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TE Mode Mixing Dynamics in Curved Multimode Optical Waveguides

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Abstract. Propagation of light through curved graded index optical waveguides supporting an arbitrary high number of modes is investigated. The discussion is restricted to optical wave fields which are well confined within the core region and losses through radiation are neglected. Using coupled mode theory formalism, two new forms for the propagation kernel for the transverse electric (TE) wave as it travels along a curved two-dimensional waveguide are presented. One form, involving the notion of "bend" modes, is shown to be attractive from a computational point of view as it allows an efficient numerical evaluation of the optical field for sharply bent waveguides.

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1 Introduction

Large multimode optical fibres are finding increasing application in many areas of applied science. In practice, macrobending occurs in a large deflection of the fibre axis such as that associated with spooling or the presence of loops. These deviations influence the signal propagation as a result of mode coupling phenomena. This, in turn influences the intermodal dispersion that may limit the achievable data transmission rate. Bends can also be imposed and designed so as to convert from fundamental to the high-order modes in optical fibres [1]. Wave propagation in a bent waveguide can be analyzed via various means ranging from numerical techniques such as the popular Beam Propagation Method (see for instance [2] and references therein) to semi-analytical approaches like

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the Ray Tracing technique [3] or the Beam Tracing Method [4]. For certain waveguide for which propagation modes are analytically known or, at least, can be easily numerically calculated, the Coupled Mode Theory (CMT) offers an efficient alternative for describing the propagation along curved waveguides [3,5]. Basically, the CMT transforms the original wave equation into a system of N first order ordinary differential equations (N is the number of modes) also called coupling matrix which is then solved using standard integration schemes. Problems arise when the number of propagation modes becomes too large: plastic optical fibres, for instance, can support several hundred thousand up to a million of modes. In these extreme cases, standard coupled mode theory becomes numerically intractable because of the computational overhead, and so there is a need for devising new strategies. The aim of this work is to shed a new light on this issue and to propose an improvement of the standard CMT by considering the coupling of the so-called "bend" modes. Bend modes are local eigenmodes that satisfy the wave equation in the curved waveguide with *constant* curvature. By construction, these modes are decoupled for circular bends and propagate almost adiabatically if the radius of curvature changes sufficiently slowly. More generally, the associated coupling matrix is nearly diagonal in most cases whereas the use of standard CMT would yield a fully populated matrix.

In this paper, the discussion is restricted to slowly varying planar waveguides with a parabolic graded index profile. Furthermore, the optical wave field is assumed to be well confined within the core region, and losses through radiation are neglected. It is further assumed that the waveguide is weakly guiding and that the paraxial approximation holds. Under these assumptions, the problem is shown to be equivalent to the classical time-independent harmonic oscillator. In this scenario, the solution admits an integral formulation involving the Feynman propagator; this is discussed in Section 2 and Section 3. Using the coupled mode theory (CMT) formalism, it is shown that the coupling matrix can be integrated analytically and the exact solution is recovered numerically via the computation of a matrix exponential. The theory is presented for both standard and improved CMT in Section 4 and Section 5. Numerical experiments carried out in Section 6 confirm the efficacy and advantages of the proposed approach.

2 Problem statement

We aim to study the propagation of a monochromatic TE (transverse electric) wave $E = \hat{E}e^{-i\omega t}$ in a weakly guiding two-dimensional dielectric waveguide whose graded-index profile n in the core of width $2a$ has the parabolic form

$$n^2(u) = n_0^2\Big(1-2\Delta\Big(\frac{u}{a}\Big)^2\Big), \qquad |u| \leq a, \tag{2.1}$$

where $\Delta = (n_0^2 - n_c^2)/2n_0^2$ denotes the usual profile height parameter, n_0 is the refractive index along the waveguide's axis and n_c is the index of the cladding. Eq. (2.1) is naturally

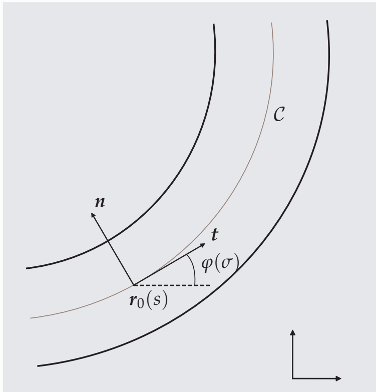

Figure 1: The geometry of the curved waveguide.

written in the local coordinate system (s,u) which follows the waveguide. This new coordinate system is defined by the transformation: $\boldsymbol{r}=\boldsymbol{r}_0(s)+u\boldsymbol{n}$, where $\boldsymbol{r}_0(s)$ is a point of a smooth curve $\mathcal{C}$ following the centre of the waveguide along its bent path as indicated in Fig. 2 and s is the arc length. The local basis vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are the usual unit tangent and normal vectors defined by

$$\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}s}=\boldsymbol{t} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}=\kappa\boldsymbol{n}. \tag{2.2}$$

Here $\kappa=\rho^{-1}$ is the curvilinear curvature and ρ the local radius of curvature. By construction, we have that (s,u) define orthogonal curvilinear coordinates. More precisely, we find that $(\mathrm{d}\boldsymbol{r})^2=h^2\mathrm{d}s^2+\mathrm{d}u^2$, where the Jacobian of the transformation is given by $h=|\partial_s\boldsymbol{r}|=1-\kappa u$. By expressing the Laplacian operator in the new coordinate system, we find that $\hat{E}$ must satisfy the following wave equation

$$\partial_s\left(h^{-1}\partial_s\hat{E}\right)+\partial_u\left(h\partial_u\hat{E}\right)+hk^2n^2(u)\hat{E}=0, \tag{2.3}$$

where $k=\omega(\varepsilon_0\mu_0)^{1/2}$ is the vacuum wavenumber and ε_0 and μ_0 denote the usual free-space dielectric and permeability constants.

