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Multiscale Nanorod Metamaterials and Realizable Permittivity Tensors

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Abstract. Our aim is to evidence new 3D composite diffractive structures whose effective permittivity tensor can exhibit very large positive or negative real eigenvalues. We use a reiterated homogenization procedure in which the first step consists in considering a bounded obstacle made of periodically disposed parallel high conducting metallic fibers of finite length and very thin cross section. As shown in [2], the resulting constitutive law is non-local. Then by reproducing periodically the same kind of obstacle at small scale, we obtain a local effective law described by a permittivity tensor that we make explicit as a function of the frequency. Due to internal resonances, the eigenvalues of this tensor have real part that change of sign and are possibly very large within some range of frequencies. Numerical simulations are shown.

AMS subject classifications: 35B27, 35Q60, 35Q61, 78M35, 78M40

Key words: Homogenization, photonic crystals, metamaterials, micro-resonators, effective tensors, non local effects.

1 Introduction

In recent years, the advent of negative index metamaterials and composites has led to increased interest in effective medium theories. In particular there is presently intense activity in constructing artificial photonic crystals made of metallo-dielectric inclusions with the goal of reaching negative bulk electric or magnetic response. An excellent example is the wire medium studied by Pendry in 1996 [7] where it is suggested that high conductivity fibers occupying a very small volume fraction could produce negative permittivity. A rigorous proof of this based on homogenization techniques appeared in [8] where, instead of letting the wavelength λ tend to infinity as customary in effective medium theories, we keep it constant (assuming a time dependence of the electromagnetic field $\exp(-i\omega t)$) and let other geometrical parameters (as the period) tend to zero.

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The advantage of this approach is that it leaves us the possibility of keeping fixed some of the geometrical parameters, in particular the subset occupied by the composite obstacle illuminated by an incident wave of a given frequency ω.

It is important to notice that in [7, 8, 12], the analysis of the wire medium problem is considerably simplified by assuming infinitely long fibers: the incident wave can be then decomposed so that reduction to polarized fields is possible and mathematically the problem is reduced to solving Helmholtz equations in the plane with suitable transmission conditions on the boundary of the wire cross sections.

In case of an $\mathbf{e_3}$-polarized electric field and suitably scaled period, filling ratio of fibers and conductivity, it is shown in [8] that the medium is characterized in the quasi static regime (i.e., for $d \ll \lambda$) by an effective relative permittivity (in the $\mathbf{e_3}$-direction) of the form

$$\varepsilon_{33}^{\text{eff}}(\omega) = 1 - \frac{\omega_c^2}{\omega^2}, \tag{1.1}$$

where ω_c represents the so called *cut-off* frequency. Furthermore the asymptotic formula (1.1) turns out to be very accurate even for small ratio $\lambda/d < 10$ as shown in Fig. 3.

Although this result pushes towards a mathematical foundation for the realizability of effective media with a negative bulk permittivity, we have however to be very careful. As will be seen later the model of a *finite* metallic wire medium ("bed-of-nails" structure) is more sophisticated and, unlike common practice in much of the metamaterials literature, it is not correct to assume that the finite structure behaves like an equivalent homogeneous medium of the same size characterized by the same constitutive relation as for the *infinite* medium, even if additional boundary conditions are imposed in order to account for the finiteness of the structure. It turns out that the extreme nonlocality of the initial structure implies automatically that the effective limit law is *non local* as well. In [2] the scatter consists of a domain of $\mathbb{R}^3$ filled by a periodic array of $\mathbf{e_3}$-parallel metallic fibers of finite length L as depicted in Fig. 1. By using a two-scale renormalization approach, we have shown that the effective constitutive equation between the displacement vector D and the electric field E involves a long range interaction kernel:

$$D(x) := \varepsilon_0 \Big(E(x) - 2\pi\gamma \mathbf{e_3} \int_{-\frac{L}{2}}^{\frac{L}{2}} g(\omega, s, x_3) E_3(x_1, x_2, s)\, ds \Big). \tag{1.2}$$

The same kind of non local constitutive law arises if we alternatively consider metallic arrays of parallel fibers disposed simultaneously in three orthogonal directions. It becomes therefore clear that the validity of an effective relation such as in (1.1) cannot be extended to a full 3D-setting, as far as composite made of metallic inclusions are concerned. Hence arises naturally the question of realizability of *a full 3D local negative effective permittivity tensor*.

In this paper we present a rigorous proof for the realizability at a given frequency of any real symmetric permittivity tensor (this includes the negative scalar ones). This result already announced in [3] is to our knowledge the first one obtained in the context of the diffraction of an electromagnetic wave by a finite obstacle in $\mathbb{R}^3$.

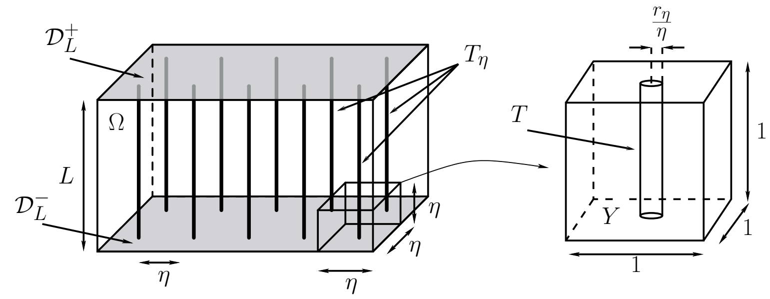

Figure 1: The "bed-of-nails" metallic wire structure.

The key issue consists in constructing a more complex multiscaled geometry (Fig. 2) involving the periodic reproduction at a larger scale of the structure shown in Fig. 1. In other words, the finite wire composite considered before becomes a micro-component of our metamaterial. In this construction the global volume fraction of metallic fibers is still vanishing and the length of fibers is infinitesimal with respect to the incident wavelength but large compared with the initial distance between fibers in each array. Applying a reiterated homogenization procedure we are led to identify the microscopic electric field as the solution of an electrostatic problem in which the densities of charges on the lower and upper base of each fibered component satisfy a spectral equation. Resonance effects occur at frequencies that are comparable to that of the incident wave (micro- resonator problem) as it was observed in a different context (artificial magnetism) in the case of a much simpler geometry (see [6, 9, 10]). As a consequence we obtain an effective permittivity tensor $\varepsilon^{\text{eff}}(\omega)$ whose eigenvalues have real part changing of sign and possibly passing through very large absolute values. By tuning the geometrical parameters of the structure, a very large class of effective permittivity tensors can then be realized.

We point out however that our model will not include artificial magnetic activity as in [4] and [5, 6, 10].

The paper is organized as follows. In Section 2 we report on the non local asymptotic model for the wire metallic medium following [2]. In Section 3, we apply the reiterated homogenization procedure to the multiscale structure described in Fig. 2 and prove the existence of an effective medium. The frequency-dependent effective permittivity is gov-

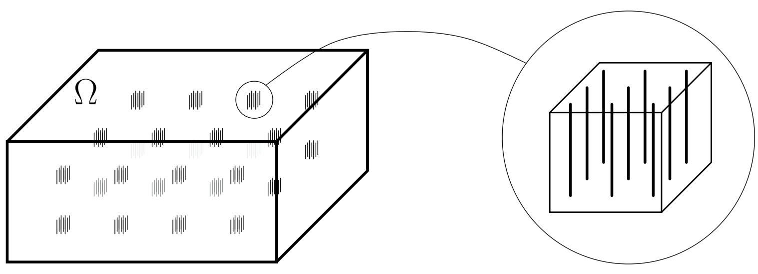

Figure 2: A multiscale structure with $\mathbf{e}_3$-oriented nanorod arrays.

erned by a spectral problem on the periodic cell that we solve numerically in Section 4. Eventually some conclusions are drawn in Section 5.

