep:*

$$\begin{aligned} \mathrm{curl}\,\mathrm{curl}\,\mathbf{u}^{\Delta t} + (\partial_t^{\Delta t})^2\mathbf{u}^{\Delta t} &= 0, \quad \text{in } \Omega^e, \\ \mathbf{u}^{\Delta t} \times \mathbf{n} &= \mathbf{g}_t, \quad \text{on } \Gamma, \end{aligned} \tag{3.4}$$

where $\mathbf{u}^{\Delta t}$ *vanishes for all* $t_n \le 0$*. Moreover, given the same assumptions as in* Theorem 3.1, *and denoting* $\mathbf{u}_n = \mathbf{u}^{\Delta t}(\cdot, n\Delta t)$*, the error is bounded by*

$$\begin{aligned} &\left(\Delta t \sum_{n=0}^{N} \|\mathbf{u}_n - \mathbf{u}(\cdot, n\Delta t)\|^2_{\mathbf{H}(\mathrm{curl},\Omega^e)} + \|\partial_t^{\Delta t}\mathbf{u}_n - \partial_t \mathbf{u}(\cdot, n\times\Delta t)\|^2_{L^2(\Omega^e)^3} \right)^{1/2} \\ &\le C\Delta t^p \|\mathbf{g}\|_{\mathbf{H}^{r,-1/2}([0,T];\mathrm{div}_\Gamma,\Gamma)}, \end{aligned} \tag{3.5}$$

where $r \ge p+3$*. The pointwise error in time bounded as follows:*

$$\begin{aligned} &\|\mathbf{u}_n - \mathbf{u}(\cdot, n\Delta t)\|_{\mathbf{H}(\mathrm{curl},\Omega^e)} + \|\partial_t^{\Delta t}\mathbf{u}_n - \partial_t\mathbf{u}(\cdot, n\Delta t)\|_{L^2(\Omega^e)} \\ \le &C\Delta t^p |\log(\Delta t)| \|\mathbf{g}\|_{\mathbf{H}^{r',-1/2}([0,T];\mathrm{div}_\Gamma,\Gamma)}, \end{aligned} \tag{3.6}$$

where $r' \ge p+7/2$*, and* C *depends on* $T = N\Delta t < \infty$ *but is independent of* $\Delta t \le \Delta t_0$.

Remark 3.3. The space

$$\begin{aligned} \mathbf{H}^{\alpha,-1/2}([0,T];\mathrm{div}_\Gamma,\Gamma) := \Big\{ &\mathbf{v}(\cdot,t) \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\, \forall t \in [0,T] \\ &\text{and } \|\mathbf{v}(\cdot,t)\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \in H^\alpha([0,T]) \Big\}. \end{aligned}$$

Proof. This is a straightforward extension of the proof of Theorem 5.2 of [19] using an appropriate integral representation for the solution of the Laplace domain Maxwell system (2.2), and Lemma 2.1. □

3.2 Dispersion and dissipation

The connection between the semi-discrete boundary integral problem and the semi-discrete Maxwell system in Theorem 3.2 shows that the method will be dispersive and probably dissipative. To analyze the dispersion and dissipation in the method, we suppose that the semi discrete Maxwell system is satisfied in all space. Note that $(\partial_t^{\Delta t})^2\mathbf{u}^{\Delta t}+\text{curl}\text{curl}\mathbf{u}^{\Delta t}=0$, implies $\text{div}((\partial_t^{\Delta t})^2\mathbf{u}^{\Delta t})=0$. Hence if $\text{div}\mathbf{u}^{\Delta t}=0$ for $t_n\le 0$, we have $\text{div}\mathbf{u}^{\Delta t}=0$ for all t_n, and so $(\partial_t^{\Delta t})^2\mathbf{u}^{\Delta t}=\Delta\mathbf{u}^{\Delta t}$. Then it suffices to consider the semi discrete wave equation

$$(\partial_t^{\Delta t})^2u^n=\Delta u^n.$$

Using the BE method this becomes

$$\frac{u^n-2u^{n-1}+u^{n-2}}{\Delta t^2}=\Delta u^n.$$

Then we seek a solution of the form $u^n=\exp(i(\mathbf{k}\cdot\mathbf{x}-\omega_{\Delta t}t_n))$, and substituting into the above equation we get

$$\frac{4\exp(i\omega_{\Delta t}\Delta t)}{\Delta t^2}\sin^2(\omega_{\Delta t}\Delta t/2)=|\mathbf{k}|^2.$$