To make some progress, we assume that the angle φ between a tangent to the centre-line and the horizontal line (see Fig. 1) is a smooth function of a slow variable $\sigma=\varepsilon s/a$, where ε is a small dimensionless parameter. With this definition, the unit tangent vector has the explicit form $\boldsymbol{t}=(\cos\varphi,\sin\varphi)$ and straightforward calculation from (2.2) shows that $\kappa a=\varepsilon|\partial_\sigma\varphi|$. Since $\partial_\sigma\varphi\sim\mathcal{O}(1)$, the small parameter ε may be thought of as a ratio of the half-width a to a typical radius of curvature, and the limiting case ε=0 corresponds to a straight waveguide. Note that, the Frenet-Serret equations (2.2) imply that $\mathcal{C}$ is a non-degenerate curve, i.e., the curvature κ must be strictly positive. A simple way to alleviate this apparent limitation is to orientate the normal vector as $\boldsymbol{n}=(-\sin\varphi,\cos\varphi)$ and let the curvature be a signed function with $\kappa a=\varepsilon\partial_\sigma\varphi$. In this work, we are interested in large multimode waveguides, that is we assume that the waveguide parameter defined as

$$V = kn_0 a\sqrt{2\Delta}$$

is a large number. Furthermore, consideration will be limited to the practically important case in which the index difference is small compared to unity (that is we consider the weak guidance limit $\Delta \to 0$). In these conditions, the main propagation direction of the electromagnetic energy is along the length of the waveguide. It is then convenient to extract from the s-dependence of the electric field a carrier wave moving in the s direction so we put $\Psi = \hat{E}\exp(-\mathrm{i}kn_0 s)$. Now, to allow a precise measure of the various terms involved in (2.3), it is judicious to rescale the longitudinal and transverse coordinates as follows

$$x = u\frac{\sqrt{V}}{a} \quad \text{and} \quad z = s\frac{\sqrt{2\Delta}}{a}. \tag{2.4}$$

The next step is to define the small parameters

$$\epsilon = \frac{\varepsilon}{\sqrt{2\Delta}} \ll 1 \quad \text{and} \quad \delta = \left(\frac{2\Delta}{V}\right)^{\frac{1}{2}} \ll 1, \tag{2.5}$$

so that the scale factor h and its inverse look like

$$h = 1 - \delta\epsilon x \partial_\sigma \varphi \quad \text{and} \quad h^{-1} = 1 + \delta\epsilon x \partial_\sigma \varphi + \mathcal{O}(\delta^2\epsilon^2), \tag{2.6}$$

which after substitution in (2.3) gives

$$\frac{\delta^2}{2}\partial^2_{zz}\Psi + \mathrm{i}\partial_z\Psi = \mathcal{H}\Psi + \gamma x\Psi + \mathcal{O}(\delta\epsilon). \tag{2.7}$$

Here, the transverse operator $\mathcal{H}$ is the z-independent Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\left(\partial^2_{xx} - v\right) \tag{2.8}$$

corresponding to the straight waveguide. The potential v stands for the quadratic well of finite depth $v(x) = x^2$, where $|x| \leq \sqrt{V}$ and $v(x) = V$ otherwise. Therefore, the interface core-cladding is now located at $|x| = \sqrt{V}$. In (2.7), function γ can be interpreted as the normalized curvature within this new coordinate system. It depends on the slow variable $\sigma = \epsilon z$ and is defined as

$$\gamma = \eta\partial_\sigma\varphi \quad \text{with } \eta = \epsilon\delta^{-1}. \tag{2.9}$$

Thus, the effect of the curvature is conveniently measured as the ratio of the curvature term ϵ with respect to the scaling parameter δ and we can already anticipate non negligible effects as soon as these two quantities are of comparable amplitude. In the present analysis, we shall limit ourselves to scenarios in which the second derivative paraxial term can be ignored. In this case one finds that Ψ must satisfy the simplified one-way equation

$$\mathrm{i}\partial_z\Psi = \mathcal{H}\Psi + \gamma x\Psi. \tag{2.10}$$

3 Formal solution

Eq. (2.10) is nothing else but the Schrödinger equation which describes the motion of a harmonic oscillator in the presence of a varying external force. Here, the parabolic well is unfortunately finite and this renders the mathematical analysis very difficult as the presence of the induced radiation field in the cladding must be carefully treated. In the present study we shall restrict ourselves to the analysis of an optical wave field which is well confined within the core region, i.e., we assume that $\Psi\approx 0$ at $x\pm\sqrt{V}$. The truncation of the graded refractive index at the core-cladding interface can be then neglected and one can assume that the parabolic profile extends to infinity. If this is done then the solution of the initial value problem (2.10) is obtained via the integral

$$\Psi(x,z)=\int_{\mathbb{R}} K(x,z;x',0)\Psi(x',0)\mathrm{d}x'. \tag{3.1}$$

In [6], the propagator kernel $K(x,z;x',0)$ is given explicitly using Feynman's path formalism. The technique is however cumbersome and an easier derivation presented in [7] shall be followed here. It suffices to look for the form $K=K_0\mathrm{e}^{\mathrm{i}S}$, where K_0 is the well known propagator for the non-perturbed simple harmonic oscillator, i.e.,

$$K_0(x,z;x',0)=\frac{1}{\sqrt{2\pi\mathrm{i}\sin z}}\exp\Big(\mathrm{i}\frac{(x^2+x'^2)\cos z-2xx'}{2\sin z}\Big). \tag{3.2}$$