2 Homogenization of the diffraction by a periodic array of finite metallic fibers

The structure under study is a square biperiodic array of thin wires of length L as depicted in Fig. 1. It is contained in the reference cylinder $\Omega := \mathcal{D}\times(-L/2,L/2)$ of height L with upper-lower bases denoted $\mathcal{D}_L^{\pm}$. The period is represented by the small parameter η and all other varying parameters will be written with subscript η, namely the radius r_η and conductivity σ_η of the fibers.

We denote $T_\eta \subset \Omega$ the volume occupied by fibers. Its horizontal section $x_3 = s$ for $|s| \le L/2$ consists of an array of disks of radius r_η periodically disposed in $\mathcal{D}$ with period η. We assume that $r_\eta \ll \eta$ so that the volume fraction $\theta_\eta := \pi r_\eta^2/\eta^2 \to 0$. Although many variants are possible we assume for simplicity that the fibers made of a unique purely ohmic metal are immersed in vacuum. The relative permittivity ε_η is then given by

$$\varepsilon_\eta := \begin{cases} 1, & \text{on } \mathbb{R}^3\setminus T_\eta, \\ 1+i\sigma_\eta, & \text{on } T_\eta, \end{cases} \tag{2.1}$$

where the conductivity coefficient $\sigma_\eta \to \infty$. The structure is illuminated by a given incident field (E^i,H^i) (wave number in the vacuum $k_0 = \sqrt{\varepsilon_0\mu_0}\omega$). For every value of parameter η, the total electromagnetic field (E_η,H_η) is solution of the time harmonic Maxwell system

$$\begin{cases} \operatorname{curl} E_\eta = i\omega\mu_0 H_\eta, \\ \operatorname{curl} H_\eta = -i\omega\varepsilon_0\varepsilon_\eta E_\eta, \end{cases} \tag{2.2}$$

and the diffracted field $(E_\eta^d,H_\eta^d) := (E_\eta - E^i,\, H_\eta - H^i)$ satisfies Silver and Müller condition:

$$(E_\eta^d,H_\eta^d) = \mathcal{O}\Big(\frac{1}{|x|}\Big), \qquad \omega\varepsilon_0\Big(\frac{x}{|x|}\wedge E_\eta^d\Big) - k_0 H_\eta^d = o\Big(\frac{1}{|x|}\Big). \tag{2.3}$$

Scaling assumptions. Our renormalization involves a limiting process governed by three quantities η, r_η, and $1/\sigma_\eta$ which tend simultaneously to zero. The asymptotic analysis brings to the fore two quantities:

- the average capacity of fibers per unit of volume $2\pi(\eta^2 \log r_\eta)^{-1}$. We will assume that this quantity remains finite positive:

$$\frac{1}{\eta^2 \log r_\eta} \to \gamma, \quad \text{where } \gamma \text{ is a suitable positive parameter;} \tag{2.4}$$

- the average conductivity of fibers per unit of volume. It is measured by the conductivity intensity factor $\kappa \in [0,+\infty]$ where

$$\kappa := \lim_{\eta \to 0} \kappa_\eta, \qquad \kappa_\eta := \sigma_\eta \theta_\eta = \sigma_\eta \frac{\pi r_\eta^2}{\eta^2}. \tag{2.5}$$

We notice that, due to $r_\eta \ll \eta$, a single fiber rescaled homothetically at the size of the period shrinks as $\eta \to 0$ to a one dimensional segment $S_0 = \{(0,0,x_3) : -1/2 \le x_3 \le 1/2\}$ in the unit cell $Y = [-1/2,1/2]^3$.

Bulk current density. We introduce the rescaled field

$$F_\eta = \kappa_\eta \frac{E_\eta}{\theta_\eta} \quad \text{on } T_\eta, \qquad F_\eta = 0 \quad \text{on } \mathbb{R}^3 \setminus T_\eta, \tag{2.6}$$

which agrees on fibers with the divergence free $\varepsilon_\eta E_\eta$. We rewrite the second equation of (2.2) as:

$$\operatorname{curl} H_\eta = -i\omega\varepsilon_0 (E_\eta + iF_\eta). \tag{2.7}$$

A two-scale analysis enables the substitution $F_\eta \sim F_0(x,x/\eta)$ where for all $x \in \Omega$, the Y-periodic vector field $F_0(x,\cdot)$ satisfies $\operatorname{div}_y F_0 = 0$ in the distributional sense and is supported on S_0. Therefore, it is $\mathbf{e_3}$-parallel and determined up to a scalar intensity factor $j(x) \in L^2(\Omega)$:

$$F_0(x,y) = j(x)\mathbf{e_3}\delta_{S_0} \quad \text{(being } \delta_{S_0} \text{ the line distribution along } S_0). \tag{2.8}$$

The resulting macroscopic field $j(x)\mathbf{e_3}$ can be interpreted as a bulk *electric polarization density* flowing parallel to $\mathbf{e_3}$ in the cylinder Ω. The possibility that $j(x) \neq 0$ indicates that the effect of fibers remains present although their volume fraction becomes infinitesimal.

Macroscopic relation between j and E. By (2.5), (2.7) and (2.8), passing to the limit in (2.2) leads to

$$\operatorname{curl} E = i\omega\mu_0 H, \qquad \operatorname{curl} H = -i\omega\varepsilon_0 (E + ij\mathbf{e_3}), \tag{2.9}$$

where the equalities hold in the sense of distributions on all $\mathbb{R}^3$ and where by convention $j(x)$ is extended by zero outside the scatter Ω. To close the system we need to say precisely how the electric polarization density j is induced by the macroscopic electric field E. This delicate task has been performed in [2] where it is shown that j satisfies the following one dimensional boundary value problem:

$$\frac{\partial^2 j}{\partial x_3^2} + \Big(k_0^2 + \frac{2i\pi\gamma}{\kappa}\Big) j = 2i\pi\gamma\, E_3 \quad \text{on } \Omega, \qquad \frac{\partial j}{\partial x_3} = 0 \quad \text{on } \mathcal{D}_L^{\pm}. \tag{2.10}$$

It is worth noticing that the polarization j satisfies Neumann conditions at the upper and lower interfaces of the slab. It is not in general continuous there because Maxwell's equations impose the continuity of the normal component of the displacement field $D \equiv \varepsilon_0 (E + ij\mathbf{e_3})$; consequently, any jump in E must be canceled by an equivalent jump in j.

Effective medium. The homogenized medium is characterized by the system (2.9) and (2.10) together with (2.3). Indeed its solution (E,H) turns out to be unique and coincides with the limit of (E_η,H_η) (see [2] for further details).

Now it is possible to express j in terms of E_3 by solving (2.10) and the second equation in (2.9) becomes $\mathrm{curl} H=-i\omega D$ with D given by (1.2) with the Neumann kernel

$$g(\omega,s,t)=\frac{1}{k\sin(kL)}\cos\Big[k\Big(s\wedge t+\frac{L}{2}\Big)\Big]\cos\Big[k\Big(s\vee t-\frac{L}{2}\Big)\Big],\quad \text{where } k^2:=\varepsilon_0\mu_0\omega^2+\frac{2i\pi\gamma}{\kappa}.$$

The effective associated law is therefore *spatially non local*.

