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Matched Asymptotic Expansions of the Eigenvalues of a 3-D Boundary-Value Problem Relative to Two Cavities Linked by a Hole of Small Size

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Abstract. In this article, we consider a domain consisting of two cavities linked by a hole of small size. We derive a numerical method to compute an approximation of the eigenvalues of an elliptic operator without refining in the neighborhood of the hole. Several convergence rates are obtained and illustrated by numerical simulations.

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1 Introduction

1.1 Motivation

In a lot of physical problems, the boundary of the computational domain is perforated. This configuration can lead to numerical difficulties when the diameter of the holes are really smaller than the other characteristic lengths. Indeed, it can be very costly to compute a sharp numerical approximation of the solution of such problems for two main reasons: With a standard method like finite elements or finite differences, a refined mesh

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cannot be avoided in the neighborhood of the hole; the mesh generation of a perforated structure can be a hard task.

Many authors have studied the effect of the perforation of the boundaries both from the theoretical and the numerical point of views, see for example [13, 16–19]. However, fewer results have been obtained for the eigenvalue problem in the case of a three dimensional domain.

In [10], Gadyl'shin considered a two dimensional domain consisting of two domains linked by a small hole. He derived a complete asymptotic expansion of the scattering frequencies of the Laplacian operator equipped with Dirichlet boundary condition. In [2], these results were extended to the eigenvalues and eigenvectors of an elliptic operator with varying coefficients. In this paper, we are interested in a three dimensional configuration with varying coefficients and Neumann boundary condition.

1.2 A Neumann eigenvalue problem

1.2.1 The geometry

Let $\Omega_{\rm int}$ and $\Omega_{\rm ext}$ be two open subsets of $\mathbb{R}^3$ with

$$\Omega_{\rm int}\cap\Omega_{\rm ext}=\emptyset \;\text{ and }\; \exists\delta_0>0: [-2\delta_0,2\delta_0]^3\cap\partial\Omega_{\rm int}\cap\partial\Omega_{\rm ext}=\big([-2\delta_0,2\delta_0]^2\times\{0\}\big). \tag{1.1}$$

Let $\Sigma\subset[-1,1]^2$ be an open subset of $\mathbb{R}^2$. For $\delta<\delta_0$, we consider the domain Ω^δ, see Fig. 1, consisting of $\Omega_{\rm ext}$ and $\Omega_{\rm int}$ linked by an iris $\Sigma_\delta=\delta\Sigma=\big\{(x,y)\in\mathbb{R}^2:\big(\frac{x}{\delta},\frac{y}{\delta}\big)\in\Sigma\big\}$

$$\Omega^\delta:=\Omega_{\rm int}\cup\Omega_{\rm ext}\cup\big(\Sigma_\delta\times\{0\}\big)\subset\mathbb{R}^3. \tag{1.2}$$

This domain tends to $\Omega:=\Omega_{\rm int}\cup\Omega_{\rm ext}\subset\mathbb{R}^2$, when $\delta\to 0$.

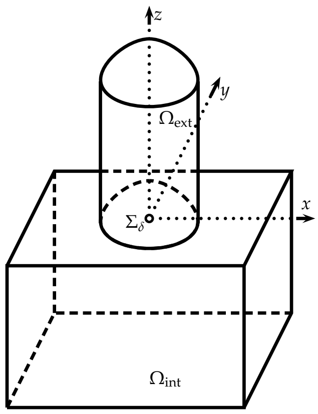

Figure 1: The computational domain Ω^δ.

1.2.2 The original problem

In these domains we consider the eigenvalue problems

$$\text{find } (\lambda_n^\delta, u_n^\delta) \in \mathbb{R} \times H^1(\Omega^\delta): \begin{cases} -\nabla\cdot(a\nabla u_n^\delta) = \lambda_n^\delta b u_n^\delta, & \text{in } \Omega^\delta, \\ \partial_{\mathbf{n}} u_n^\delta = 0, & \text{on } \partial\Omega^\delta, \end{cases} \tag{1.3a}$$

$$\text{find } (\lambda_n, u_n) \in \mathbb{R} \times H^1(\Omega): \begin{cases} -\nabla\cdot(a\nabla u_n) = \lambda_n b u_n, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3b}$$

with $a\in L^\infty(\Omega)$ and $b\in L^\infty(\Omega)$ two functions of Ω with $\inf_{\mathbf{x}\in\Omega} a(\mathbf{x})>0$, $\inf_{\mathbf{x}\in\Omega} b(\mathbf{x})>0$ and whose restrictions to Ω_{int} and Ω_{ext} are regular and can be expanded in the neighborhood of $\mathbf{0}$ with the form

$$a|_{\Omega_{\text{ext}}}(\mathbf{x}) = \sum_{i,j,k\geq 0} a^{\text{ext}}_{i,j,k} x^i y^j z^k, \quad b|_{\Omega_{\text{ext}}}(\mathbf{x}) = \sum_{i,j,k\geq 0} b^{\text{ext}}_{i,j,k} x^i y^j z^k, \quad \text{with } a^{\text{ext}}_{i,j,k}, b^{\text{ext}}_{i,j,k} \in \mathbb{R}, \tag{1.4a}$$

$$a|_{\Omega_{\text{int}}}(\mathbf{x}) = \sum_{i,j,k\geq 0} a^{\text{int}}_{i,j,k} x^i y^j z^k, \quad b|_{\Omega_{\text{int}}}(\mathbf{x}) = \sum_{i,j,k\geq 0} b^{\text{int}}_{i,j,k} x^i y^j z^k, \quad \text{with } a^{\text{int}}_{i,j,k}, b^{\text{int}}_{i,j,k} \in \mathbb{R}. \tag{1.4b}$$

In order to shorten the expressions, we adopt the notations $a_0^{\cdot} = a_{0,0,0}^{\cdot}$ and $b_0^{\cdot} = b_{0,0,0}^{\cdot}$. The discrete sets of eigenmodes $(u_n^\delta, \lambda_n^\delta)_{n\geq 0}$ (resp. $(u_n, \lambda_n)_{n\geq 0}$) can be chosen to be a bi-orthogonal basis of $L^2(\Omega^\delta)$ and $H^1(\Omega^\delta)$ (resp. $L^2(\Omega)$ and $H^1(\Omega)$) and to satisfy

$$0 = \lambda_0^\delta \leq \lambda_1^\delta \leq \lambda_2^\delta \leq \cdots \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n^\delta = +\infty, \tag{1.5}$$

respectively,

$$0 = \lambda_0 = \lambda_1 \leq \lambda_2 \leq \cdots \text{ and } \lim_{n\to+\infty} \lambda_n = +\infty \text{ and } \forall n\in\mathbb{N},\ u_n|_{\Omega_{\text{int}}} = 0 \text{ or } u_n|_{\Omega_{\text{ext}}} = 0. \tag{1.6}$$

Some natural questions arise: Does the eigenvalue λ_n^δ converge to λ_n? Is it possible to obtain an asymptotic expansion of λ_n^δ? With this asymptotic expansion, is it possible to compute a numerical approximation of λ_n^δ with a small computation cost?