Now, using the fact that $\partial_x K_0=\mathrm{i}f(x,x',z)K_0$, where

$$f(x,x',z)=\frac{x\cos z-x'}{\sin z}, \tag{3.3}$$

we find the equation for the phase as

$$-\partial_z S=f(x,x',z)\partial_x S+\gamma x-\frac{\mathrm{i}}{2}\partial^2_{xx}S+\frac{1}{2}(\partial_x S)^2 \tag{3.4}$$

with the initial condition that $S=0$ at $z=0$. Thanks to linearity of f with respect to x and x' the real-valued phase function S admits the separable form $S=\hat{a}(z)x+\hat{b}(z)x'+\hat{c}(z)$. The closed form solution for these functions is given in [6,7] and reminded here for completeness:

$$\hat{a}(z)=-\frac{1}{\sin z}\int_0^z \gamma(\epsilon z')\sin z'\mathrm{d}z', \quad \hat{b}(z)=\int_0^z \frac{\hat{a}(z')}{\sin z'}\mathrm{d}z', \quad \hat{c}(z)=-\frac{1}{2}\int_0^z \hat{a}^2(z')\mathrm{d}z'. \tag{3.5}$$

Despite the elegance of the integral approach, the numerical evaluation of (3.1) is not trivial as both K as well as the input optical field are highly oscillatory functions of the transverse coordinate x, with the number of oscillations growing linearly with V. There is, however, a scenario for which (3.1) admits a closed form solution: when the input field is an off-axis Gaussian beam of width w_0 centered on τ_0. In this particular case, standard calculations will show that the light intensity distribution along the transverse section remains Gaussian, that is

$$\Psi(x,z)=\frac{A}{\sqrt{q(z)}}\exp\Big(\mathrm{i}\chi(x,z)-\frac{(x-\tau(z))^2}{w^2(z)}\Big), \tag{3.6}$$

where function q describes an elliptical path in the complex plane as we have $q(z)=\cos z+2\mathrm{i}\sin z/w_0^2$. The width of the beam, which is given by $w(z)=w_0|q(z)|$ is a π-periodic function with respect to the normalized coordinate axis z and χ is a real-valued phase function. Its exact form is not essential for our discussion and this is left to the reader. The main point is to observe that the gaussian distribution is centered about

$$\tau(z)=\int_0^z\gamma(\epsilon z')\sin(z'-z)\mathrm{d}z'+\tau_0\cos z. \tag{3.7}$$

In the simplest scenario, where γ is a constant, the bend path is circular and the shift has the oscillatory behavior $\tau(z)=-\gamma+(\gamma+\tau_0)\cos z$. So a simple way to avoid the oscillation is to inject a gaussian field centered on $\tau_0=-\gamma$. The beamwidth variation can also be avoided for the specific value $w_0=\sqrt{2}$, which is precisely the one of the fundamental mode of the straight waveguide. In a more general case, if we assume that the geometry of the waveguide, i.e., the normalized curvature γ, can be described via discrete superposition of its Fourier components $\gamma_\Omega=A_\Omega\cos(\Omega z+\phi)$, we can easily anticipate that the τ-Fourier components will behave like $A_\Omega/(\Omega^2-1)$. At the resonant frequency $|\Omega|=1$, τ grows linearly with z and the light energy will eventually escape the core region to be scattered in the cladding. In practice, any fluctuations with a strong spatial frequency component around the resonant frequency will, via radiation loss, have their light energy filtered out. This resonance effect is precisely that used in fibre Bragg grating structures to block certain wavelengths. In the context of the present paper, it is required that γ is a slowly varying function so this scenario is obviously not encountered, but we shall comment on this further when necessary. Note the highly oscillating case, i.e., $\Omega\gg 1$, can be handled via multiple-scale asymptotic analysis and this is discussed in [8].

4 Standard coupled mode theory

Another classical method for solving the one-way equation (2.10) is to express the field distribution in the waveguide by using standard coupled mode theory, i.e., Ψ is expanded in the eigenfunction basis ψ_ν of the unperturbed waveguide via the scalar product

$$\Psi(x,z)=\mathsf{\Psi}^{\mathrm{T}}(x)\mathsf{D}_\beta(z)\mathsf{a}(z), \tag{4.1}$$

with $\mathsf{\Psi}=(\psi_0,\psi_1,\cdots)^{\mathrm{T}}$. The diagonal matrix D_β contains the propagation constants,

$$(\mathsf{D}_\beta(z))_{\nu\nu}=\mathrm{e}^{-\mathrm{i}\beta_\nu z}$$

and the vector $\mathsf{a}(z)=(a_0(z),a_1(z),\cdots)^{\mathrm{T}}$ contains the modal amplitudes. In (4.1), the summation extends over the discrete spectrum of the guided modes of the straight waveguide satisfying the eigenvalue problem

$$\mathcal{H}\psi_\nu=\beta_\nu\psi_\nu, \tag{4.2}$$

with the boundary condition $\psi_\nu(x)\to 0$ as $|x|\to\infty$. Eigenvalues of (4.2) are real positive quantities. They characterize the number of oscillations of the guided modes across the transverse section of the waveguide. Due to the finiteness of the waveguide we have necessarily $\beta_\nu<V/2$. Above this cut-off value, modes are not guided anymore and belong to the continuum set of radiation modes [9]. As in the previous section, we shall consider optical wave fields which are well confined within the core, so that, in the summation (4.1), the modes of the parabolic waveguide are essentially the same as the modes of an infinitely extended square-law medium. These ideal modes are given by