Case of infinitely long wires. The case of infinitely long wires $(L=+\infty)$ can be easily handled by decomposing the incident wave by means of a Fourier transform in x_3. We are then reduced to look for solutions (E,H,j) of (2.9) and (2.10) having a multiplicative x_3-dependence $\exp(i\beta x_3)$. The solution j reads as

$$j=\frac{2i\pi\gamma}{k_0^2-\beta^2+\frac{2i\pi\gamma}{\kappa}}E\cdot\mathbf{e_3}.$$

Thus, after plugging in (2.9), we obtain the same response as a medium with diagonal permittivity tensor ε^{eff} given by:

$$\varepsilon^{\text{eff}}_{1,1}=\varepsilon^{\text{eff}}_{2,2}=1,\qquad \varepsilon^{\text{eff}}_{3,3}=1-\frac{2\pi\gamma}{k_0^2}\Big[\frac{k_0^2-\beta^2}{k_0^2-\beta^2+\frac{2i\pi\gamma}{\kappa}}\Big]. \tag{2.11}$$

The dependence of $\varepsilon^{\text{eff}}(\beta)$ with respect to β confirms that the limit behavior is nonlocal (it involves a convolution with respect to the inverse Fourier transform of ε^{eff} as a function of β). However, for $\kappa=+\infty$, this effect disappears and we recover (1.1) with the cut-off frequency $\omega_c=\sqrt{2\pi\gamma/\varepsilon_0\mu_0}$.

Domain of validity. The accuracy of the formula (2.11) has been checked in [2] where a stack of parallel diffraction gratings infinite in extent in the horizontal direction was considered. In this numerical test, the fibers are infinitely long and infinitely conductings $(\kappa=+\infty)$ and the period d is chosen equal to the distance between each grating. The radius of the fibers is $r=d/200$ so that the theoretical coefficient given in (2.4) is $\gamma\sim 0.2$. Actually the value of γ can be optimized more precisely by studying correctors [12]. The numerical test depicted in Fig. 3 was done for $\gamma=0.25$ and describes the transmitted energy obtained when the structure is illuminated by a plane wave with wavelength λ under normal incidence (dashed line). It is compared with the transmitted energy for a homogeneous slab of height $10d$ with permittivity ε^{eff} (solid line). We found an excellent fit between the two curves which almost coincide for values of the wavelength that are not very high, i.e., for $\lambda/d>4$.

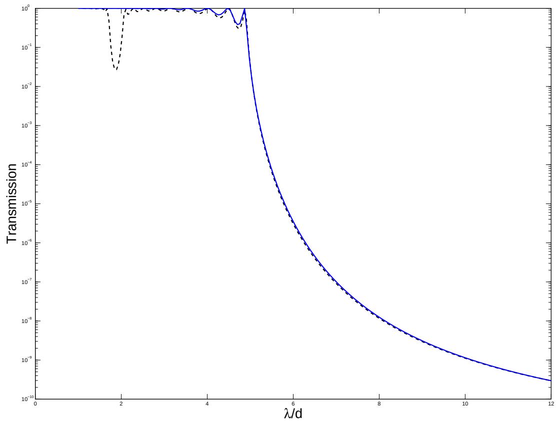

Figure 3: Transmitted energy through the real structure (dashed line) and the homogenized structure (solid line).

3 Reiterated homogenization, band gaps and resonances

As sketched in the introduction, we consider in this second step a periodic structure Σ_η disposed in a finite domain Ω and whose electromagnetic properties are characterized by the system of Eqs. (2.9) and (2.10). In other words, we use the first homogenization process depicted in Section 2 in order to build a metamaterial with non local macroscopic law. Then we create a second kind of device by placing it periodically with a small period η. Alternatively we may complete both operations simultaneously by using two different scales as shown in Fig. 2. The common issue falls in the theory of reiterated homogenization [13].

For simplicity we will focus only on the case where initial metallic fibers are all disposed in the x_3-direction. The periodic obstacle Σ_η represented in Fig. 4 is given by

$$\Sigma_\eta = \Omega \cap \Big(\bigcup_{i\in\mathbb{Z}^3} \eta(i+\Sigma) \Big), \tag{3.1}$$

where $\Sigma = D\times(-h/2,h/2)$ is a cylinder strictly contained in the unit cell Y. Here D is a connected subdomain of $(-1/2,1/2)^2$ and $h<1$ denotes the height. The upper and lower basis of Σ are denoted $D^\pm$. Their rescaled periodic counterparts are

$$D^\pm_\eta = \Omega \cap \Big(\bigcup_{i\in\mathbb{Z}^3} \eta(i+D^\pm) \Big). \tag{3.2}$$

Plugging the system of effective equations (2.9) and (2.10) (once Ω is substituted with Σ_η and $\mathcal{D}^\pm_L$ with $D^\pm_\eta$), we obtain a global diffraction problem described by a triple (E_η,H_η,j_η)

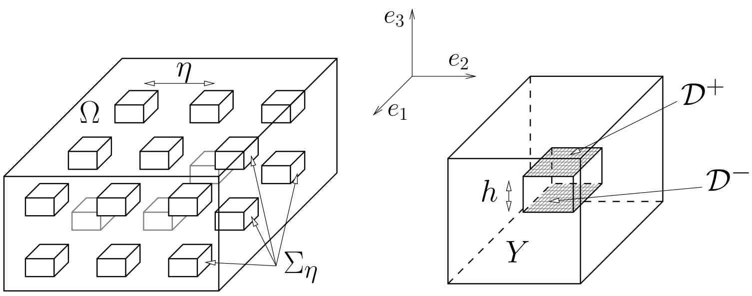

Figure 4: The unit cell in the reiterated process.

in $L^2_{\rm loc}$ such that:

$$\begin{cases} {\rm curl} E_\eta = i\omega\mu_0 H_\eta, & \text{on } \mathbb{R}^3, \\ {\rm curl} H_\eta = -i\omega\varepsilon_0 (E_\eta + i j_\eta \mathbf{e_3}), & \text{on } \mathbb{R}^3, \\ \dfrac{\partial^2 j_\eta}{\partial x_3^2} + \Big(k_0^2 + \dfrac{2i\pi\gamma}{\kappa}\Big) j_\eta = 2i\pi\gamma E_\eta \cdot \mathbf{e_3}, & \text{on } \Sigma_\eta, \\ \dfrac{\partial j_\eta}{\partial x_3} = 0, & \text{on } D_\eta^\pm, \\ (E_\eta - E^i, H_\eta - H^i) \quad \text{satisfies the radiation condition (2.3)}, \end{cases} \tag{3.3}$$

where the scalar function j_η has been extended by setting $j_\eta = 0$ on $\mathbb{R}^3 \setminus \Sigma_\eta$.

Estimates in $L^2_{\rm loc}$ and two-scale convergence. In the perspective of passing to the limit as $\eta \to 0$, we consider a reference ball $B = \{x \in \mathbb{R}^3 : |x| < R\}$ such that $\Omega \subset B$ and we assume in a first step that

$$\sup_\eta \int_B |E_\eta|^2 + |H_\eta|^2 < +\infty. \tag{3.4}$$

Then, up to subsequences, $(E_\eta(x), H_\eta(x))$ converges weakly in $L^2(B)$ to some limit electromagnetic field $(E(x), H(x))$. We denote by $\mathcal{P}_\eta$, $\mathcal{P}$ the outgoing flux of the Poynting vector through ∂B orientated by its exterior normal $n(x)$:

$$\mathcal{P}_\eta := \int_{\partial B} (E_\eta \wedge \overline{H_\eta}) \cdot n(x) d\sigma, \qquad \mathcal{P} := \int_{\partial B} (E \wedge \overline{H}) \cdot n(x) d\sigma. \tag{3.5}$$

Since $j_\eta = 0$ in the complementary of Ω, we infer from (3.3) that all the components of the fields E_η and H_η satisfy the Helmholtz equation $\Delta u + k_0^2 u = 0$ on the open set $\mathbb{R}^3 \setminus \overline{\Omega}$. Therefore the convergence of $(E_\eta(x), H_\eta(x))$ on the whole $\mathbb{R}^3 \setminus B$ can be established thanks to the following Lemma (see for instance [2] for a detailed proof).