1.3 Main theorem

The next Theorem gives positive answers to these three questions.

Theorem 1.1. *Let $n\in\mathbb{N}$ and let α be the positive real defined in Section 3.1.*

(i) If λ_n is a simple eigenvalue of the limit problem, then λ_n^δ can be expanded as follows

$$\lambda_n^\delta = \lambda_n + 2\pi\alpha \frac{a_0^{int} a_0^{ext}}{a_0^{int} + a_0^{ext}} \frac{\left(\overline{u}_n(\mathbf{0})\right)^2}{\int_\Omega b(u_n)^2} \delta + \mathcal{O}_{\delta\to 0}(\delta^2 \ln\delta), \tag{1.7}$$

with the notation

$$\overline{u}(\mathbf{0}) = u\big|_{\Omega_{ext}}(\mathbf{0}) \quad \text{if} \quad u|_{\Omega_{int}} = 0 \quad \text{and} \quad \overline{u}(\mathbf{0}) = u\big|_{\Omega_{int}}(\mathbf{0}) \quad \text{if} \quad u|_{\Omega_{ext}} = 0. \tag{1.8}$$

(ii) If λ_n is a double eigenvalue of the limit problem ($\lambda_n = \lambda_{n+1}$), then $\lambda_n^\delta = \lambda_n + \mathcal{O}_{\delta\to 0}(\delta^2)$ and λ_{n+1}^δ can be expanded as follows

$$\lambda_{n+1}^\delta = \lambda_n + 2\pi\alpha \frac{a_0^{int} a_0^{ext}}{a_0^{int} + a_0^{ext}} \Big(\frac{\big(\overline{u}_n(\mathbf{0})\big)^2}{\int_\Omega \big(bu_n\big)^2} + \frac{\big(\overline{u}_{n+1}(\mathbf{0})\big)^2}{\int_\Omega \big(bu_{n+1}\big)^2} \Big) \delta + \mathcal{O}_{\delta\to 0}(\delta^2 \ln \delta). \tag{1.9}$$

Remark 1.1. Formulas (1.7) and (1.9) involve only quantities which are independent of δ. Consequently truncating these expressions by eliminating the remainders furnish an approximation of λ_n^δ which does not require any mesh refinement.

Remark 1.2. Due to (1.6), one has either $u_n|_{\Omega_{\rm int}} = 0$ or $u_n|_{\Omega_{\rm ext}} = 0$. Consequently, (1.8) always defines $\overline{u}(\mathbf{0})$.

1.4 Matched asymptotic expansions

The first order asymptotic expansions of an eigenvalue λ_n^δ reads

$$\lambda_n^\delta = \lambda_n^0 + \delta \lambda_n^1 + o_{\delta\to 0}(\delta) \tag{1.10}$$

and has been derived in parallel to the derivation of the first order asymptotic expansion of the eigenfunction u_n^δ. The model (1.3a) involves two characteristic lengths of different magnitude: the size of the hole δ which is much smaller than the diameter of the cavity. Multiple scalings should be used to obtain an approximation of the eigenvector u_n^δ uniformly valid. The first scaling corresponds to the $\mathbf{x}$-variable and takes care of the cavity phenomena. The second scaling $\mathbf{X} = \mathbf{x}/\delta$ permits to describe the boundary layer phenomenons located in the neighborhood of the hole. Guided by the well known method of Matching of Asymptotic Expansions, see [11] and [20], we look for the asymptotic expansions of the two functions $\delta \mapsto u_n^\delta(\mathbf{x})$ and $\delta \mapsto \Pi_n^\delta(\mathbf{X}) := u_n^\delta(\delta \mathbf{X})$. At first order, they take the form

$$u_n^\delta(\mathbf{x}) = u_n^0(\mathbf{x}) + \delta u_n^1(\mathbf{x}) + o_{\delta\to 0}(\delta), \tag{1.11a}$$
$$\Pi_n^\delta(\mathbf{X}) := u_n^\delta(\delta\mathbf{X}) = \Pi_n^0(\mathbf{X}) + \delta \Pi_n^{1,0}(\mathbf{X}) + \delta \ln\delta\, \Pi_n^{1,1}(\mathbf{X}) + o_{\delta\to 0}(\delta). \tag{1.11b}$$

The functions u_n^i are defined on the domain Ω and are possibly singular in the neighborhood of the origin. The functions Π_n^i are defined on a normalized version of the neighborhood of the hole

$$\widehat{\Omega} := \mathbb{R}^3 \setminus \{(x,y,0) : (x,y) \in \mathbb{R}^2 \setminus \Sigma\}. \tag{1.12}$$

These two asymptotic expansions match asymptotically in an intermediate region.

Remark 1.3. The presence of poly-logarithmic gauge functions $\delta^n \ln^p \delta$ is rather not classical for three dimensional problems. They are due to the non constancy of the coefficient

a. Indeed in the case of a constant a we have $\Pi^{1,1}=0$. Moreover in the case of non constant coefficient, it is possible to show that Logarithms appear also in the eigenvalue and far-field expansions of second order which take the form

$$\lambda_n^\delta = \lambda_n^0 + \delta\lambda_n^1 + \delta^2\lambda_n^{2,0} + \delta^2 \ln\delta \lambda_n^{2,1} + o_{\delta\to 0}(\delta^2), \tag{1.13a}$$

$$u_n^\delta(\mathbf{x}) = u_n^0(\mathbf{x}) + \delta u_n^1(\mathbf{x}) + \delta^2 u_n^{2,0}(\mathbf{x}) + \delta^2 \ln\delta u_n^{2,1}(\mathbf{x}) + o_{\delta\to 0}(\delta^2). \tag{1.13b}$$

1.5 Content

In this paper we will not give the complete proof of Theorem 1.1 which is mainly based on the third order matched asymptotic expansions, on the min-max principle [14] and on a quasi-mode approach [8]. A complete proof for a two dimensional Dirichlet-Laplacian can be found in [3].

This paper will be focused on the description of the coefficients of the asymptotic expansions (1.10), (1.11a) and (1.11b) and on the question of their existence and uniqueness in the case of a simple eigenvalue. Moreover, the formulas (1.7) and (1.9) will be shown to be in good agreement with some direct numerical simulations.

2 Matched asymptotic expansions of simple eigenvalues

In this section, we suppose that λ_n is a simple eigenvalue. We will only deal with the case $u_n|_{\Omega_{\rm ext}}=0$. The case $u_n|_{\Omega_{\rm int}}=0$ can be deduced by symmetry.

The coefficients of the three asymptotic expansions (1.10), (1.11a) and (1.11b) have been formally derived using the Van Dyke matching principle [20]. The problems defining these coefficients will be proved to be well-posed in Section 3.