$$\psi_\nu(x)=\frac{1}{\sqrt{\pi^{\frac{1}{2}}2^\nu\nu!}}H_\nu(x)\mathrm{e}^{-\frac{x^2}{2}}, \tag{4.3}$$

where H_ν denotes the usual Hermite polynomial. It can be shown that for a sufficiently large number of modes (V large enough), the finiteness of the waveguide will have little effect on most of the guided modes and (4.3) is a very good approximation [10]. The associated eigenvalues are the discrete energy levels of the harmonic oscillator: $\beta_\nu=\nu+1/2$. Now, from the recurrence relations for the Hermite polynomials (see [11]) we find that

$$x\psi_\nu=2^{-\frac{1}{2}}\left(\sqrt{\nu}\psi_{\nu-1}+\sqrt{\nu+1}\psi_{\nu+1}\right) \tag{4.4}$$

and by using orthogonality properties of guided modes, (2.10) is transformed into the system of ordinary differential equations

$$\frac{\mathrm{d}\mathsf{a}}{\mathrm{d}z}=\mathsf{Q}(z)\mathsf{a}=-\mathrm{i}\gamma\left(\mathrm{e}^{\mathrm{i}z}\mathsf{A}+\mathrm{e}^{-\mathrm{i}z}\mathsf{A}^{\mathrm{T}}\right)\mathsf{a}, \tag{4.5}$$

where the lower diagonal matrix A is the algebraic representation of the creation operator [12]:

$$\mathsf{A}=\frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & & & & \\ 1 & 0 & 0 & & & \\ & \sqrt{2} & 0 & 0 & & \\ & & \sqrt{3} & 0 & 0 & \\ & & & \ddots & \ddots & \end{bmatrix}. \tag{4.6}$$

The system (4.5) is the standard discrete form of the (time-dependent) forced harmonic oscillator in one dimension [7]. Note that the total energy in the system is conserved (i.e., $\partial_z\|\mathsf{a}\|^2=0$) since the coupling matrix is skew-Hermitian. Now, provided that the (infinite) Magnus series converges

$$\hat{\mathsf{Q}}(z)=\int_0^z \mathsf{Q}(z)\mathrm{d}z-\frac{1}{2}\int_0^z\Big[\int_0^{z'}\mathsf{Q}(z'')\mathrm{d}z'',\mathsf{Q}(z')\Big]\mathrm{d}z' \\ +\frac{1}{4}\int_0^z\Big[\int_0^{z'}\Big[\int_0^{z''}\mathsf{Q}(z''')\mathrm{d}z''',\mathsf{Q}(z'')\Big]\mathrm{d}z'',\mathsf{Q}(z')\Big]\mathrm{d}z'+\cdots, \tag{4.7}$$

(here $[\cdot,\cdot]$ denotes the matrix commutator: $[\mathsf{X},\mathsf{Y}]=\mathsf{XY}-\mathsf{YX}$) then the solution can be written in the exponential form $\mathsf{a}(z)=\exp(\hat{\mathsf{Q}}(z))\mathsf{a}(0)$. The problem simplifies if we exploit the well known commutating properties of the creation and annihilation matrix operators, i.e.,

$$[\mathsf{A}^{\mathrm{T}},\mathsf{A}]=\frac{\mathsf{I}}{2}, \tag{4.8}$$

where I is the identity matrix. Thanks to the relation (4.8), the Magnus series (4.7) reduces only to the first two terms. This allows us to write the explicit solution as

$$\mathsf{a}(z)=\mathrm{e}^{\mathrm{i}\Theta(z)}\mathrm{e}^{-\mathrm{i}\eta\Omega(z)}\mathsf{a}(0), \tag{4.9}$$

where η is the curvature measure given in (2.9), Θ is the phase function

$$\Theta(z)=\frac{1}{2}\int_0^z\int_0^{z'}\gamma(\epsilon z')\gamma(\epsilon z'')\sin(z'-z'')\mathrm{d}z'\mathrm{d}z'' \tag{4.10}$$

and Ω stands for the bidiagonal matrix

$$\Omega(z)=g(z)\mathsf{A}+\overline{g(z)}\mathsf{A}^{\mathrm{T}}. \tag{4.11}$$

Here, g contains the "history" of the curvature along the waveguide axis, i.e.,

$$g(z)=\int_0^z\partial_\sigma\varphi(\epsilon z')\mathrm{e}^{\mathrm{i}z'}\mathrm{d}z'. \tag{4.12}$$

In the case of a circular path, the coupled system admits 2π-periodic solutions in z (up to a phase). In physical dimensions, this corresponds to a periodicity length, sometimes called the beat length, of

$$L=\frac{2\pi a}{\sqrt{2\Delta}},$$

which is also predicted by ray-theory [3]. We can also observe that this is precisely the perturbation period corresponding to the resonance described earlier. Finally, combining (3.1) together with (4.1) and (4.9) yields the separable form of the solution

$$K(x,z;x',0)=\mathrm{e}^{\mathrm{i}\Theta(z)}\Psi^{\mathrm{T}}(x)\mathsf{D}_\beta(z)\mathrm{e}^{-\mathrm{i}\eta\Omega(z)}\Psi(x'). \tag{4.13}$$