Lemma 3.1. *Let (E_η, H_η, j_η) be a solution of* (3.3) *such that $(E_\eta, H_\eta) \rightharpoonup (E, H)$ weakly in $L^2(B)$ and assume that $(E_\eta - E_\eta^i, H_\eta - H_\eta^i)$ satisfies the outgoing wave condition* (2.3) *for a suitable sequence of incident waves (E_η^i, H_η^i) converging uniformly to (E^i, H^i). Then the convergence* *of* (E_η,H_η) *holds in* $\mathcal{C}^\infty(K)$ *for every compact subset* $K\subset\mathbb{R}^3\setminus\overline{\Omega}$. *The limit field* (E,H) *solves the Helmoltz equation* $\Delta u+k_0^2u=0$ *in* $\mathbb{R}^3\setminus\overline{\Omega}$ *and is such that* $(E-E^i,H-H^i)$ *satisfies* (2.3). *Furthermore we have the convergence* $\mathcal{P}_\eta\to\mathcal{P}$.

As a consequence of the boundedness of the energy flux $\mathcal{P}_\eta$ defined in (3.5) we deduce

Lemma 3.2. *Under the conditions of Lemma* 3.1, *the sequences* $\{j_\eta\}$ *and* $\{H_\eta\}$ *are bounded respectively in* $L^2(B)$ *and in* $W^{1,2}(B)$. *In particular* $H_\eta\to H$ *strongly in* $L^2(B)$. *In addition, if* $E=0$ *on* B, *then* $\lim_{\eta\to0}\int_B|E_\eta|^2=0$.

Proof. By Lemma 3.1, the field (E_η,H_η) is smooth on ∂B. We may integrate by parts and by using (3.3), we obtain

$$\mathcal{P}_\eta=\int_B\left(\mathrm{curl}E_\eta\cdot\overline{H_\eta}-\mathrm{curl}\overline{H_\eta}\cdot E_\eta\right)dx=i\omega\int_B\left(\mu_0|H_\eta|^2-\varepsilon_0|E_\eta|^2\right)-\omega\varepsilon_0\int_\Omega\overline{j_\eta}E_\eta\cdot\mathbf{e_3}dx. \quad (3.6)$$

Now, by using the propagation equation for j_η in (3.3) together with the Neumann boundary condition and after integrating by parts with respect to x_3 on $[-h/2,h/2]$, we are led to

$$-\int_{\Sigma_\eta}\left|\frac{\partial j_\eta}{\partial x_3}\right|^2dx+\int_{\Sigma_\eta}\left(k_0^2+\frac{2i\pi\gamma}{\kappa}\right)|j_\eta|^2dx=\int_{\Sigma_\eta}2i\pi\gamma\, E_\eta\cdot\mathbf{e_3}\overline{j_\eta}dx. \quad (3.7)$$

Thus by equating the imaginary parts in (3.6) and the real parts in (3.7), we are lead to the following energy estimate (which corresponds to the dissipation by Joule's effect)

$$\Re(\mathcal{P}_\eta)=\omega\varepsilon_0\Re\Big(\int_\Omega E_\eta\cdot\mathbf{e_3}\overline{j_\eta}dx\Big)=\frac{\omega\varepsilon_0}{\kappa}\int_\Omega|j_\eta|^2dx. \quad (3.8)$$

Here we used that $j_\eta=0$ on $\mathbb{R}^3\setminus\Sigma_\eta$. Thus the sequence $\{j_\eta\}$ is bounded in $L^2(B)$. By the second equation in (3.3), the divergence free field H_η is such that $\mathrm{curl}H_\eta$ remains uniformly bounded in $L^2(B)$. It follows from the C^∞ convergence on a neighborhood of ∂B (see Lemma 3.1) that $\{H_\eta\}$ remain bounded in the Sobolev space $W^{1,2}(B)$, thus converges strongly in $L^2(B)$ by Rellich's Theorem.

Let us prove the last statement. If $E=0$ on B, then $\lim_\eta\mathcal{P}_\eta=\mathcal{P}=0$ and by (3.8) $j_\eta\to0$ strongly in $L^2(B)$. By applying the divergence operator to both sides of Eq. (3.3), we derive that $\mathrm{div}E_\eta=-i\omega\varepsilon_0\partial j_\eta/\partial x_3$ converges strongly to zero in the norm of $W^{-1,2}(B)$. On the same way as $H_\eta\to0$ in $L^2(B)$, by the first Maxwell equation in (2.2), we have $\mathrm{curl}E_\eta\to0$ strongly in $W^{-1,2}(B)$. We may therefore apply the div-curl Lemma (see [16]) to the sequence (E_η,E_η). Thus, for every $\varphi\in C_c^\infty(B;[0,1])$, there holds $\lim_{\eta\to0}\int_B\varphi|E_\eta|^2dx=0$. Taking $\varphi=1$ on a smaller ball B' such that $\Omega\subset B'\subset B$ and as, by Lemma 3.1, the convergence of E_η to 0 is uniform on $B\setminus B'$, we immediately derive that also $\lim_{\eta\to0}\int_B|E_\eta|^2dx=0$. The proof is finished. □

By Lemma 3.2, the sequence $\{j_\eta\}$ is bounded in $L^2(B)$. In contrast with the magnetic field H_η, it turns out that j_η and E_η are not strongly relatively compact in $L^2(\Omega)$. They show oscillations at scale η that we will describe by identifying their two scale limits

$(E_0(x,y),j_0(x,y))$, which are nothing else but the zero order term in the formal two-scale expansions

$$E_\eta = E_0\Big(x,\frac{x}{\eta}\Big) + \eta E_1\Big(x,\frac{x}{\eta}\Big) + \cdots, \qquad j_\eta = j_0\Big(x,\frac{x}{\eta}\Big) + \eta j_1\Big(x,\frac{x}{\eta}\Big) + \cdots.$$

We refer to [1,11] for the definition of the two-scale convergence and its main properties. The macroscopic limit field (E,j) on Ω (i.e., the weak limit of (E_η,j_η) in $L^2(\Omega)$ can be recast simply as

$$E(x) = \int_Y E_0(x,y)dy, \qquad j(x) = \int_Y j_0(x,y)dy. \tag{3.9}$$

Electrostatic problem on the unit periodic cell. The unit cell is $Y = [-1/2,1/2]^3$ and $\mathbb{T} := \mathbb{R}^3 \setminus \mathbb{Z}^3$ denotes the three dimensional torus. Thanks to the above $L^2(B)$ bounds, we may assume, possibly after extracting a subsequence, that the sequence (E_η,j_η) two-scale converges to (E_0,j_0) whose components are elements of $L^2(B\times\mathbb{T})$. As $E_\eta \to E$ uniformly on compact subsets of $\mathbb{R}^3\setminus\Omega$ where $j_\eta = 0$, we have of course $E_0(x,\cdot) = E(x)$, $j_0(x,\cdot) = j(x) = 0$ for a.e. $x \in B\setminus\Omega$.