2.1 The limit coefficients

The far-field and eigenvalue limit coefficients are rather naturally defined by

$$u_n^0 = u_n \quad \text{and} \quad \lambda_n^0 = \lambda_n. \tag{2.1}$$

Moreover, the near-field limit coefficient Π_n^0 is defined on $\widehat{\Omega}$ by

$$\begin{cases} -\nabla\cdot\left(a_0 \nabla \Pi_n^0\right) = 0, \text{ in } \widehat{\Omega} \quad \text{and} \quad \partial_{\mathbf{n}}\Pi_n^0 = 0, \text{ on } \partial\widehat{\Omega}, \\ \Pi_n^0|_{\widehat{\Omega}_{\rm int}}(\mathbf{X}) = u_n^0|_{\Omega_{\rm int}}(\mathbf{0}) + o_{R\to+\infty}(1) \quad \text{and} \quad \Pi_n^0|_{\widehat{\Omega}_{\rm ext}}(\mathbf{X}) = o_{R\to+\infty}(1), \end{cases} \tag{2.2}$$

with $R=\sqrt{X^2+Y^2+Z^2}$, $\widehat{\Omega}_{\rm int}$ the lower half space and $\widehat{\Omega}_{\rm ext}$ the upper half space. As it will be proved in Section 3.1, see Remark 3.1 and Lemma 3.1, this coefficient can be expanded in the neighborhood of $+\infty$ with the form

$$\begin{cases} \Pi_n^0|_{\widehat{\Omega}_{\rm int}}(\mathbf{X}) = u_n^0|_{\Omega_{\rm int}}(\mathbf{0}) - \dfrac{a_0^{\rm ext}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} u_n|_{\Omega_{\rm int}}(\mathbf{0})\dfrac{1}{R} + \mathcal{O}_{R\to+\infty}\Big(\dfrac{1}{R^2}\Big), \\ \Pi_n^0|_{\widehat{\Omega}_{\rm int}}(\mathbf{X}) = \dfrac{a_0^{\rm int}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} u_n|_{\Omega_{\rm int}}(\mathbf{0})\dfrac{1}{R} + \mathcal{O}_{R\to+\infty}\Big(\dfrac{1}{R^2}\Big). \end{cases} \tag{2.3}$$

2.2 The first order coefficients

The first order asymptotic expansions are given by (1.10), (1.11a) and (1.11b) where the first order coefficients $\lambda_n^1 \in \mathbb{R}$, $u_n^1:\Omega\to\mathbb{R}$, $\Pi_n^{1,0}:\widehat{\Omega}\to\mathbb{R}$ and $\Pi_n^{1,1}:\widehat{\Omega}\to\mathbb{R}$ remain to be defined. The terms u_n^1 and λ_n^1 are solutions of the well-posed coupled problem

$$\begin{cases} \nabla\cdot\big(a\nabla u_n^1\big) + \lambda_n b u_n^1 = -\lambda_n^1 b u_n, \text{ in } \Omega \quad \text{and} \quad \partial_{\mathbf{n}} u_n^1 = 0, \text{ on } \partial\Omega\setminus\{\mathbf{0}\}, \\ u_n^1 + \dfrac{a_0^{\rm ext}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} u_n|_{\Omega_{\rm int}}(\mathbf{0})\dfrac{1}{r} \in H^1(\Omega_{\rm int}), \\ u_n^1 - \dfrac{a_0^{\rm int}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} u_n|_{\Omega_{\rm int}}(\mathbf{0})\dfrac{1}{r} \in H^1(\Omega_{\rm ext}), \end{cases} \tag{2.4}$$

with $r=\sqrt{x^2+y^2+z^2}$. The coefficient u_n^1 can then be expanded in the neighborhood of $\mathbf{0}$ with the help of the Kondratiev's theory, see [7,12],

$$\begin{cases} u_n^1|_{\Omega_{\rm int}}(\mathbf{x}) = -\dfrac{a_0^{\rm ext}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} u_n|_{\Omega_{\rm int}}(\mathbf{0})\dfrac{1}{r} + \mathsf{s}_n^1(\mathbf{x}) + \mathsf{r}_n^1(\mathbf{x}), \\ u_n^1|_{\Omega_{\rm ext}}(\mathbf{x}) = \dfrac{a_0^{\rm int}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} u_n|_{\Omega_{\rm int}}(\mathbf{0})\dfrac{1}{r} + \mathsf{s}_n^1(\mathbf{x}) + \mathsf{r}_n^1(\mathbf{x}), \end{cases} \tag{2.5}$$

with

$$\begin{aligned} \mathsf{s}_n^1|_{\Omega_{\rm int}}(\mathbf{x}) &= \dfrac{a_0^{\rm ext}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} \dfrac{u_n|_{\Omega_{\rm int}}(\mathbf{0})}{2}\Big(\dfrac{a_{1,0,0}^{\rm int}}{a_0^{\rm int}}\dfrac{x}{r} + \dfrac{a_{0,1,0}^{\rm int}}{a_0^{\rm int}}\dfrac{y}{r} + \dfrac{a_{0,0,1}^{\rm int}}{a_0^{\rm int}}\Big(\dfrac{z}{r} + \ln\dfrac{r-z}{2}\Big)\Big), \\ \mathsf{s}_n^1|_{\Omega_{\rm ext}}(\mathbf{x}) &= -\dfrac{a_0^{\rm int}\alpha}{a_0^{\rm int}+a_0^{\rm ext}} \dfrac{u_n|_{\Omega_{\rm int}}(\mathbf{0})}{2}\Big(\dfrac{a_{1,0,0}^{\rm ext}}{a_0^{\rm ext}}\dfrac{x}{r} + \dfrac{a_{0,1,0}^{\rm ext}}{a_0^{\rm ext}}\dfrac{y}{r} + \dfrac{a_{0,0,1}^{\rm ext}}{a_0^{\rm ext}}\Big(\dfrac{z}{r} - \ln\dfrac{r+z}{2}\Big)\Big) \end{aligned}$$

and $r_n^1:\Omega\to\mathbf{R}$ such that the two restrictions $r_n^1|_{\Omega_{\rm int}}$ and $r_n^1|_{\Omega_{\rm ext}}$ are continuous in the neighborhood of $\mathbf{0}$.

The coefficients $\Pi_n^{1,0}$ and $\Pi_n^{1,1}$ are defined in the infinite domain $\widehat{\Omega}$ by

$$\begin{cases} -\nabla\cdot\big(a_0\nabla\Pi_n^{1,0}\big) = \nabla\cdot\big((a_{1,0,0}X + a_{0,1,0}Y + a_{0,0,1}Z)\nabla\Pi_n^0\big), & \text{in } \widehat{\Omega}, \\ \partial_{\mathbf{n}}\Pi_n^{1,0} = 0, & \text{on } \partial\widehat{\Omega}, \\ \Pi_n^{1,0}|_{\widehat{\Omega}_{\rm int}}(\mathbf{X}) = \partial_x u_n^0|_{\Omega_{\rm int}}(\mathbf{0})X + \partial_y u_n^0|_{\Omega_{\rm int}}(\mathbf{0})Y + \mathsf{s}_n^1|_{\Omega_{\rm int}}(\mathbf{X}) + \mathsf{r}_n^1|_{\Omega_{\rm int}}(\mathbf{0}) + o_{r\to0}(1), \\ \Pi_n^{1,0}|_{\widehat{\Omega}_{\rm ext}}(\mathbf{X}) = \mathsf{s}_n^1|_{\Omega_{\rm ext}}(\mathbf{X}) + \mathsf{r}_n^1|_{\Omega_{\rm ext}}(\mathbf{0}) + o_{r\to0}(1), \end{cases} \tag{2.6a}$$