Eq. (4.13) can be interpreted as the spectral decomposition of the Feynman propagator in the straight waveguide modal basis and in the absence of external perturbation, $K=K_0$

and (4.13) reduces to the Mehler's Hermite polynomial formula. Physically, the scattering matrix coefficient $(\mathrm{e}^{-\mathrm{i}\eta\Omega(z)})_{\zeta\nu}$ gives the amplitude of mode ζ at z if only mode ν is present at the waveguide input $z=0$. Although an alternative representation can be found in [6,7], the exponential form in (4.13) allows a simple description of the mode mixing dynamics. Indeed, for sufficiently small curvature, η is a small quantity and the first terms of the infinite series

$$\mathrm{e}^{-\mathrm{i}\eta\Omega} = \mathsf{I} - \mathrm{i}\eta\Omega + \frac{(-\mathrm{i}\eta\Omega)^2}{2!} + \cdots \tag{4.14}$$

should then provide a good approximation. Because of the bidiagonal structure of the coupling matrix, the integer power of Ω corresponds to the number of modes coupled with a given mode at the waveguide input. For instance, truncating the series (4.14) up to $K+1$ terms means that mode ν will be scattered into modes ranging from $\nu-K$ to $\nu+K$. For any arbitrary curvature, the numerical evaluation of the scattering matrix involves computing the exponential of a large bidiagonal matrix whose size grows linearly with V. Though there are numerous techniques available for this purpose [13], these are likely to become ineffective in the limit of large V due to the computational overhead. We shall take another approach and exploit the algebraic structure of Ω: we first expand Ω^p (p is an integer) in terms of integer powers of the matrix A. This yields the explicit form the matrix coefficients as (see [14] for more details):

$$(\Omega^p)_{\zeta\nu} = \sum_{q=0}^{\lfloor p/2 \rfloor} g^{p-q}\overline{g}^q (\mathsf{A}^{p-2q})_{\zeta\nu} \sum_{0\le i_1\le\cdots\le i_q\le p-q} \prod_{l=1}^{q} (\mathsf{D}_{i_l-l})_{\nu\nu}, \tag{4.15}$$

where $\mathsf{D}_0 = \mathsf{A}^{\mathrm{T}}\mathsf{A}$ and D_i stands for the "shifted" diagonal matrix whose non-zero entries are

$$(\mathsf{D}_i)_{\nu\nu} = \frac{\nu+i+1}{2}.$$

Since A has a 1-band structure, its power is also a 1-band matrix; more precisely we find that (we take $\zeta\ge\nu$)

$$(\mathsf{A}^{p-2q})_{\zeta\nu} = \frac{1}{2^{\zeta-\nu}}\sqrt{\frac{\zeta!}{\nu!}}, \quad \text{if } \zeta-\nu = p-2q, \tag{4.16}$$

and $(\mathsf{A}^{p-2q})_{\zeta\nu} = 0$, otherwise. Now, expanding the exponential in its power series, we finally obtain, after some algebra, the explicit form for the scattering matrix coefficient:

$$(\mathrm{e}^{-\mathrm{i}\eta\Omega})_{\zeta\nu} = \frac{(-\mathrm{i}\eta g)^m}{m!\sqrt{2^m}}\sqrt{\frac{\zeta!}{\nu!}}\Big\{1+\sum_{n=1}^{\infty}\mathsf{C}_{\nu mn}|\eta g|^{2n}\Big\}, \tag{4.17}$$

where coefficients of the series are given explicitly as

$$\mathsf{C}_{\nu mn} = \Big(-\frac{1}{2}\Big)^n \frac{m!}{(2n+m)!} \sum_{0\le i_1\le\cdots\le i_n\le m+n} \prod_{l=1}^{n} (\nu+1+i_l-l). \tag{4.18}$$

Note that the mode number separation $m=\zeta-\nu$ is assumed positive in (4.17) and negative values are simply recovered by observing that the matrix Ω is Hermitian. Coefficients associated with the second order correction term (in η) have the simple expression

$$C_{\nu m1}=-\frac{\nu}{2(m+1)}-\frac{1}{4}, \tag{4.19}$$

and if we define the $M+1$-term series $S_M^d=\sum_{i=0}^M i^d$, we obtain, for the next order correction term

$$C_{\nu m2}=\frac{m!}{4(m+4)!}\left(S_{m+2}^0\nu(\nu-1)+\frac{1}{2}\left[S_{m+2}^1(2\nu^2+\nu-1)+S_{m+2}^2 3\nu+S_{m+2}^3\right]\right). \tag{4.20}$$

Higher order terms become too cumbersome to be included in the paper. Despite the apparent complexity of (4.18), coefficients $C_{\nu mn}$ can be computed at a negligible cost and tabulated once for all; only functions $\Theta(z)$ and $g(z)$ needs to be computed at each z-step. Here again, if we assume that the geometry of the waveguide is described via a discrete superposition of its Fourier components, then the scattering matrix will be fully populated whenever there is a non zero component associated with a nearly-resonant frequency $|\Omega|\approx 1$ since we have in this case $|g|\sim|\Omega^2-1|^{-1}$. Away from the resonance, (i.e., $\Omega\ll 1$), then $g\sim\mathcal{O}(1)$ and the numerical convergence of the infinite series in (4.17) is controlled by the curvature strength η. In practice, we have observed that, for the low order mode scattering, η should be at most of order $\mathcal{O}(1)$ to avoid very expensive numerical calculations. This will be made more precise in the last section of this paper.

5 Coupled bend mode theory

Despite the progress made so far, we have to bear in mind that the normalized curvature measure behaves as $\eta\sim\kappa a\Delta^{-1}\sqrt{V}$, and so the condition that η should not exceed unity does not necessarily hold. This limitation can be explained very simply: for sufficiently large curvature, a bend tends to shift a beam (consisting of a packet of modes) off the centre axis towards the outside of the bend. Thus, expanding the beam in the straight waveguide eigenmode basis can turn out to be very cumbersome as this may give rise to very strong coupling among a large number of "straight" modes. This effect is highlighted by the need to consider a large number of terms in the series (4.17).