Lemma 3.3. *For a.e. $x\in\Omega$, the Y-periodic fields $E_0(x,\cdot)$, $j_0(x,\cdot)$ satisfy*

$$\mathrm{curl}_y E_0(x,\cdot) = 0 \quad on \ \ Y, \qquad \mathrm{div}_y\big(E_0(x,\cdot) + ij_0(x,\cdot)\mathbf{e_3}\big) = 0 \quad on \ \ Y, \tag{3.10a}$$

$$j_0(x,\cdot) = 0 \quad on \ \ Y\setminus\Sigma, \qquad \frac{\partial j_0}{\partial y_3}(x,\cdot) = 0 \quad on \ \ \Sigma, \tag{3.10b}$$

$$j_0(x,y) = j_0(x,y_1,y_2) = \frac{2i\pi\gamma}{k_0^2 + \frac{2i\pi\gamma}{\kappa}} \langle E_0\rangle_h(x,y_1,y_2), \quad for \ \ y\in\Sigma, \tag{3.10c}$$

where $\langle E_0\rangle_h(x,y_1,y_2) := \frac{1}{h}\int_{-h/2}^{h/2} E_0(x,y_1,y_2,s)\cdot\mathbf{e_3}ds$ (normalized circulation of $E_0(x,\cdot)$ along vertical lines joining the bases $D_h^\pm$ of the cylinder Σ).

Proof. i) By the first equation in (3.3), $\mathrm{curl}E_\eta$ is bounded in $L^2(\Omega;\mathbb{C}^3)$, whereas, by the second equation, $F_\eta := E_\eta + ij_\eta\mathbf{e_3}$ is divergence free and two-scale converges to $F_0 := E_0 + ij_0\mathbf{e_3}$. It is then standard (see [1]) that $\mathrm{curl}_y E_0(x,\cdot) = 0$ and $\mathrm{div}_y F_0(x,\cdot) = 0$ holds for a.e. x, which gives precisely (3.10a).

ii) The first assertion in (3.10b) is obtained by applying the two-scale convergence of j_η to j_0 for test functions of the kind $\varphi(x,y) = \theta(x)\phi(y)$, where $\theta\in C_c^\infty(\Omega)$ and $\phi\in C^\infty(\mathbb{T})$ vanishing on Σ (thus $j_\eta\varphi(x,x/\eta) = 0$).

To prove that $\frac{\partial j_0}{\partial y_3}(x,\cdot) = 0$ in Σ, we simply take the previous ϕ compactly supported in Σ and exploit the fact that, by (3.7), the function $\partial j_\eta/\partial x_3$ is uniformly bounded in $L^2(\Sigma_\eta)$. Therefore, integrating by parts in x_3, we obtain

$$\begin{aligned} 0 &= \lim_\eta \int_{\Sigma_\eta} \eta \frac{\partial j_\eta}{\partial x_3}\varphi\Big(x,\frac{x}{\eta}\Big) = -\lim_\eta \int_\Omega j_\eta \Big(\theta(x)\frac{\partial\phi}{\partial y_3}\Big(\frac{x}{\eta}\Big) + \eta\frac{\partial\theta}{\partial x_3}\phi\Big(\frac{x}{\eta}\Big)\Big) \\ &= -\int_{\Omega\times Y} \theta(x) j_0(x,y)\frac{\partial\phi}{\partial y_3}. \end{aligned}$$

Localizing with respect to the x variable, we deduce that $\int_Y j_0(x,y)\frac{\partial\phi}{\partial y_3}dy$ vanishes for a.e. $x\in\Omega$. Therefore, $j_0(x,y)=j_0(x,y_1,y_2)$ on $\Omega\times\Sigma$.

iii) To show (3.10c), we chose

$$\varphi(x,y)=\theta(x)\phi(y')1_{\{|y_3|<\frac{h}{2}\}},$$

where we denote $y'=(y_1,y_2)$ and $\theta(x)$, $\phi(y')$ are smooth compactly supported in Ω and D respectively. Then we multiply the third equation in (3.3) by $\varphi(x,x/\eta)$ and integrate over Ω. Exploiting (3.10b), the homogeneous Neumann condition on $D_\eta^{\pm}$, we obtain after passing to the limit $\eta\to 0$

$$\left(k_0^2+\frac{2i\pi\gamma}{\kappa}\right)\int_\Omega\int_D \theta(x)hj_0(x,y')\phi(y')dxdy'=2i\pi\gamma\int_\Omega\int_\Sigma \theta(x)E_0(x,y)\cdot\mathbf{e_3}dxdy,$$

hence the conclusion, as $\int_\Sigma E_0(x,y)\cdot\mathbf{e_3}dy=h\int_D\langle E_0\rangle_h(x,y')dy'$. □

Micro-resonator problem. By (3.10a), we can write $E_0(x,\cdot)$ in term of a suitable periodic scalar potential $\Phi(x,\cdot)$:

$$E_0(x,y)=E(x)+\nabla_y\Phi(x,y).$$

By (3.10a), (3.10b) and (3.10c), Φ satisfies

$$\Delta_y\Phi=ij_0(\delta_{D^+}-\delta_{D^-}),\qquad j_0=\frac{2i\pi\gamma}{k_0^2+\frac{2i\pi\gamma}{\kappa}}(E_3+[\Phi]_h),\tag{3.11}$$

where

$$[\Phi]_h(x,y_1,y_2):=\frac{1}{h}\left(\Phi\left(x,y_1,y_2,\frac{h}{2}\right)-\Phi\left(x,y_1,y_2,-\frac{h}{2}\right)\right).$$

We introduce the operator $B_h:w\in L^2(D)\mapsto[\varphi]_h(y_1,y_2)$ where φ is the unique solution in $H^1(\mathbb{T})$ of the equation $-\Delta\varphi=w(\delta_{D^+}-\delta_{D^-})$.

Lemma 3.4. *The linear operator B_h is positive compact self-adjoint. Let $\nu_0^2>\nu_1^2\geq\nu_2^2\cdots\geq\nu_n^2\geq\cdots$ be its real positive eigenvalues ($\nu_n^2\to 0$ as $n\to\infty$). Then the fundamental eigenvalue ν_0^2 is simple and satisfies the inequality*

$$|D|(1-h)\leq\nu_0^2\leq 1.\tag{3.12}$$

Proof. By integration by parts, for every (v,w) in $(L^2(D))^2$, there holds

$$(B_hv,w)=\frac{1}{h}\int_Y\nabla\varphi_v\cdot\nabla\varphi_w dx.$$

Thus B_h is non negative self-adjoint. It is compact as the composition of the continuous map $v\in L^2(D)\mapsto\varphi_v\in H^1(\mathbb{T})$ by the compact map $\varphi\in H^1(\mathbb{T})\mapsto[\varphi]_h\in L^2(D)$ (here we used the compact embedding of the trace space $H^{1/2}(D)$ into $L^2(D)$). Next we use, for every

$w\in L^2(D)$, the following variational characterizations of $(B_h(w)|w)$ (in primal and dual form)

$$(B_h w|w)=\sup\Big\{2\int_D w[\varphi]_h-\frac{1}{h}\int_Y|\nabla\varphi|^2:\varphi\in H^1(\mathbb{T})\Big\} \tag{3.13}$$

$$=\inf\Big\{\frac{1}{h}\int_Y|\sigma|^2:\sigma\in(L^2(\mathbb{T}))^3,\ -\mathrm{div}\,\sigma=w(\delta_{D^+}-\delta_{D^-})\Big\}, \tag{3.14}$$

where in the last equality the divergence is taken in the sense of distributions on the torus (in particular admissible σ have normal traces changing of sign on opposite faces of ∂Y). Furthermore, $B_h w$ coincides with $[\varphi_w]_h$ being φ_w the unique maximizer in (3.13). By exploiting the symmetry with respect to y_3, it is easy to see that $\varphi_w(y',-y_3)=-\varphi_w(y',y_3)$ so that, setting $Y^+:=Y\cap\{y_3>0\}$

$$\frac{h}{2}(B_h w|w)=\sup\Big\{2\int_D w(y')\varphi\Big(y',\frac{h}{2}\Big)dy'-\int_{Y^+}|\nabla\varphi|^2dx\Big\}, \tag{3.15}$$

where the supremum is taken with respect to functions $\varphi(y',y_3)\in H^1(Y^+)$ which are periodic in y' and whose trace vanishes on $y_3\in\{0,1/2\}$.