$$\begin{cases} -\nabla\cdot\big(a_0\nabla\Pi_n^{1,1}\big)=0, \text{ in } \widehat{\Omega} \quad\text{and}\quad \partial_{\mathbf{n}}\Pi_n^{1,1}=0, \text{ on } \partial\widehat{\Omega}, \\ \Pi_n^{1,1}|_{\widehat{\Omega}_{\rm int}}(\mathbf{X})=\dfrac{u_n|_{\Omega_{\rm int}}(\mathbf{0})}{2}\dfrac{a_0^{\rm ext}\alpha}{a_0^{\rm int}+a_0^{\rm ext}}\dfrac{a_{0,0,1}^{\rm int}}{a_0^{\rm int}}+o_{r\to 0}(1), \\ \Pi_n^{1,1}|_{\widehat{\Omega}_{\rm ext}}(\mathbf{X})=\dfrac{u_n|_{\Omega_{\rm int}}(\mathbf{0})}{2}\dfrac{a_0^{\rm int}\alpha}{a_0^{\rm int}+a_0^{\rm ext}}\dfrac{a_{0,0,1}^{\rm ext}}{a_0^{\rm ext}}+o_{r\to 0}(1). \end{cases} \tag{2.6b}$$

3 Existence of the coefficients of the asymptotic expansions

3.1 Existence and uniqueness of the Π_n^i

The question of existence and uniqueness of the Laplacian problems equipped with Neumann boundary condition is a rather well understood topic, see for example [1]. For $F\in L^2(\widehat{\Omega})$ compactly supported, let us recall that there exists a unique $\Pi\in H^1_{\rm loc}(\widehat{\Omega})$ satisfying

$$\nabla\cdot\big(a_0\nabla\Pi\big)=F, \text{ in } \widehat{\Omega}, \quad \partial_{\mathbf{n}}\Pi=0, \text{ on } \partial\widehat{\Omega}, \quad \Pi=o_{R\to+\infty}(1). \tag{3.1}$$

This is mainly due to the Hardy inequality

$$\exists\gamma>0:\ \gamma\Big(\|\nabla\Pi\|_{L^2(\widehat{\Omega})}+\Big\|\frac{\Pi}{1+R}\Big\|_{L^2(\widehat{\Omega})}\Big)\le\|\nabla\Pi\|_{L^2(\widehat{\Omega})}, \quad \forall\Pi\in K_0^1 \tag{3.2}$$

and to the equivalence of the last problem with the variational formulation

$$\text{Find } \Pi\in K_0^1:\ \int a_0\nabla\Pi\cdot\nabla\Pi'=\int_{\widehat{\Omega}}F\Pi', \quad \text{for all } \Pi'\in K_0^1 \tag{3.3}$$

with the Kondratiev's space

$$K_0^1:=\Big\{\Pi:\widehat{\Omega}\to\mathbb{R}:\nabla\Pi\in L^2\big(\widehat{\Omega}\big) \text{ and } \frac{\Pi}{1+R}\in L^2\big(\widehat{\Omega}\big)\Big\}. \tag{3.4}$$

We will now prove the existence and uniqueness of Π^0 and $\Pi^{1,1}$. Let $\Psi_{\rm int}$ and $\Psi_{\rm ext}$ be two regular cut-off functions satisfying

$$\begin{cases} \Psi_{\rm int}(\mathbf{X})=0, & \text{in } \widehat{\Omega}_{\rm ext}, \\ \Psi_{\rm int}(\mathbf{X})=\varphi(R), & \text{in } \widehat{\Omega}_{\rm int}, \end{cases} \tag{3.5a}$$

$$\begin{cases} \Psi_{\rm ext}(\mathbf{X})=0, & \text{in } \widehat{\Omega}_{\rm int}, \\ \Psi_{\rm ext}(\mathbf{X})=\varphi(R), & \text{in } \widehat{\Omega}_{\rm ext}, \end{cases} \tag{3.5b}$$

with $\varphi(R)=0$ for $R<1$ and $\varphi(R)=1$ for $R>2$.

Theorem 3.1. *For all reals A and B, there exists a unique $\Pi_{A,B}\in H^1_{\rm loc}(\widehat{\Omega})$ satisfying*

$$\begin{cases} -\nabla\cdot\big(a_0\nabla\Pi_{A,B}\big)=0, \ \textit{in } \widehat{\Omega} \quad\textit{and}\quad \partial_{\mathbf{n}}\Pi_{A,B}=0, \ \textit{on } \partial\widehat{\Omega}, \\ \Pi_{A,B}|_{\widehat{\Omega}_{\rm int}}(\mathbf{X})=A+o_{R\to+\infty}(1) \quad\textit{and}\quad \Pi_{A,B}|_{\widehat{\Omega}_{\rm ext}}(\mathbf{X})=B+o_{R\to+\infty}(1). \end{cases} \tag{3.6}$$

Proof. We consider the function $\Pi(\mathbf{X}) = \Pi_{A,B}(\mathbf{X}) - \Psi_{\rm int}(\mathbf{X})A - \Psi_{\rm ext}(\mathbf{X})B$ which satisfies (3.1) with

$$F(\mathbf{X}) = -\nabla\cdot\big(a_0(A\nabla\Psi_{\rm int}(\mathbf{X}) + B\nabla\Psi_{\rm ext}(\mathbf{X}))\big). \tag{3.7}$$

Since the function F is compactly supported, the problem (3.6) is well posed. $\square$

Remark 3.1. The coefficients Π_n^0 and $\Pi_n^{1,1}$, defined by (2.2) and (2.6a), can be expressed as

$$\Pi_n^0 = \Pi_{A,B} \ \text{ with } \ A = u_n|_{\Omega_{\rm int}}(\mathbf{0}) \ \text{ and } \ B = 0,$$

$$\Pi_n^{1,1} = \Pi_{A,B} \ \text{ with } \ A = \frac{1}{2}\frac{a_0^{\rm ext}\alpha}{a_0^{\rm int}+a_0^{\rm ext}}\frac{a_{0,0,1}^{\rm int}}{a_0^{\rm int}}u_n|_{\Omega_{\rm int}}(\mathbf{0}) \ \text{ and } \ B = \frac{1}{2}\frac{a_0^{\rm int}\alpha}{a_0^{\rm int}+a_0^{\rm ext}}\frac{a_{0,0,1}^{\rm ext}}{a_0^{\rm ext}}u_n|_{\Omega_{\rm int}}(\mathbf{0}).$$

Remark 3.2. The existence and uniqueness of $\Pi^{1,0}$ can be as well demonstrated. This requires extra arguments that are not central in our study. We have chosen not to give the details in this article.