We then obtain the asymptotic expansion

$$\omega_{\Delta t}=\pm|\mathbf{k}|-\frac{1}{2}i|\mathbf{k}|^2\Delta t\mp\frac{1}{2}|\mathbf{k}|^3\Delta t^2+\cdots\quad\text{as }\Delta t\to 0.$$

The method would have no dispersion or dissipation error if $|\omega|=\pm|\mathbf{k}|$. But in our case, we conclude the method does show both dispersion and dissipation. From the second term on the right hand side we see the method is dissipative and from the third term we see it has dispersion with a phase error of $\mathcal{O}(\Delta t^2)$. However the method is isotropic ($\omega_{\Delta t}$ only depends on $|\mathbf{k}|$ not the direction of $\mathbf{k}$).

By similar calculations, we have the relations of $\omega_{\Delta t}$ and $\mathbf{k}$ for BDF2, BDF3 and the trapezoidal rule respectively

$$\begin{aligned}
\text{BDF2}:&\quad \omega_{\Delta t}=\pm|\mathbf{k}|\mp\frac{1}{3}|\mathbf{k}|^3\Delta t^2+\frac{i}{4}|\mathbf{k}|^4\Delta t^3+\cdots,\\
\text{BDF3}:&\quad \omega_{\Delta t}=\pm|\mathbf{k}|+\frac{i}{4}|\mathbf{k}|^4\Delta t^3\mp\frac{3}{10}|\mathbf{k}|^5\Delta t^4+\cdots,\\
\text{Trapezoidal}:&\quad \omega_{\Delta t}=\pm|\mathbf{k}|\mp\frac{1}{12}|\mathbf{k}|^3\Delta t^2\pm\frac{1}{80}|\mathbf{k}|^5\Delta t^4+\cdots.
\end{aligned}$$

Interestingly even through that the trapezoidal rule does not fit with the assumption of Theorem 3.1 it gives rise to a dispersive but not dissipative method (see [4] for further comments on how to use the trapezoidal rule). BDF3 also does not fit the theory, but can perhaps be used in practice provided the time of integration is not too long.

4 Full discretization

4.1 Space discretization

To obtain a fully discrete problem, we use Raviart-Thomas elements [5] of degree k denoted $\mathcal{RT}_k(\Gamma_h)$ to approximate $\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$, where Γ_h denotes a non-degenerate regular and quasi-uniform triangulation of Γ with mesh size $h>0$. The Galerkin discretization of the Laplace domain variational problem (2.6) is to find $\hat{\mathbf{J}}_h\in\mathcal{RT}_k(\Gamma_h)$, for any $\xi_h\in\mathcal{RT}_k(\Gamma_h)$

$$\begin{aligned} b_s(\hat{\mathbf{J}}_h,\xi_h)=&\int_\Gamma\int_\Gamma s^2\Phi(\mathbf{x},\mathbf{y})\hat{\mathbf{J}}_h(\mathbf{y})\cdot\overline{\xi_h(\mathbf{x})}d\sigma_y d\sigma_x+\int_\Gamma\int_\Gamma\Phi(\mathbf{x},\mathbf{y})\mathrm{div}_\Gamma\hat{\mathbf{J}}_h(\mathbf{y})\overline{\mathrm{div}_\Gamma\xi_h(\mathbf{x})}d\sigma_y d\sigma_x\\ =&s\int_\Gamma\mathbf{n}(\mathbf{x})\times\hat{\mathbf{g}}(\mathbf{x})\cdot\overline{\xi_h(\mathbf{x})}d\sigma_x. \end{aligned}\tag{4.1}$$

Lemma 4.1. *The discrete Laplace domain Galerkin variational problem* (4.1) *has a unique solution* $\hat{\mathbf{J}}_h$ *in* $\mathcal{RT}_k(\Gamma_h)$ *with quasi-optimal convergence*

$$\|\hat{\mathbf{J}}-\hat{\mathbf{J}}_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\le C(\Gamma,\sigma_0)\inf_{\mathbf{v}_h\in\mathcal{RT}_k(\Gamma)}|s|^6\|\hat{\mathbf{J}}-\mathbf{v}_h\|_{H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)},\tag{4.2}$$

and stability

$$\|\hat{\mathbf{J}}_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\le C(\Gamma,\sigma_0)\frac{1}{\sigma}|s|^4\|\mathbf{n}\times\hat{\mathbf{g}}\|_{\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)},\tag{4.3}$$

where the constant $C(\Gamma,\sigma_0)$ *is independent of* s, $\hat{\mathbf{J}}$ *and* h.