We claim that $B_h w\geq 0$ whenever $w\geq 0$. Indeed, in that case, the right hand member of (3.15) increases while substituting φ with φ^+. By the uniqueness of the maximizer, it follows that φ_w is non negative for $y_3>0$, thus in particular on D_h^+ and the claim. As a consequence, by Krein-Rutman criterium, B_h admits a unique positive eigenvector associated with ν_0^2 which is simple.

Eventually, we establish (3.12). For every $w\in L^2(D)$, the vector field defined by $\sigma(y)=w(y')\mathbf{e_3}$ on Σ and extended by zero outside is admissible for (3.14). Therefore $B_h(w,w)\leq\int_D|w|^2dy'$ thus $\nu_0^2\leq 1$. Conversely, by taking $w=1$ and $\varphi=\varphi(y_3)$ in (3.13), where φ is a odd function in $H_0^1(-1/2,1/2)$, we obtain

$$|D|\nu_0^2\geq(B_h(1)|1)\geq\frac{1}{h}\sup\Big\{4|D|\varphi\Big(\frac{h}{2}\Big)-2\int_0^{\frac{1}{2}}\varphi'^2ds,\ \ \varphi(0)=\varphi\Big(\frac{1}{2}\Big)=0\Big\}.$$

The optimum in the right hand side is obtained for φ such that $\varphi'=|D|(1-h)$ for $|s|<h/2$ and $\varphi'=-|D|h$ otherwise and we get: $\nu_0^2\geq|D|(1-h)$. □

Coming back to (3.11), we see that $j_0(x,\cdot)$ satisfies a spectral equation of Freedholm type:

$$B_h(j_0)-\Big(\frac{k_0^2}{2\pi\gamma}+\frac{i}{\kappa}\Big)j_0=-iE_3(x). \tag{3.16}$$

Let $\{\varphi_n:n\in\mathbb{N}\}$ be an orthonormal basis of $L^2(D)$ such that $B_h\varphi_n=\nu_n^2\varphi_n$. Then the solution of (3.16) is given by

$$j_0(x,y_1,y_2)=iE_3(x)\chi(\omega,y_1,y_2), \tag{3.17}$$

where

$$\chi(\omega,y_1,y_2):=\sum_n c_n\varphi_n, \qquad c_n=\frac{\int_D \varphi_n}{\frac{k_0^2}{2\pi\gamma}-\nu_n^2+\frac{i}{\kappa}}.$$

Effective permittivity. Recalling the relation $k_0^2=\omega^2\varepsilon_0\mu_0$, we introduce

$$\Lambda(\kappa,\gamma,\omega):=\int_D \chi(\omega,y_1,y_2)dy_1dy_2=\sum_n \frac{(\int_D \varphi_n)^2}{\frac{k_0^2}{2\pi\gamma}-\nu_n^2+\frac{i}{\kappa}}. \tag{3.18}$$

Then exploiting (3.17) and (3.18), the weak limit j of $\{j_\eta\}$ in $L^2(\Omega)$ can be computed thanks to (3.9):

$$j(x)=\int_\Sigma j_0(x,y)dy=ih\Lambda(\kappa,\gamma,\omega)E_3(x). \tag{3.19}$$

Accordingly the displacement field $\mathcal{D}_\eta(x):=\varepsilon_0(E_\eta+ij_\eta\mathbf{e_3})$, which appears in the right hand member of the second equation in (3.3) converges weakly on Ω to $\mathcal{D}(x):=E(x)+ij(x)\mathbf{e_3}=\varepsilon^{eff}E(x)$, where the diagonal tensor ε^{eff}, given below, represents the *local effective permittivity law* on Ω.

$$\varepsilon_{11}^{eff}=\varepsilon_{22}^{eff}=1, \qquad \varepsilon_{33}^{eff}=1-h\Lambda(\kappa,\gamma,\omega). \tag{3.20}$$

Its third component ε_{33}^{eff} depends on the frequency and has a strictly positive imaginary part. Its real part changes sign when $k_0^2(\omega)=\varepsilon_0\mu_0\omega^2$ passes through the eigenvalues (resonances) and becomes very large if $\kappa\gg 1$.

Theorem 3.1 (Homogenization). *Let (E^i,H^i) be an incident wave and let (E_η,H_η) be the unique solution of* (3.3). *Assume that $\kappa,\gamma>0$ and set*

$$\hat{\varepsilon}(x)=\begin{cases}\varepsilon^{eff}, & \text{if } x\in\Omega,\\ 1, & \text{if } x\in\mathbb{R}^3\setminus\Omega,\end{cases} \qquad \varepsilon^{eff} \text{ defined by (3.20)}.$$

Then $(E_\eta,H_\eta)\to(E,H)$ in $L^2_{\rm loc}(\mathbb{R}^3)$ where (E,H) is the unique solution of the following diffraction problem:

$$\begin{cases}\mathrm{curl}E=i\omega\mu_0H, & \text{on } \mathbb{R}^3,\\ \mathrm{curl}H=-i\omega\varepsilon_0\hat{\varepsilon}(x)E, & \text{on } \mathbb{R}^3,\\ (E-E^i,H-H^i) \text{ satisfies the outgoing wave condition (2.3).}\end{cases} \tag{3.21}$$

Proof. As a preliminary step, we show that the solution of the limit problem is unique. By linearity, this reduces to check that (E,H) vanishes whenever it solves (3.21) and satisfy (2.3) for $(E^i,H^i)=(0,0)$. As $\hat{\varepsilon}=1$ outside Ω, the real part of the outgoing flux of the Poynting vector through the boundary of a ball B_R satisfies

$$\Re\Big(\int_{\partial B_R}(E\wedge\bar{H})\cdot n(x)\Big)=\lim_{R\to\infty}\Re\Big(\int_{\partial B_R}(E\wedge\bar{H})\cdot n(x)\Big)=0.$$

On the other hand, by exploiting (3.21) and integrating by parts over B_R

$$
\begin{aligned}
0=\Re\Big(\int_{\partial B_R}(E\wedge\bar{H})\cdot n(x)\Big)&=\Re\Big(\int_{B_R}\mathrm{curl}E\cdot\overline{H}-\mathrm{curl}\overline{H}\cdot E\Big)\\
&=\omega\Im\Big(\int_{B_R}\Big\{-\mu_0|H|^2+\varepsilon_0\overline{\hat{\varepsilon}(x)}E\cdot\overline{E}\Big\}\Big)\\
&=-\omega\varepsilon_0\int_\Omega\Im(\varepsilon_{33}^{\mathrm{eff}})|E_3|^2.
\end{aligned}
$$

Thus E_3 vanishes on Ω. It follows that (E,H) satisfies Maxwell equations in the vacuum in all $\mathbb{R}^3$ without incident wave. It is then standard to deduce that (E,H) vanishes in all $\mathbb{R}^3$.