Now we will be interested in the obtention of the asymptotic expansion of $\Pi_{A,B}$ in the neighborhood of $R = +\infty$. Let us introduce the function $\Pi_\star \in H^1_{\rm loc}(\widehat{\Omega})$ satisfying

$$\begin{cases} \nabla\cdot\big(a_0\nabla\Pi_\star\big) = 0, \ \text{in } \widehat{\Omega} \ \text{ and } \ \partial_{\mathbf{n}}\Pi_\star = 0, \ \text{on } \partial\widehat{\Omega}, \\ \Pi_\star|_{\widehat{\Omega}_{\rm int}} = 1 + o_{R\to+\infty}(1) \ \text{ and } \ \Pi_\star|_{\widehat{\Omega}_{\rm ext}} = o_{R\to+\infty}(1). \end{cases} \tag{3.8}$$

The function $\Pi_\star$ is related to $\Pi_{A,B}$ by $\Pi_{A,B} = B + (A-B)\Pi_\star$. In order to solve this problem, we will use a simple layer formulation based on the operator $\mathbf{S}$

$$\mathbf{S}\colon \big(H^{\frac{1}{2}}(\Sigma)\big)^\star \mapsto H^{\frac{1}{2}}(\Sigma), \quad \lambda \mapsto \mathbf{S}\lambda(\mathbf{X}) = \frac{1}{4\pi}\int_\Sigma \frac{\lambda(\mathbf{X}')}{\|\mathbf{X}-\mathbf{X}'\|}dX'. \tag{3.9}$$

Taking into account the two transmission conditions $\Pi_\star|_{\Sigma_{\rm int}} = \Pi_\star|_{\Sigma_{\rm ext}}$ and $a_0^{\rm int}\partial_z\Pi_\star|_{\Sigma_{\rm int}} = a_0^{\rm ext}\partial_z\Pi_\star|_{\Sigma_{\rm ext}}$ and the representation formulas, see [9],

$$\Pi_\star|_{\Sigma_{\rm int}} = 1 + 2\mathbf{S}\partial_z\Pi_\star|_{\Sigma_{\rm int}} \quad \text{and} \quad \Pi_\star|_{\Sigma_{\rm ext}} = -2\mathbf{S}\partial_z\Pi_\star|_{\Sigma_{\rm ext}}, \tag{3.10}$$

we get the formulation

$$\text{Find } \lambda_\star \in \big(H^{\frac{1}{2}}(\Sigma)\big)^\star \text{ such that } \mathbf{S}\lambda_\star = 1, \ \text{on } \Sigma, \tag{3.11}$$

with λ_* related to the two normal derivatives by

$$\partial_z\Pi_\star|_{\Sigma_{\rm int}} = -\frac{a_0^{\rm ext}}{a_0^{\rm ext}+a_0^{\rm int}}\frac{\lambda_\star}{2} \quad \text{and} \quad \partial_z\Pi_\star|_{\Sigma_{\rm ext}} = -\frac{a_0^{\rm int}}{a_0^{\rm int}+a_0^{\rm ext}}\frac{\lambda_\star}{2}. \tag{3.12}$$

Then, the restrictions of Π_* to $\widehat{\Omega}_{\rm int}$ and $\widehat{\Omega}_{\rm ext}$ are given by the representation formula

$$\Pi_\star = \begin{cases} 1 - \dfrac{a_0^{\rm ext}}{a_0^{\rm ext}+a_0^{\rm int}}\Big(\dfrac{1}{4\pi}\displaystyle\int_\Sigma \dfrac{\lambda_\star(\mathbf{X}')}{\|\mathbf{X}-\mathbf{X}'\|}d\mathbf{X}'\Big), & \text{in } \widehat{\Omega}_{\rm int}, \\ +\dfrac{a_0^{\rm int}}{a_0^{\rm ext}+a_0^{\rm int}}\Big(\dfrac{1}{4\pi}\displaystyle\int_\Sigma \dfrac{\lambda_\star(\mathbf{X}')}{\|\mathbf{X}-\mathbf{X}'\|}d\mathbf{X}'\Big), & \text{in } \widehat{\Omega}_{\rm ext}. \end{cases} \tag{3.13}$$

Expanding (3.13), the first order asymptotic expansion of $\Pi_\star$ for $R=\|\mathbf{X}\|\to+\infty$ reads

$$\Pi_\star = \begin{cases} 1 - \dfrac{a_0^{\rm ext}}{a_0^{\rm ext}+a_0^{\rm int}}\dfrac{\alpha}{R} + \mathcal{O}_{R\to+\infty}\Big(\dfrac{1}{R^2}\Big), & \text{in } \widehat{\Omega}_{\rm int}, \\ +\dfrac{a_0^{\rm int}}{a_0^{\rm ext}+a_0^{\rm int}}\dfrac{\alpha}{R} + \mathcal{O}_{R\to+\infty}\Big(\dfrac{1}{R^2}\Big), & \text{in } \widehat{\Omega}_{\rm ext}, \end{cases} \tag{3.14}$$

where $\alpha=(4\pi)^{-1}\int_\Sigma \lambda_\star(\mathbf{X}')d\mathbf{X}'$ with $\lambda_\star$ defined by (3.11). The next Lemma follows.

Lemma 3.1. $\Pi_{A,B}$ *can be expanded with the form*

$$\begin{cases} \Pi_{A,B}|_{\widehat{\Omega}_{int}}(\mathbf{X}) = A + (B-A)\dfrac{a_0^{ext}}{a_0^{int}+a_0^{ext}}\dfrac{\alpha}{R} + \mathcal{O}_{R\to+\infty}\Big(\dfrac{1}{R^2}\Big), \\ \Pi_{A,B}|_{\widehat{\Omega}_{ext}}(\mathbf{X}) = B + (A-B)\dfrac{a_0^{int}}{a_0^{int}+a_0^{ext}}\dfrac{\alpha}{R} + \mathcal{O}_{R\to+\infty}\Big(\dfrac{1}{R^2}\Big). \end{cases} \tag{3.15}$$

Remark 3.3. The coefficient α depends on the shape of Σ but is independent of the value of $a_0^{\rm int}$ and $a_0^{\rm ext}$. This quantity is related to the so called acoustic conductivity of the hole c and to the effective size of the hole s by $s=c/\pi=\alpha\delta$, see [19]. It can be numerically computed for every Σ, see Section 4. Moreover it has been analytically computed for some simple Σ with area $\mathcal{A}$, see [15]

- $\alpha=\frac{2\rho}{\pi}=\frac{2}{\pi}\sqrt{\frac{\mathcal{A}}{\pi}}$ for Σ a circle with radius ρ;
- $\alpha=\frac{2}{\pi}\sqrt{\frac{\mathcal{A}}{\pi}\frac{(\pi/2)\sqrt{b/a}}{K(1-b^2/a^2)}}$ for Σ an ellipse with minor axes a and b. The function K denotes the complete elliptic integral of first kind.