Proof. Because this is a Galerkin method, we have

$$b_s(\hat{\mathbf{J}}-\hat{\mathbf{J}}_h,s(\hat{\mathbf{J}}-\hat{\mathbf{J}}_h))=b_s(\hat{\mathbf{J}}-\hat{\mathbf{J}}_h,s(\hat{\mathbf{J}}-\xi_h)),\quad\forall\xi_h\in\mathcal{RT}_k(\Gamma_h),$$

and using the continuity and coercivity of b_s from Theorem 2.1, we get the inequality

$$\begin{aligned} C(\Gamma,\sigma_0)\frac{\sigma}{|s|^2}\|\hat{\mathbf{J}}-\hat{\mathbf{J}}_h\|^2_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\le&|b_s(\hat{\mathbf{J}}-\hat{\mathbf{J}}_h,s(\hat{\mathbf{J}}-\hat{\mathbf{J}}_h))|=|b_s(\hat{\mathbf{J}}-\hat{\mathbf{J}}_h,s(\hat{\mathbf{J}}-\xi_h))|\\ \le&C(\Gamma,\sigma_0)|s|^4\|\hat{\mathbf{J}}-\hat{\mathbf{J}}_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\|\hat{\mathbf{J}}-\xi_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}. \end{aligned}\tag{4.4}$$

To prove the stability, we consider the discretized sesquilinear form

$$b_s(\hat{\mathbf{J}}_h,s\hat{\mathbf{J}}_h)=\int_\Gamma s\mathbf{n}(\mathbf{x})\times\hat{\mathbf{g}}(\mathbf{x})\cdot s\hat{\mathbf{J}}_h(\mathbf{x})d\sigma_x,\tag{4.5}$$

then by coercivity of b_s again

$$\|\hat{\mathbf{J}}_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\le C(\Gamma,\sigma_0)\frac{1}{\sigma}|s|^4\|\mathbf{n}\times\hat{\mathbf{g}}\|_{\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)}.\tag{4.6}$$

This completes the proof. □

Using the error estimates [13, Lemma 8.1], we obtain the following theorem:

Theorem 4.1. *Suppose* $\mathbf{J}\in\mathbf{H}^\alpha(\mathrm{div}_\Gamma,\Gamma)$ *for some* $\alpha>0$*. Let* $\varepsilon>0$ *and* $h\le h_0$ *for some* $h_0>0$*. Then if* $\hat{\mathbf{J}}_h$ *is the solution of discrete Galerkin variational problem* (4.1) *and* $\hat{\mathbf{J}}$ *is the solution of Galerkin variational formulation* (2.6)

$$\|\hat{\mathbf{J}}-\hat{\mathbf{J}}_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}\le C(\Gamma,\sigma_0)h^\beta|s|^6\|\hat{\mathbf{J}}\|_{\mathbf{H}^\alpha(\mathrm{div}_\Gamma,\Gamma)}, \tag{4.7}$$

where C *is the constant independent of* $\hat{\mathbf{J}}$*,* s*,* h*, and* $\beta=\min\{3/2-\varepsilon,\alpha+1/2-\varepsilon,1+k^*,\alpha+k^*\}$*, and* $k^*>0$*.*

Remark 4.1. We refer to [7,13] for the definition of k^*, which depends on the shape of Γ. From [13], $k^*\ge 1/2$ if at most three edges of Γ meet at a vertex. Otherwise it depends on how faces meet at vertices and can be arbitrarily small (and positive) in adverse structures [7]. A quasi-uniform mesh is required to use the results of [13].