In a second step, we prove Theorem 3.1 assuming the energy bound (3.4). By Lemma 3.2, possibly after extracting a subsequence, the triple (E_η,H_η,j_η) converges weakly in $L2(B)$ so some triple (E,H,j). By Lemma 3.1, the convergence of (E_η,H_η) can be improved and extended to all $\mathbb{R}^3\setminus\overline{\Omega}$ so that (E,H) satisfies Helmholtz equation in the vacuum and satisfies the radiation condition (2.3). By passing to the limit in (3.3) we are led to the system of equations

$$
\mathrm{curl}E=i\omega\mu_0H,\qquad \mathrm{curl}H=-i\omega\varepsilon_0(E+ij\mathbf{e_3})
$$

holding in the distributional sense on $\mathbb{R}^3$. By inserting the expression of j given in (3.19), we derive that (E,H) solves (3.21). The uniqueness of the solution implies that the whole sequence (E_η,H_η) does converge to (E,H).

In a last step, we establish the energy bound (3.4) by using a contradiction argument. Assume that the sequence (E_η,H_η) is not bounded in $L^2(B)$. Then possibly after extracting a subsequence, we may assume that

$$
t_\eta:=\Big(\int_B|E_\eta|^2+|H_\eta|^2\Big)^{\frac{1}{2}}\to\infty.
$$

We normalize the fields and define

$$
\tilde{E}_\eta:=\frac{1}{t_\eta}E_\eta,\quad \tilde{H}_\eta:=\frac{1}{t_\eta}H_\eta,\quad \text{with}\ \int_B|\tilde{E}_\eta|^2+|\tilde{H}_\eta|^2=1. \tag{3.22}
$$

Then, by applying the previous step, substituting (E^i,H^i) with $(E^i/t_\eta,H^i/t_\eta)$, we obtain that $(\tilde{E}_\eta,\tilde{H}_\eta)$ converges to the unique solution $(\tilde{E},\tilde{H})$ of the effective diffraction problem (3.21) in which the radiation condition (2.3) holds with $(E^i,H^i)=0$. Therefore $\tilde{E}=\tilde{H}=0$. In addition, by Lemma 3.2, the convergence of $(\tilde{E}_\eta,\tilde{H}_\eta)$ is strong in $L^2(B)$. This is in contradiction with the normalization condition in (3.22). $\square$

We finish this Section by two remarks related to some straightforward variants of the main result.

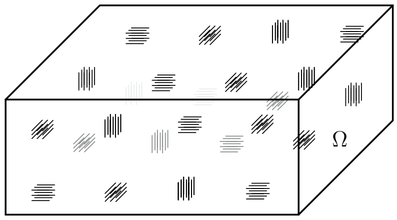

Figure 5: A multiscale structure with multi-oriented nanorods arrays.

Remark 3.1. [Infinitely conducting case] For $\kappa=+\infty$ (fibers of infinite conductivity), the function Λ in (3.18) becomes real and $\varepsilon_{33}^{eff}(\omega)$ is real as well and satisfies, for every n, $\varepsilon_{33}^{eff}(\omega_n\pm 0)=\mp\infty$, where $\omega_n:=\nu_n\sqrt{2\pi\gamma/\varepsilon_0\mu_0}$.

Accordingly the associated diffraction problem is characterized by infinitely many band gaps of frequencies (accumulating at zero) where ε_{33}^{eff} is negative. An important feature of the limiting case $\kappa=+\infty$ is that the metamaterial we obtain is *non-dissipative*.

Remark 3.2. [Variant with three directions of fibers] We may construct a metamaterial by mixing three families of metallic fibers components each of them being disposed alternatively in the three directions of axis as depicted in Fig. 5. That way we will reach all effective diagonal tensors of the kind

$$\varepsilon^{\rm eff}(\omega)=\mathrm{diag}\big(1-h_1\Lambda_1(\omega),1-h_2\Lambda_2(\omega),1-h_3\Lambda_3(\omega)\big). \tag{3.23}$$

Here the parameters h_l and functions $\Lambda_l=\Lambda_l(\kappa_l,\gamma_l,\omega)$ can be tuned playing with the particular geometry and electromagnetic properties of each family of inclusions.

This is a very useful variant of Theorem 3.1 that we do not develop in details in order to avoid a too lengthy paper. However the proof follows exactly the same scheme with straightforward modifications that we sketch hereafter. The periodic obstacle Σ_η is given by (3.1) where Σ is now the union of three disjoint cylinders $\Sigma^{(l)}$, $l\in\{1,2,3\}$, each having cross section D_l, height h_l and direction e_l. Accordingly the wire components structure Σ_η is split in three subfamilies $\Sigma_\eta^{(l)}$. The system (3.3) has to be written with a three components electric polarization vector $J_\eta=(j_\eta^{(1)},j_\eta^{(2)},j_\eta^{(3)})$. The second equation in (3.3) becomes $\mathrm{curl}H_\eta=-i\omega\varepsilon_0(E_\eta+iJ_\eta)$ and the third and fourth equations are substituted with their counterparts ($l\in\{1,2,3\}$):

$$\frac{\partial^2 j_\eta^{(l)}}{\partial x_l^2}+\Big(k_0^2+\frac{2i\pi\gamma^{(l)}}{\kappa^{(l)}}\Big)j_\eta^{(l)}=2i\pi\gamma^{(l)}E_\eta\cdot\mathbf{e_1}\quad\text{on }\Sigma_\eta^{(l)},\qquad \frac{\partial j_\eta^{(l)}}{\partial x_l}=0\quad\text{on }D^\pm_{l,\eta}.$$

The two-scale limit (E_0,J_0) $(J_0=(j_0^{(l)}))$ of (E_η,J_η) is then characterized in term of a scalar periodic potential $\Phi(x,\cdot)$ such that

$$E_0(x,y)=E(x)+\nabla_y\Phi(x,y),\quad \Delta_y\Phi=i\sum_{l=1}^{l=3}j_l(\delta_{D_l^+}-\delta_{D_l^-}),\quad j_0^{(l)}=\frac{2i\pi\gamma_l}{k_0^2+\frac{2i\pi\gamma_l}{\kappa_l}}(E_l+[\Phi]_{h_l}^{(l)}),$$

where $[\Phi]_{h_l}^{(l)}$, as a function on D_l denotes the difference of the values taken by potential ϕ between D_l^+ and D_l^- (up to multiplicative factor h_l^{-1}). By linearity it is easy to check that these equations lead to the analogous of (3.19) and it holds

$$J(x)=\int_\Sigma J_0(x,y)dy=i\sum_{l=1}^{l=3} h_l\Lambda_l(\kappa_l,\gamma_l,\omega)E_l(x)\mathbf{e}_l,$$

from which follows the constitutive relation

$$D(x)=\varepsilon_0(E(x)+iJ(x))=\varepsilon^{\text{eff}}(\omega)E(x)$$

with $\varepsilon^{\text{eff}}(\omega)$ given by (3.23).