The coefficient α can be rather easily approximated by the coefficient α of the circle with same area

$$\alpha_{app} = \frac{2}{\pi}\sqrt{\frac{\mathcal{A}}{\pi}}. \tag{3.16}$$

This approximation does not require any numerical computation and is rather accurate for not too elongated holes. In Fig. 2 we illustrate the accuracy of this approximation in the case of an ellipse of minor axes a and b and a rectangle $[0,a]\times[0,b]$. The relative error is less than 5% for $0.4<b/a<1$.

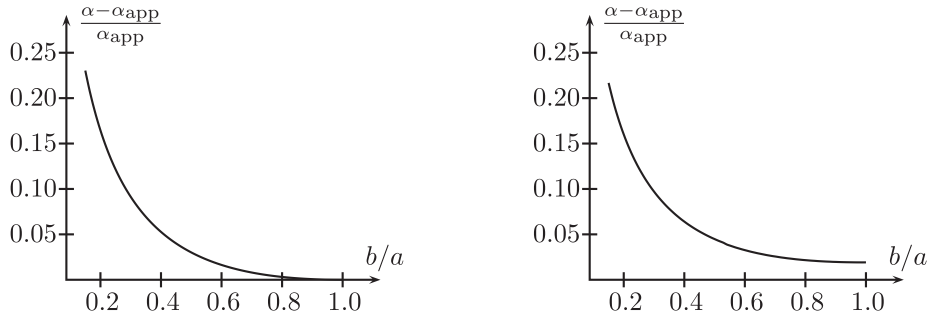

Figure 2: Efficiency of the approximation of α by $\alpha_{app}=\frac{\pi}{2}\sqrt{\frac{\mathcal{A}}{\pi}}$ for ellipses (left) and for rectangles (right).

3.2 Existence and uniqueness of u_n^1 and λ_n^1

The two coefficients u_n^1 and λ_n^1 do have to solve the problem (2.4). The following Lemma ensures the existence and uniqueness of u_n^1 and λ_n^1 up to the knowledge of the u_n-component of u_n^1. This component can be chosen arbitrarily.

Lemma 3.2. *Problem* (2.4) *has solutions. Moreover if* (u_n^1,λ_n^1) *and* $(u_{n,*}^1,\lambda_{n,*}^1)$ *are solutions, one has* $\lambda_n^1=\lambda_{n,*}^1$ *and*

$$\lambda_n^1=2\pi\alpha\frac{a_0^{int}a_0^{ext}}{a_0^{int}+a_0^{ext}}\frac{\left(u_n\big|_{\Omega_{int}}(\mathbf{0})\right)^2}{\int_{\Omega_{int}}b\left(u_n\right)^2}\quad and\quad \exists\gamma\in\mathbb{R}:\ u_{n,*}^1-u_n^1=\gamma u_n. \tag{3.17}$$

Proof. The function u_n^1 does not belong to $H^1(\Omega)$. Consequently, the Fredholm alternative cannot be directly applied. For this reason, we introduce the auxiliary function $\omega_n^1\in H^1(\Omega)$

$$\left\{\begin{aligned}&\omega_n^1|_{\Omega_{\rm int}}(\mathbf{x})=u_n^1(\mathbf{x})+\chi(r)u_n|_{\Omega_{\rm int}}(\mathbf{0})\frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}}\frac{\alpha}{r},\\&\omega_n^1|_{\Omega_{\rm ext}}(\mathbf{x})=u_n^1(\mathbf{x})-\chi(r)u_n|_{\Omega_{\rm int}}(\mathbf{0})\frac{a_0^{\rm int}}{a_0^{\rm int}+a_0^{\rm ext}}\frac{\alpha}{r},\end{aligned}\right. \tag{3.18}$$

with χ a regular cut-off function satisfying (see (1.1))

$$\chi(z)=1,\ \text{if}\ z\le\delta_0\ \text{and}\ \chi(z)=0,\ \text{if}\ z\ge2\delta_0. \tag{3.19}$$

Using (2.4), ω_n^1 belongs to $H^1(\Omega)$ and satisfies

$$\nabla\cdot\left(a\nabla\omega_n^1\right)+\lambda_n b\omega_n^1=F_n^1,\ \text{in}\ \Omega\ \text{and}\ \partial_{\mathbf{n}}\omega_n^1=0,\ \text{on}\ \partial\Omega, \tag{3.20}$$

with $F_n^1\in\left(H^1(\Omega)\right)^\star$ defined by

$$\left\{\begin{aligned}&F_n^1|_{\Omega_{\rm int}}(\mathbf{x})=-\lambda_n^1 bu_n(\mathbf{x})+\left(\nabla\cdot(a\nabla)+\lambda_n b\right)\left(\chi(r)u_n|_{\Omega_{\rm int}}(\mathbf{0})\frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}}\frac{\alpha}{r}\right),\\&F_n^1|_{\Omega_{\rm ext}}(\mathbf{x})=-\left(\nabla\cdot(a\nabla)+\lambda_n b\right)\left(\chi(r)u_n|_{\Omega_{\rm int}}(\mathbf{0})\frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}}\frac{\alpha}{r}\right).\end{aligned}\right.$$

Since λ_n^0 is a simple eigenvalue of the elliptic operator on $\Omega_{\rm int}$, the problem (3.20) defining ω_n^1 has solutions if and only if

$$\int_{\Omega_{\rm int}} F_n^1 u_n = 0. \tag{3.21}$$

Moreover this solution is determined up to its u_n-component, i.e., if ω_n^1 and $\omega_n^{1,*}$ are two solutions of (3.20) then

$$\exists \gamma \in \mathbb{R}:\ \omega_n^1 = \omega_n^{1,*} + \gamma u_n. \tag{3.22}$$

The necessary and sufficient condition (3.21) for existence of u_n^1 will now be made explicit. Since $u_n = 0$, in $\Omega_{\rm ext}$ and $\nabla\cdot\big(a\nabla u_n\big) + \lambda_n b u_n = 0$, in $\Omega_{\rm int}$, (3.21) takes the form

$$\begin{aligned}\lambda_n^1 \int_{\Omega_{int}} b(u_n)^2 =& \int_{\Omega_{\rm int}} \nabla\cdot\left(a\nabla\left(\chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\right)\right) u_n \\ &- \int_{\Omega_{\rm int}} \chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\Big(\nabla\cdot\big(a\nabla u_n\big)\Big).\end{aligned} \tag{3.23}$$

Let us denote by B_η the ball of center $\mathbf{0}$ and of radius η. Since the domain $\Omega_{\rm int}\setminus B_\eta$ tends to $\Omega_{\rm int}$ when $\eta\to 0$, we have due to Lebesgues Theorem

$$\begin{aligned}\lambda_n^1 \int_{\Omega_{int}} b(u_n)^2 = \lim_{\eta\to 0^+} \Bigg[& \int_{\Omega_{\rm int}\setminus B_\eta} \nabla\cdot\left(a\nabla\left(\chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\right)\right) u_n \\ &- \int_{\Omega_{\rm int}\setminus B_\eta} \chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\big(\nabla\cdot\big(a\nabla u_n\big)\big)\Bigg].\end{aligned} \tag{3.24}$$