Proof. Following the proof of [13, Lemma 8.1, Theorem 8.2], $r=\min\{k^*,\max\{0,1/2-\varepsilon\}\}$, and using the quasi-uniformity of the mesh, there exists $\mathbf{v}_h\in\mathcal{RT}_k(\Gamma_h)$, such that

$$\begin{aligned}\|\hat{\mathbf{J}}-\mathbf{v}_h\|_{\mathbf{H}^{-1/2}(\Gamma)}&\le Ch^r\|\hat{\mathbf{J}}-\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div}_\Gamma,\Gamma)}\le Ch^{r+\min\{1,\alpha\}}\|\hat{\mathbf{J}}\|_{\mathbf{H}^\alpha(\mathrm{div}_\Gamma,\Gamma)},\\ \|\mathrm{div}_\Gamma(\hat{\mathbf{J}}-\mathbf{v}_h)\|_{H^{-1/2}(\Gamma)}&\le Ch^{1/2}\|\mathrm{div}_\Gamma(\hat{\mathbf{J}}-\mathbf{v}_h)\|_{L^2(\Gamma)}\le Ch^{1/2}\|\hat{\mathbf{J}}-\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div}_\Gamma,\Gamma)}\\ &\le Ch^{1/2+\min\{1,\alpha\}}\|\hat{\mathbf{J}}\|_{\mathbf{H}^\alpha(\mathrm{div}_\Gamma,\Gamma)}.\end{aligned}$$

The error estimate follows from the above result. □

Denote $I_h\mathbf{J}:=\mathscr{L}^{-1}(\hat{\mathbf{J}}_h)$ the inverse Laplace transform of $\hat{\mathbf{J}}_h$, then we have the stability and convergence theorem:

Theorem 4.2. *Suppose* $\mathbf{n}\times\mathbf{g}\in\mathbf{H}^{l+4}([0,T];\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma))$ *for* $l\in\mathbb{R}$*, and* $h\le h_0$ *for some* $h_0>0$*, then*

$$\begin{aligned}&\|I_h\mathbf{J}\|^2_{H^l([0,T];\mathbf{H}^{-1/2}(\Gamma))}+\|\mathrm{div}_\Gamma I_h\mathbf{J}\|^2_{H^l([0,T];\mathbf{H}^{-1/2}(\Gamma))}\\ \le& C(\Gamma,\sigma_0,T)\Big\{\|\mathbf{n}\times\mathbf{g}\|^2_{H^{l+4}([0,T];\mathbf{H}^{-1/2}(\Gamma))}+\|\mathrm{curl}_\Gamma(\mathbf{n}\times\mathbf{g})\|^2_{H^{l+4}([0,T];\mathbf{H}^{-1/2}(\Gamma))}\Big\},\end{aligned} \tag{4.8}$$

and if $\mathbf{J}\in\mathbf{H}^{l+6}([0,T];\mathbf{H}^\alpha(\mathrm{div}_\Gamma,\Gamma))$ *for some constant* $\alpha>0$*, the*

$$\begin{aligned}&\|\mathbf{J}-I_h\mathbf{J}\|^2_{H^l([0,T];\mathbf{H}^{-1/2}(\Gamma))}+\|\mathrm{div}_\Gamma(\mathbf{J}-I_h\mathbf{J})\|^2_{H^l([0,T];\mathbf{H}^{-1/2}(\Gamma))}\\ \le& C(\Gamma,\sigma_0,T)h^{2\beta}\|\Big\{\|\mathbf{J}\|^2_{H^{l+6}([0,T];\mathbf{H}^\alpha(\Gamma))}+\|\mathrm{div}_\Gamma\mathbf{J}\|^2_{H^{l+6}([0,T];\mathbf{H}^\alpha(\Gamma))}\Big\},\end{aligned} \tag{4.9}$$

where $\beta=\min\{3/2-\varepsilon,\alpha+1/2-\varepsilon,1+k^*,\alpha+k^*\}$*, and* k^* *is the constant in* Theorem 4.1.

4.2 Full discretization

We now consider the full space-time discrete problem of finding $\mathbf{J}_h^{\Delta t}$ in $\mathcal{RT}_k(\Gamma_h)$ at each time step, such that

$$\langle V(\partial_t^{\Delta t})\mathbf{J}_h^{\Delta t},\xi_h\rangle = \langle \mathbf{n}\times\mathbf{g}_t^{\Delta t},\xi_h\rangle, \quad \forall \xi_h \in \mathcal{RT}_k(\Gamma_h) \subset \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma). \tag{4.10}$$

Here again $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ and $H^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$. By splitting $\mathbf{J}-\mathbf{J}_h^{\Delta t}=(\mathbf{J}-I_h\mathbf{J})+(I_h\mathbf{J}-\mathbf{J}_h^{\Delta t})$, where $\mathbf{J}_h^{\Delta t}$ is the full spatial-time approximation, we have the following theorem:

Theorem 4.3. *Assume the conditions of* Theorem 3.1 *on the CQ scheme and assume a regular and quasi-uniform mesh on* Γ. *Given compatible boundary data* $\mathbf{g}$, *and the Raviart-Thomas method* $\mathcal{RT}_k(\Gamma_h)$ *for the space discretization, the fully discrete problem* (4.10) *has a unique* $\mathbf{J}_h^{\Delta t}$ *that converges unconditionally as follows*

$$\|\mathbf{J}(\cdot,n\Delta t)-\mathbf{J}_h^{n\Delta t}\|^2_{\mathbf{H}^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma(\mathbf{J}(\cdot,n\Delta t)-\mathbf{J}_h^{n\Delta t})\|^2_{H^{-1/2}(\Gamma)} \le \mathcal{O}(\Delta t^{2p})+\mathcal{O}(h^{2\beta}). \tag{4.11}$$

Proof. This result follows from Theorems 3.1 and 4.2 immediately. □

5 Numerical experiments

We consider a sphere with diameter 0.5m. The scatterer is illuminated by a Gaussian plane wave with the center frequency f_0=120MHz and band width f_{bw}=40MHz (see [24]) for details. This choice of parameters is not specially significant, but typical of those used to test engineering algorithms. For spatial discretization, we use $\mathcal{RT}_1(\Gamma_h)$ with curved patches. For the sphere, if we use the EFIE, due to the well known low frequency instability, we observe a long term increase in the solution at later times using the basic CQ EFIE, see Fig. 1. Note that the error bounds in Theorem 4.3 depend on T so allowing growth in the solution that would be controlled as $\Delta t \to 0$ for fixed T. This instability can be removed by using a loop-tree decomposition [25]. All subsequent results are computed by the stabilized method. Next the surface current is computed by three methods: BE, BDF2 and BDF3 shown in Fig. 2. Even though BDF3 is not covered by our theory, it works well in this case.

Fig. 3 shows the radar cross section (RCS) of the sphere at 100MHz computed from the surface current found by CQ with BE, BDF2 and BDF3 with the oversampling factor $\psi=5$ and $\psi=10$, where the oversampling factor $\psi=1/(2(f_0+f_{bw})\Delta t)$. To compute the RCS, the Fourier transform of the surface current is used in appropriate surface integrals (see [2]). The error is computed by comparing the CQ RCS to the RCS found by the standard frequency domain method EFIE on the same mesh called the method of moments (MoM) using an identical spatial discretization. So the reported error only includes time discretization error. The convergence of BE, BDF2 and BDF3 are of order 1, 2 and 3 respectively as shown in the Fig. 4. They agree with the theoretically predicted error order.

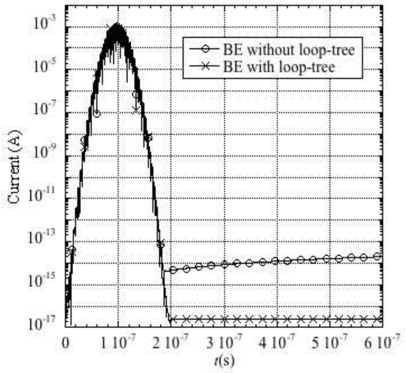

Figure 1: The average surface current computed by EFIE with/without the stabilization.

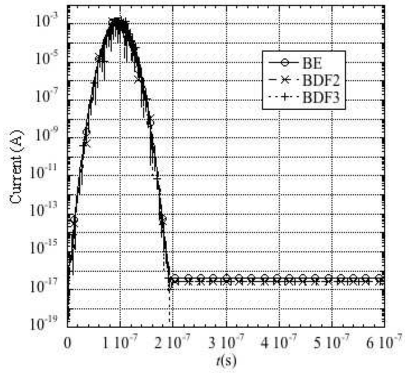

Figure 2: The average surface current by BE, BDF2 and BDF3 with stabilization. Note that the y-axis differs from that Fig. 1 to accommodate the undershoot of BDF3.