4 Numerical analysis and results

The effective permittivity is computed numerically in the case of inclusions with a rectangular section $D=[-l_1/2;l_1/2]\times[-l_2/2;l_2/2]$. In view of (3.18), (3.20), we need only to determine the eigenvalues and eigenvectors of the operator involved in the micro-resonator problem (3.16). To that aim we express B_h as an integral operator on $L^2(D)$ by means of the 3D-periodic Green kernel of the Laplace operator:

$$\begin{aligned}K(x;y)=&\frac{1}{4\pi}\sum_{(i,j,k)\in\mathbb{Z}^3}\text{erfc}\Big(\frac{\|x-y+(i,j,k)\|}{2\sqrt{\beta}}\Big)\big(\|x-y+(i,j,k)\|\big)^{-1}\\ &-\beta+8\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty}\gamma_{ijk}\frac{e^{-4\beta\pi^2\|(i,j,k)\|^2}}{4\pi^2\|(i,j,k)\|^2}\\ &\times\cos\big(2i\pi(x_1-y_1)\big)\cos\big(2j\pi(x_2-y_2)\big)\cos\big(2k\pi(x_3-y_3)\big),\end{aligned}\tag{4.1}$$

where for all $(i,j,k)\in(\mathbb{Z}\setminus\{0\})^3$ we have set:

$$\gamma_{000}=0,\qquad \gamma_{i00}=\gamma_{0j0}=\gamma_{00k}=\frac{1}{4},\qquad \gamma_{ij0}=\gamma_{i0k}=\gamma_{0jk}=\frac{1}{2},\qquad \gamma_{ijk}=1.$$

It turns out that the latter expression does not depend of parameter β. The value of β is set in order to improve the speed of convergence of the series (we took $\beta=0.072$ as in [15]). The unique periodic solution of $-\Delta\varphi=w(\delta_{D^+}-\delta_{D^-})$ reads

$$\varphi(y)=\int_D w(z')\Big(K\Big(y;z',\frac{h}{2}\Big)-K\Big(y;z',\frac{-h}{2}\Big)\Big)dz'.$$

Since such a solution is odd in y_3, we have

$$[\varphi]_h(y')=\frac{2}{h}\varphi\Big(y',\frac{h}{2}\Big),$$

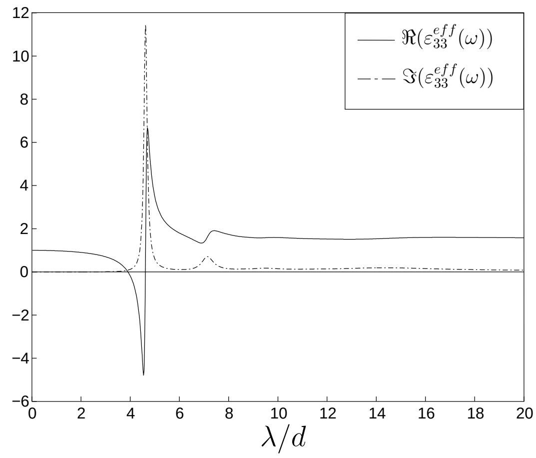

Figure 6: ε^{eff} for $\kappa=100$.

thus

$$B_h w(y') = \frac{2}{h}\int_D w(z')\big(K(y',0;z',0) - K(y',h;z',0)\big)dz'. \tag{4.2}$$

Recall that it is a compact and self adjoint operator. We consider a Galerkin approximation by considering a partition $\{C_i^N\}$ of D made of N^2 rectangles of dimension $\frac{l_1}{N}\times\frac{l_2}{N}$ and the associated orthonormal basis. Set

$$e_i^N = \frac{1}{|C_i^N|^{\frac{1}{2}}}1_{C_i^N}, \qquad V^N = vect\{e_i^N : 1\le i\le N^2\},$$

where $1_{C_i^N}$ denotes the characteristic function of the set C_i^N.

Then an approximation of finite rank of B_h is given by $B_h^N := P^N\circ B_h\circ P^N$, where P^N denotes the orthogonal projection on V_N. As B_h is compact while P_N is uniformly bounded and pointwise convergent to the identity, we clearly have that B_h^N is uniformly compact and converges strongly to B_h. The convergence of eigenvalues and eigenvectors of B_h^N to that of B_h is then standard (see for instance [14]). An approximation of ν_n^2 and $(1|\varphi_n)$ in the expression (3.18) is then determined by searching the eigenvalues and eigenvectors of the matrix A^N whose N^4 entries are given by:

$$A_{ij}^N = \langle B_h e_i^N, e_j^N\rangle = \frac{2N^2}{hl_1l_2}\iint_{C_i^N\times C_j^N}\big(K(y,0;z,0) - K(y,h;z,0)\big)dydz. \tag{4.3}$$

The computation of A_{ij}^N requires some attention due to the singularity of the term corresponding to $i=j=k=0$ in the expansion (4.1). To that aim, we decompose $K=K_r+S$ where

$$S(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|}\mathrm{erfc}\Big(\frac{|x-y|}{2\sqrt{\beta}}\Big), \qquad K_r(x,y) = K(x,y) - S(x,y).$$

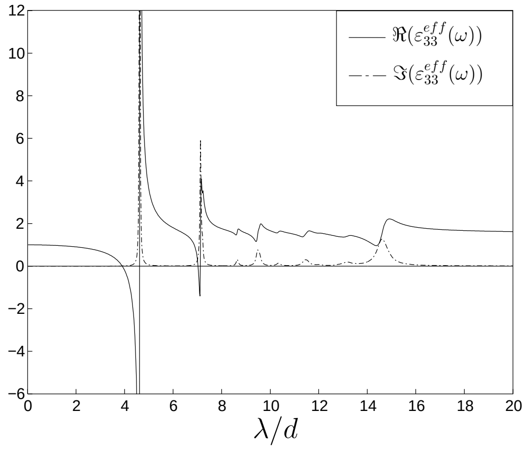

Figure 7: ε^{eff} for $\kappa = 1000$.

For C_i^N close to C_j^N, the double integration of S on $C_i^N \times C_j^N$ in (4.3) is evaluated as

$$\int_{C_i^N \times C_j^N} S(y,z)\,dydz \simeq |C_j^N| \int_{C_i^N - y_j^N} \frac{1}{|z'|} \mathrm{erfc}\Big(\frac{|z'|}{2\sqrt{\beta}}\Big) dz'dx$$

being y_j^N the center of C_j^N.

The numerical tests presented in Fig. 6 and Fig. 7 have been performed for $N=50$. We draw the real and imaginary parts of the function ε_{33}^{eff} as functions of d/λ (period over wavelength) choosing successively conductivity parameter κ to be 100 and 1000. The resonant effect shows up with an important enhancement around the highest eigenvalue ν_0^2 (for which the eigenvector has constant positive sign). The dissipation (imaginary part) is concentrated in a frequency window which becomes very small when κ increases.

5 Conclusions

We have demonstrated in a full 3D setting that metamaterials with negative permittivity can be constructed by placing an assembly of periodically disposed metallic wire components ("bed-of-nails" structures). The volume fraction of metallic wires has to be kept very small.

In a first step we consider a single "bed-of-nails" structure made of an array of parallel nanorods with *finite* length but however large with respect to the period a. Such a structure is shown to be described asymptotically by a *non-local* permittivity law: the polarization current density depends on the electric field over a region of finite size.

In a second step which to our knowledge is completely new, we consider a slightly more complex structure consisting of periodically disposed (period η) systems of small arrays (period $a \ll \eta$) of parallel nanorods of length $l \sim \eta$. Surprisingly such a metamaterial evidences micro-resonances effects and can be characterized by a local effective permittivity tensor with possibly real negative eigenvalues (band gaps of frequencies).

By exploiting an effective relation in closed form (see (3.23) and (3.18)) and by tuning the geometrical parameters of the nanorods (filling ratio, conductivity), we can reach theoretically a wide range of metamaterials including any one characterized by *an arbitrary real symmetric permittivity tensor*.

Acknowledgments

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