In order to take advantage of the weak dependence of the curvature with respect to the arc length, a better option is to treat γ as a fixed parameter and consider the eigenmode of a curved waveguide $\psi_\nu^{\mathrm{b}}=\psi_\nu^{\mathrm{b}}(x;\gamma)$ (the superscript "b" refers to the bend modes). These eigenmodes satisfy

$$(\mathcal{H}+\gamma x)\psi_\nu^{\mathrm{b}}=\beta_\nu^{\mathrm{b}}\psi_\nu^{\mathrm{b}}, \tag{5.1}$$

where the eigenvalue depends implicitly on the local radius of curvature, $\beta_\nu^{\mathrm{b}}=\beta_\nu^{\mathrm{b}}(\gamma)$. Propagation constants associated with highest order modes are expected to have a small imaginary part due to some leakage of light energy into the outer cladding. Here again, we shall simplify the analysis by restricting ourselves to bend modes which are well confined in the core region so that their value at the core-cladding interface is negligible. These modes are therefore purely real and can be normalized so that they satisfy the orthogonality condition

$$\int_{\mathbb{R}} \psi_\nu^{\mathrm{b}} \psi_\mu^{\mathrm{b}} \mathrm{d}x = \delta_{\nu\mu}. \tag{5.2}$$

Now, let us assume for the moment that these bend modes are known and expand Ψ in the eigenfunction basis of the curved waveguide as

$$\Psi(x,z) = \Psi_{\mathrm{b}}^{\mathrm{T}}(x;\gamma)\mathsf{D}_{\beta^{\mathrm{b}}}(z)\mathsf{b}(z) \tag{5.3}$$

with $\Psi_{\mathrm{b}} = (\psi_0^{\mathrm{b}}, \psi_1^{\mathrm{b}}, \cdots)^{\mathrm{T}}$. The diagonal matrix $\mathsf{D}_{\beta^{\mathrm{b}}}$ contains the phase of each mode,

$$(\mathsf{D}_{\beta^{\mathrm{b}}}(z))_{\nu\nu} = \exp\Big(-\mathrm{i}\int_0^z \beta_\nu^{\mathrm{b}} \mathrm{d}z\Big),$$

and the vector $\mathsf{b}(z) = (b_0(z), b_1(z), \cdots)^{\mathrm{T}}$ contains the bend mode amplitudes. Employing the bend mode expansion in Eq. (2.10), yields a new coupled system

$$\frac{\mathrm{d}\mathsf{b}}{\mathrm{d}z} = \mathsf{Q}\mathsf{b} \tag{5.4}$$

with coupling coefficients given by the overlap integrals

$$(\mathsf{Q})_{\nu\mu} = -\exp\Big(\mathrm{i}\int_0^z (\beta_\nu^{\mathrm{b}} - \beta_\mu^{\mathrm{b}})\mathrm{d}z\Big)\int_{\mathbb{R}} \psi_\nu^{\mathrm{b}} (\partial_z \psi_\mu^{\mathrm{b}})\mathrm{d}x. \tag{5.5}$$

Applying the chain rule $\partial_z \psi_\mu^{\mathrm{b}} = \epsilon(\partial_\sigma \gamma)\partial_\gamma \psi_\mu^{\mathrm{b}}$, we find that the coupling terms are now proportional to $\epsilon\eta$ and we can anticipate that in many situations, $\epsilon\eta$ is sufficiently small so that the first terms of the infinite Feynman-Dyson series:

$$\mathsf{b}(z) = \Big\{\mathsf{I} + \int_0^z \mathsf{Q}(z')\mathrm{d}z' + \int_0^z \mathsf{Q}(z')\int_0^{z'} \mathsf{Q}(z'')\mathrm{d}z'\mathrm{d}z'' + \cdots\Big\}\mathsf{b}(0) \tag{5.6}$$

should provide a good approximation.

The price to pay for such improvement is the evaluation of the bend modes. In general, (5.1) does not yield closed form solutions for a given index profile and one must resort to numerical techniques [15]. However, for the parabolic index grading the ideal eigenmodes of (5.1) admit an analytical form. Indeed, by doing the simple change of variable $\tilde{x} = x + \gamma$, it is easy to show that

$$\mathcal{H}\Psi + \gamma x \Psi = -\frac{1}{2}(\partial^2_{\tilde{x}\tilde{x}} - \tilde{x}^2)\Psi - \frac{\gamma^2}{2}\Psi, \tag{5.7}$$

and we get $\psi_\nu^{\mathrm{b}}(x)=\psi_\nu(\tilde{x})$, thus bend mode shapes are simply given by the equivalent straight modes with a shift of the waveguide axis. Furthermore, $\beta_\nu^{\mathrm{b}}=\beta_\nu-\gamma^2/2$, so

$$\mathsf{D}_{\beta^{\mathrm{b}}}(z)=\exp\Big(\mathrm{i}\int_0^z\frac{\gamma^2}{2}\mathrm{d}z\Big)\mathsf{D}_\beta(z). \tag{5.8}$$

Finally, we can calculate explicitly the z-derivative of a bend mode and the resulting coupling matrix has the simple form:

$$\mathsf{Q}=\epsilon\partial_\sigma\gamma\big(\mathrm{e}^{\mathrm{i}z}\mathsf{A}-\mathrm{e}^{-\mathrm{i}z}\mathsf{A}^{\mathrm{T}}\big). \tag{5.9}$$