Two Green formulas lead to

$$\begin{aligned}&\int_{\Omega_{\rm int}\setminus B_\eta} \nabla\cdot\left(a\nabla\left(\chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\right)\right) u_n \\ =& -\left[\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} a\partial_r\left(\chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\right) u_n r^2 \sin(\theta) d\theta d\varphi\right](r=\eta) \\ &- \int_{\Omega_{\rm int}\setminus B_\eta} a\nabla\left(\chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\right)\nabla u_n r^2 \sin(\theta) dr d\theta d\varphi,\end{aligned} \tag{3.25a}$$

$$\begin{aligned}&\int_{\Omega_{\rm int}\setminus B_\eta} \chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\big(\nabla\cdot\big(a\nabla u_n\big)\big) \\ =& -\left[\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} a\chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\partial_r u_n r^2 \sin(\theta) d\theta d\varphi\right](r=\eta) \\ &- \int_{\Omega_{\rm int}\setminus B_\eta} a\nabla\left(\chi(r) u_n|_{\Omega_{\rm int}}(\mathbf{0}) \frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}} \frac{\alpha}{r}\right)\nabla u_n r^2 \sin(\theta) dr d\theta d\varphi,\end{aligned} \tag{3.25b}$$

with the spherical coordinates (r,θ,φ) defined by $x = r\sin(\theta)\cos(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$ and $z = r\cos(\theta)$. Inserting (3.25a), and (3.25b) in (3.24), we obtain $(\chi(\eta) = 1$ and $\partial_r\chi(\eta) = 0$, for small η)

$$\lambda_n^1\int_{\Omega_{int}} b(u_n)^2 = -\lim_{\eta\to 0}\left\{\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} a(\mathbf{x})\partial_r\left(u_n|_{\Omega_{\rm int}}(\mathbf{0})\frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}}\frac{\alpha}{r}\right)u_n(\mathbf{x})r^2\sin(\theta)d\theta\, d\varphi\right.$$
$$\left. -\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} a(\mathbf{x})u_n|_{\Omega_{\rm int}}(\mathbf{0})\frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}}\frac{\alpha}{r}\partial_r u_n(\mathbf{x})r^2\sin(\theta)d\theta d\varphi\right\}(r=\eta). \quad (3.26)$$

Using the first order Taylor expansions of u_n, $\partial_r u_n$ in $\Omega_{\rm int}$ and (1.4b), we obtain

$$u_n(\mathbf{x}) = u_n|_{\Omega_{\rm int}}(\mathbf{0}) + \mathcal{O}_{r\to 0}(r), \quad \partial_r u_n(\mathbf{x}) = \mathcal{O}_{r\to 0}(1) \quad \text{and} \quad a(\mathbf{x}) = a_0^{\rm int} + \mathcal{O}_{r\to 0}(r). \quad (3.27)$$

Inserting (3.27) in (3.26), we have

$$\lambda_n^1\int_{\Omega_{int}} b(u_n)^2 = \lim_{\eta\to 0}\left\{a_0^{\rm int}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}\left(u_n|_{\Omega_{\rm int}}(\mathbf{0})\right)^2\frac{a_0^{\rm ext}}{a_0^{\rm int}+a_0^{\rm ext}}\alpha\sin(\theta)d\theta d\varphi + \mathcal{O}_{\eta\to 0}(\eta)\right\}. \quad (3.28)$$

Taking the limit, get (3.17). □

4 Numerical simulations

In this section, we will present two series of numeral experiments illustrating Theorem 1.1. The numerical computations are based on the parallel version of the CESC library of CERFACS (boundary element and finite element code) and of the ARPACK library (large scale eigenvalue problems solver).

For both series of experiments, λ_n^δ, λ_n and u_n are evaluated with a P_1-continuous (piecewise linear continuous approximation on a tetrahedral mesh) finite element approximation. The parameter δ takes 11 values going from 1 to 10^{-2}: $\delta = 10^{-k/5}$ with $0 \le k \le 10$. The coefficient $\alpha = (4\pi)^{-1}\int_\Sigma \lambda_\star(\mathbf{X}')d\mathbf{X}'$ is either numerically computed by solving (3.11) or approximated by $\alpha_{app} = (2/\pi)\sqrt{\mathcal{A}/\pi}$. The numerical computation of α relies on P_1-continuous (piecewise linear continuous approximation on a triangular mesh) boundary element approximation. The computation of λ_n^δ requires a refined mesh in the neighborhood of the hole. Even if the geometry is simple, 2.5 million degrees of freedom are required for the smallest δ. The computation of u_n and λ_n is achieved on a coarse mesh and is therefore less costly and easier to handle (one does not have to face some errors of the mesher: For very small mesh step our mesher was simply not able to generate a regular mesh without dividing the computational domain in many regions).

We report in Figs. 5, 6, 7 and 8 the results of our simulations for $n=1$ to 4. For $n=0$, we do not show the results since $\lambda_0^\delta = \lambda_0 = \lambda_0^1 = 0$.

During all the computations, we have tried to diminish as much as possible our numerical errors. For the smallest values of δ the encountered linear systems become rather large (millions of unknowns) and the errors committed by the eigenvalue solver ARPACK cannot be completely neglected.

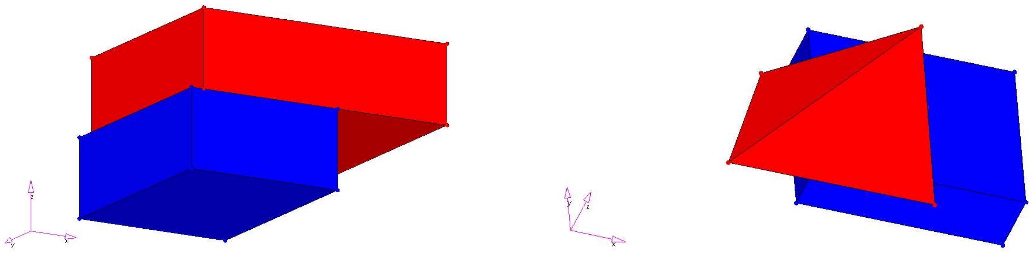

Figure 3: The geometry for both experiments.

Figure 4: The shape of the holes Σ for both experiments.

4.1 First experiment

Let $\ell_x^{\text{int}}=0.6$, $\ell_x^{\text{ext}}=1.0$, $\ell_y=0.8$, $\ell_z=0.3$. The two cavities, see Fig. 3, are defined by

$$\Omega_{\text{int}}=\left[-\frac{\ell_x^{\text{int}}}{2},\frac{\ell_x^{\text{int}}}{2}\right]\times\left[-\frac{\ell_y}{3},\frac{2\ell_y}{3}\right]\times[-\ell_z,0],$$
$$\Omega_{\text{ext}}=\left[-\frac{\ell_x^{\text{ext}}}{4},\frac{3\ell_x^{\text{ext}}}{4}\right]\times\left[-\frac{\ell_y}{3},\frac{2\ell_y}{3}\right]\times[0,\ell_z].$$

The shape of the hole Σ, see Fig. 4 is a polygon with vertexes $A=(0,0)$, $B=(0.1,0)$, $C=(0.1,-0.08)$, $D=(-0.08,-0.08)$, $E=(-0.08,0.1)$ and $F=(0,0.1)$ with

$$\alpha=0.0578\cdots \quad \text{and} \quad \alpha_{app}=0.0538\cdots. \tag{4.1}$$

The coefficient functions a and b are constant and equal to 1.