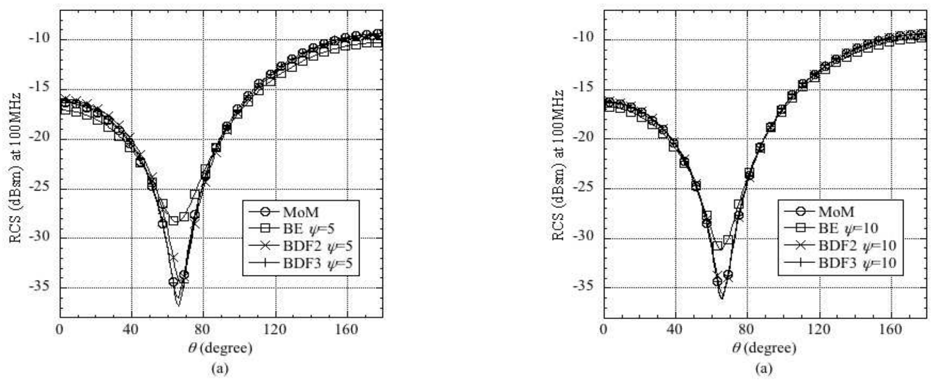

Figure 3: The RCS with oversampling factor $\psi=5$ and $\psi=10$ for the sphere as a function of elevation angle.

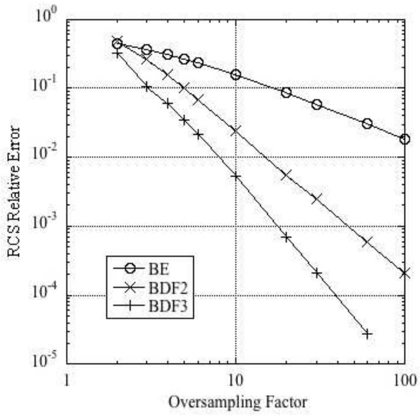

Figure 4: Relative least square RCS error as a function of the oversampling factor. The convergence rates agree with our theory for BE and BDF2.

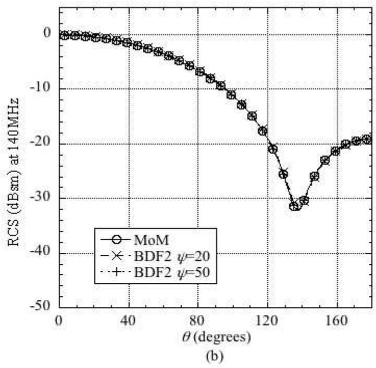

Figure 5: The RCS for dielectric cube by MoM and CQ using BDF2 (with two different oversampling factors).

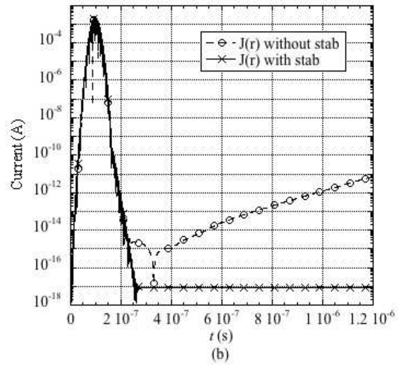

Figure 6: The surface current computed by EFIE using CQ with BDF2 for the cube scatterer with oversampling factor $\psi = 10$. Results with and without stabilization are shown.

The CQ method is applicable to more general problems, for example, scattering by a penetrable dielectric. This involves a system of time dependent integral equations [23]. This problem has not yet been covered by theory, but we have positive numerical results. For example consider the cube scatterer with side length 0.5m filled with a non-dispersive dielectric having relative permittivity $\varepsilon_r = 3$ and permeability $\mu_r = 2$. The incident field is same as for the previous example. Fig. 5 shows the RCS, and Fig. 6 shows the current computed by BDF2 with and without stabilization. The results show good agreement with the RCS computed by a frequency domain (MoM) code, and suggestion that stabilization is more necessary in the case for a perfect conductor (compare Figs. 1 and 6)

6 Conclusion

For the time domain electromagnetic scattering problem exterior to a perfectly conducting Lipschitz polyhedral domain, we can derive an unconditionally stable time domain method by using Lubich's CQ technique for time discretization based on an A-stable multistep method. We could also use CQ with an implicit Runge-Kutta method, which can give a higher order of convergence and similar stability. The corresponding error estimates have not been done (for acoustic wave problems see [4, 20]). For transmission problems, we may also use CQ, as we have seen in Figs. 5 and 6. The case of acoustic waves has been investigated by Laliena and Sayas [16]. We believe that similar error estimates can be achieved for Maxwell's equations, and this is currently under investigation. Other future work will also include a proof of convergence of the method for a general physical dispersive and dissipative medium.

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