Here again, we can exploit the commutating properties of the creation and annihilation matrices to obtain the explicit solution as

$$\mathsf{b}(z)=\mathrm{e}^{\mathrm{i}\Theta_{\mathrm{b}}(z)}\mathrm{e}^{\eta\epsilon\Omega_{\mathrm{b}}(z)}\mathsf{b}(0), \tag{5.10}$$

where Θ_{b} is the phase function

$$\Theta_{\mathrm{b}}(z)=\frac{\epsilon^2}{2}\int_0^z\int_0^{z'}\partial_\sigma\gamma(\epsilon z')\partial_\sigma\gamma(\epsilon z'')\sin(z'-z'')\mathrm{d}z'\mathrm{d}z'', \tag{5.11}$$

and Ω_{b} stands for the z-dependent bidiagonal matrix

$$\Omega_{\mathrm{b}}(z)=g'(z)\mathsf{A}-\overline{g'(z)}\mathsf{A}^{\mathrm{T}}. \tag{5.12}$$

Here, g' contains the "history" of the rate of change of curvature along the waveguide axis

$$g'(z)=\int_0^z\partial^2_{\sigma\sigma}\varphi(\epsilon z')\mathrm{e}^{\mathrm{i}z'}\mathrm{d}z'. \tag{5.13}$$

If the curvature grows at a constant rate (spiral-like path), then the coupled system admits 2π-periodic solutions in z, which, in physical dimensions, corresponds to the same periodicity as the circular beat length introduced in the previous section. By exploiting the band factorization of Ω_{b}^p, we finally find the general form for the bend mode scattering matrix coefficient:

$$(\mathrm{e}^{\eta\epsilon\Omega_{\mathrm{b}}})_{\zeta\nu}=\frac{(\epsilon\eta g')^m}{m!\sqrt{2^m}}\sqrt{\frac{\zeta!}{\nu!}}\Big\{1+\sum_{n=1}^{\infty}\mathsf{C}_{\nu mn}|\epsilon\eta g'|^{2n}\Big\}, \tag{5.14}$$

where coefficients of the series are identical to those associated with the scattering matrix of the straight modes (see (4.18)). Finally the envelope of the transverse field can be recombined as the combination of bend modes via the new spectral decomposition

$$K(x,z;x',0)=\mathrm{e}^{\mathrm{i}\Theta_{\mathrm{b}}(z)}\Psi_{\mathrm{b}}^{\mathrm{T}}(x;\gamma(\epsilon z))\mathsf{D}_{\beta^{\mathrm{b}}}(z)\mathrm{e}^{\eta\epsilon\Omega_{\mathrm{b}}(z)}\Psi_{\mathrm{b}}(x';\gamma(0)). \tag{5.15}$$

Clearly the new decomposition (5.15) offers a substantial improvement over the one associated with the straight mode coupling (4.13) since the numerical convergence of the series (5.14) is now controlled by the order of magnitude of $\epsilon\eta$.

6 Numerical examples and concluding remarks

In this last section, we aim to investigate the accuracy of the straight and bend mode scattering matrix coefficients on two specific examples. We shall study light beam propagation along sinusoidal bends described by the trigonometric function

$$\varphi = -\cos(\epsilon z). \tag{6.1}$$

The choice for the cosine is simply to ensure that the curvature is zero at the waveguide input $z=0$ so that bend modes coincide with straight modes. To simplify the analysis, we consider that the input is excited by the fundamental mode only, with $a_0(0)=b_0(0)=1$. From (4.9) and (5.10), only the first column of the scattering matrix is populated and we have simply

$$a_\zeta(z) = \mathrm{e}^{\mathrm{i}\Theta(z)}(\mathrm{e}^{-\mathrm{i}\eta\Omega(z)})_{\zeta 0} \quad \text{and} \quad b_\zeta(z) = \mathrm{e}^{\mathrm{i}\Theta_\mathrm{b}(z)}(\mathrm{e}^{\eta\epsilon\Omega_\mathrm{b}(z)})_{\zeta 0}. \tag{6.2}$$

Fig. 2 (left) shows the evolution of the power $|a_\zeta(z)|^2$ carried by the first four straight modes $\zeta=0,1,2,3$. These results correspond to the waveguide A which parameters are indicated on Table 1 (in each case the number of guided modes can be estimated as $N \approx V/2$). In all calculations, the first 8 terms in the infinite series have been taken into account. We checked the accuracy of these results with those computed by applying a standard 4th order Runge-Kutta method directly to the coupled mode system (4.5). The good agreement is conveniently displayed in Table 2 where the real part of $a_\zeta(z)$ is shown for the first four modes at $z=8\pi$ where the scattering of the fundamental mode is strong. Here, 2 to 3 digits are recovered and better accuracy can easily be obtained by taking more terms in the series. Table 3 shows similar results at $z=14\pi$; here the scattering is weak so the series is quickly convergent and the agreement is even better. As expected, strong scattering occurs in the region of highest curvature (the exact location of the maximum (i.e., $\gamma=\eta$) is indicated in the last column of Table 1). Fig. 2 (right) shows the evolution of the power $|b_\zeta(z)|^2$ carried by the first four bend modes $\zeta=0,1,2,3$ for the same waveguide. These results are computed using only the first order correction term, i.e.,

$$b_\zeta \approx \mathrm{e}^{\mathrm{i}\Theta_\mathrm{b}(z)} \frac{(\epsilon\eta g')^\zeta}{\sqrt{2^\zeta \zeta!}} \left\{1 + \frac{1}{4}|g'|^2(\epsilon\eta)^2\right\}. \tag{6.3}$$

This shows weak coupling, where only the first two bend modes are excited, the amplitudes of the other modes being negligible. Here, the fundamental mode propagates

Table 1: Waveguide parameters.