4.2 Second experiment

Let $\ell_x=1.0$, $\ell_y=0.8$ and $\ell_z=0.3$. The interior cavity is

$$\Omega_{\text{int}}=\left[-\frac{\ell_x}{2},\frac{\ell_x}{2}\right]\times\left[-\frac{\ell_y}{3},\frac{2\ell_y}{3}\right]\times[-\ell_z,0].$$

The exterior cavity is a pyramid with basis the polygon linking $(0.2,-0.4,0)$, $(-0.7,-0.4,0)$, $(-0.5,0.2,0)$ and $(0.2,0.2)$ and with upper vertex $(0.1,0.1,0.7)$. The shape of the hole is a centered circle of radius 0.1. It corresponds to

$$\alpha=\alpha_{app}=0.0637\cdots. \tag{4.2}$$

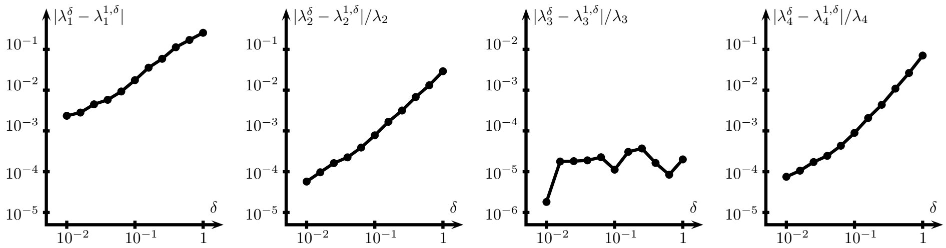

Figure 5: Error $\lambda_n^\delta - (\lambda_n + \delta\lambda_1^n)$ in log-log scale for the first experiment.

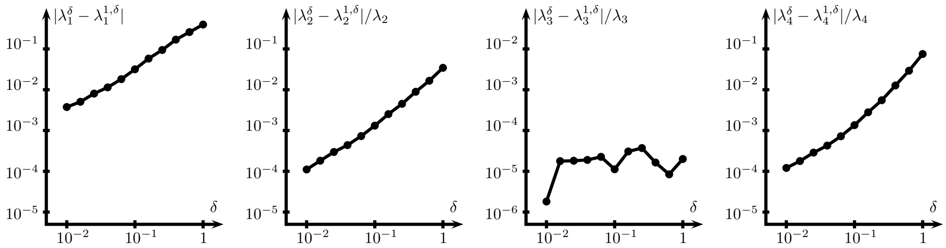

Figure 6: Error $\lambda_n^\delta - (\lambda_n + \delta\lambda_1^n)$ in log-log scale for the first experiment with α replaced by α_{app}.

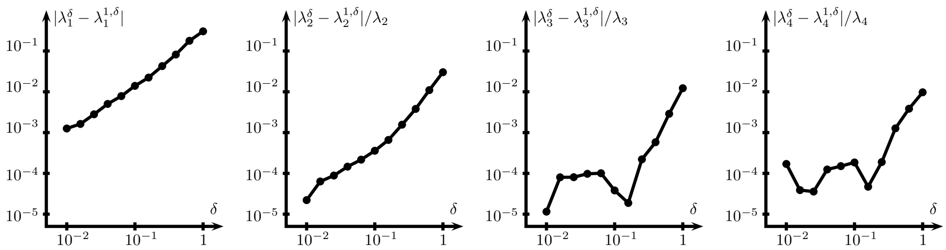

Figure 7: Error $\lambda_n^\delta - (\lambda_n + \delta\lambda_1^n)$ in log-log scale for the second experiment.

Table 1: The values of λ_n and λ_n^1 (see (1.10)).

	1st experiment		2nd experiment	
n	λ_n	λ_n^1	λ_n	λ_n^1
0	0	0	0	0
1	0	2.02	0	2.17
2	9.87	0.757	9.24	0.426
3	15.42	0	16.7	0.251
4	15.42	1.01	19.7	1.11

The functions a and b are piecewise constant and given by

$$a|_{\Omega_{\text{int}}} = 2, \quad a|_{\Omega_{\text{ext}}} = 1, \quad b|_{\Omega_{\text{int}}} = 1, \quad b|_{\Omega_{\text{ext}}} = 2. \tag{4.3}$$

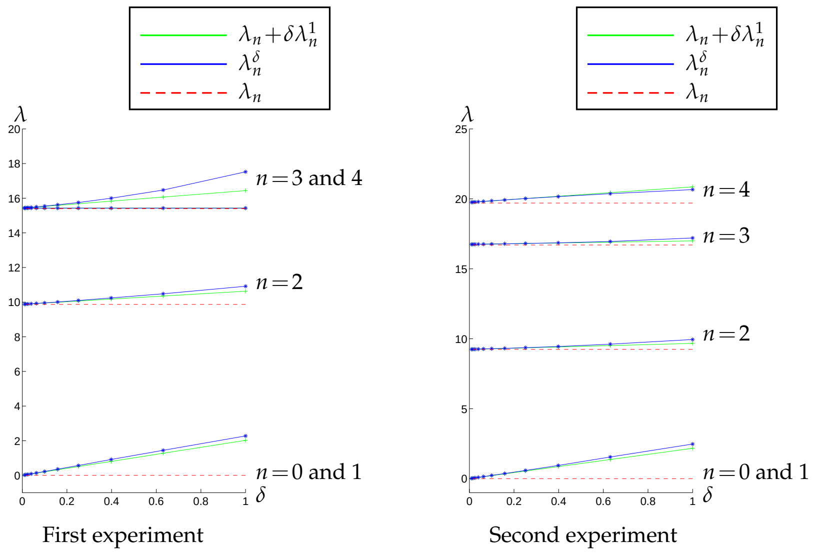

Figure 8: The eigenvalue λ_n^δ, its limit λ_n and its approximation $\lambda_n+\delta\lambda_n^1$ with respect to δ.

5 Conclusions

In this article we derived the first order asymptotic expansion of the eigenvalues and eigenvectors of a three dimensional elliptic operator equipped with Neumann boundary condition. This expansion allows to compute with a small computation cost a numerical approximation of these eigenvalues.

The reader can also remark that this work can easily be adapted to deal with a multiperforated straight structure if one can manage the boundary homogenisation, see [6, 17, 19]. It would be of interest to see the impact of varying coefficients on the blockage coefficient C which measures the permeability of the wall and is related to the effective size of the hole $s=\alpha\delta$ of Remark 3.3 and to the area A of the cell containing one hole by

$$C=\frac{\mathsf{A}}{2\pi s}.$$

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