	V	Δ	ε	η	$\epsilon\eta$	$\sqrt{V}$	$(z/2\pi)_{\text{max curv.}}$
Waveguide A	100	0.01	0.005	2.5	0.17	10	3.5355
Waveguide B	1000	0.001	0.001	15.8	0.35	31.6	5.5902

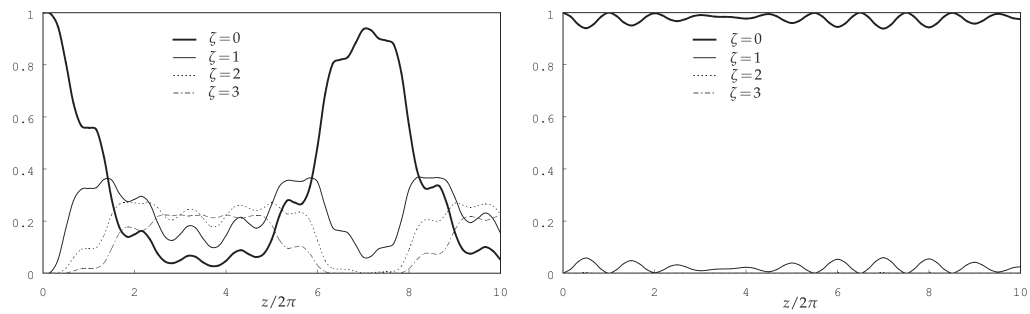

Figure 2: Evolution of the modal power carried by the straight modes (left) and bend modes (right) along the waveguide A.

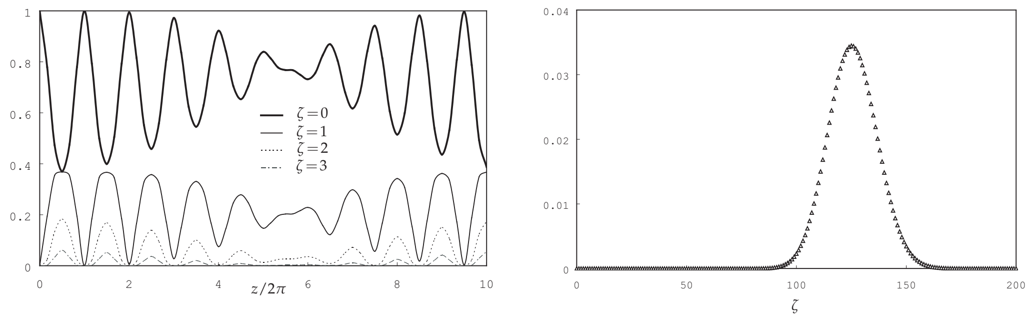

Figure 3: Left: Evolution of the modal power carried by the bend modes along the waveguide B. Right: Distribution of the straight mode amplitudes at $z=11\pi$.

almost adiabatically as no more than 5% of its power is transferred to other modes. In this example, the bend mode scattering matrix is nearly bidiagonal and the first two terms of the Feynman-Dyson series (5.6) is a good approximation.

In the next example, we consider an extreme scenario (waveguide B) where the curvature perturbation is high ($\eta=15.8$). This value would require a very large number of terms in the straight mode scattering series to ensure convergence. Instead, we computed the amplitudes of the bend modes, the evolution of which are displayed in Fig. 3 (left). As illustrated in Table 4, comparisons with results from the Runge-Kutta method are excellent. We can observe the quasi-periodic nature of the coupling except in the region of high curvature where $\partial_\sigma\gamma\approx 0$. Fig. 3 (right) illustrates the repartition of the amplitudes of the straight modes in that region (these were computed using the Runge-Kutta method), clearly indicating the very large number of modes, and associated high accuracy, required using the traditional approach. By contrast, the same optical field can be recovered using just 3 or 4 bend modes.

Through the above numerical experiments we have confirmed the accuracy of our approach. Moreover, we were able to identify an intermediate regime when $\eta\sim 1$, that is

$$\kappa a\sim\Delta V^{-\frac{1}{2}},$$

Table 2: Comparison of real part of $a_\zeta(z)$; analytical vs 4th order Runge-Kutta at $z=8\pi$ for waveguide A.

ζ	Analytical	Runge-Kutta
0	0.2145	0.2152
1	−0.3789	−0.3795
2	0.4690	0.4695
3	−0.4704	−0.4707

Table 3: Comparison (real part of $a_\zeta(z)$) analytical vs Runge-Kutta (4th order) at $z=14\pi$ for waveguide A.

ζ	Analytical	Runge-Kutta
0	0.763998231	0.763998233
1	−0.191927322	−0.191927321
2	−0.020563261	−0.020563261
3	0.006522258	0.006522258

Table 4: Comparison (real part of $b_\zeta(z)$) analytical vs Runge-Kutta (4th order) at $z=\pi/2$ for waveguide B.

ζ	Analytical	Runge-Kutta
0	0.43234937	0.43234944
1	−0.42530277	−0.42530280
2	−0.30581691	−0.30581693
3	0.17150603	0.17150604

for which straight modes are found to be moderately coupled. In the bend mode basis, the condition is relaxed as we found that moderate coupling occurs when $\eta\epsilon \sim 1$. In physical dimensions, this means

$$(\kappa a)^2 \sim \Delta^{\frac{3}{2}} V^{-\frac{1}{2}}.$$

In practice, the coupled "bend" mode theory is shown to be attractive from a computational point of view as it allows an efficient numerical evaluation of the optical field for sharply bent waveguides. Of course, these regimes have been identified in the context of parabolic index profile waveguides but there are reasons to believe that these should remain valid for a wider range of graded index profiles. Finally, we are confident that the methodology presented here should serve as a basis for the analysis of curved fibres of circular cross-section; this work is ongoing by the authors.

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