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A Discussion on Two Stochastic Elliptic Modeling Strategies

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Abstract. Based on the study of two commonly used stochastic elliptic models: I: $-\nabla\cdot(a(\boldsymbol{x},\omega)\cdot\nabla u(\boldsymbol{x},\omega))=f(\boldsymbol{x})$ and II: $-\nabla\cdot(a(\boldsymbol{x},\omega)\diamond\nabla u(\boldsymbol{x},\omega))=f(\boldsymbol{x})$, we constructed a new stochastic elliptic model III: $-\nabla\cdot\big((a^{-1})^{\diamond(-1)}\diamond\nabla u(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x})$, in [20]. The difference between models I and II is twofold: a scaling factor induced by the way of applying the Wick product and the regularization induced by the Wick product itself. In [20], we showed that model III has the same scaling factor as model I. In this paper we present a detailed discussion about the difference between models I and III with respect to the two characteristic parameters of the random coefficient, i.e., the standard deviation σ and the correlation length l_c. Numerical results are presented for both one- and two-dimensional cases.

AMS subject classifications: 60H15, 65C20, 65C30

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1 Introduction

Stochastic elliptic models are of fundamental importance for the stochastic modeling of physical and engineering applications [9, 15]. The two commonly studied stochastic elliptic models in literature include

$$\text{Model I:} \qquad -\nabla\cdot\big(a(\boldsymbol{x},\omega)\nabla u_I(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x}), \tag{1.1a}$$

$$\text{Model II:} \qquad -\nabla\cdot\big(a(\boldsymbol{x},\omega)\diamond\nabla u_{II}(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x}), \tag{1.1b}$$

where $\boldsymbol{x}\in\mathbb{R}^d$, $d=1,2,3$, ω indicates randomness, $a(\boldsymbol{x},\omega)$ a non-negative random process and $\diamond$ the Wick product. Based on the properties of $a(\boldsymbol{x},\omega)$, models I and II can be adapted

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for different applications. For example, if the random coefficient $a(\boldsymbol{x},\omega)$ is ergodic and has two wildly separated scales, model I becomes a typical stochastic multi-scale elliptic model. In this work, we consider a general case, where we assume that $a(\boldsymbol{x},\omega)$ is log-normal and the underlying Gaussian random process is homogeneous stationary and ergodicity is not required. For such a set-up, we refer to [1, 2, 6–8, 14] and references therein for theoretical and numerical studies for model I and [9–11, 17–19] and references therein for model II.

The difference between models I and II is twofold: a scaling factor induced by the way of applying the Wick product and the regularization induced by the Wick product itself. It was shown in [20] that the scaling factor is an exponential function of the variance of the underlying Gaussian random process of $a(\boldsymbol{x},\omega)$. By applying the Wick product in a different way, a new stochastic elliptic model

$$\text{Model III:} \quad -\nabla\cdot\left((a^{-1})^{\diamond(-1)}\diamond\nabla u_{III}(\boldsymbol{x},\omega)\right)=f(\boldsymbol{x}) \tag{1.2}$$

was proposed in [20], whose solution has the same scaling factor as model I. Numerical experiments showed that for one-dimensional problems the solutions of models I and III can be very close to each other, which implies that the regularization effect induced by the Wick product is relatively small.

In this work, we continue the study on the two stochastic modeling strategies based on the regular product and the Wick product. We will focus on the regularization effect induced by the Wick product by examining the difference between models I and III with respect to the standard deviation σ and the correlation length l_c of the underlying Gaussian process of a log-normal random coefficient $a(\boldsymbol{x},\omega)$. Asymptotic analysis shows that the difference between the solutions of models I and III is of second order with respect to σ, i.e.,

$$\|u_I-u_{III}\|\sim C(l_c)\sigma^2.$$

Such a fact is independent of the physical dimension d. Thus model III can provide most of the information given by model I when σ is relatively small. In particular, when l_c goes to infinity, the constant $C(l_c)$ will decay to zero. It is shown that the solutions of models I and III converge to each other as l_c goes to zero, which is a fact that is only true for one-dimensional problems. Analysis and numerical results also show that the solutions of models I and III are almost linear with respect to each other in a statistical sense if σ is relatively small.

This paper is organized as follows: in Section 2 we introduce the weighted Wiener chaos space, which is a uniform theoretical framework for models I-III. A detailed description of the three stochastic elliptic models is given in Section 3. We present some theoretical studies about the difference between models I and III in Section 4. Numerical results for two-dimensional problems are given in Section 5 followed by a summary section.

2 Wiener chaos space

We consider all stochastic elliptic models I-III in the weighted Wiener chaos space [10,13] since we assume that the random coefficient is log-normal.

Definition 2.1. Let $\{\mathrm{u}_k(\boldsymbol{x})\}_{k=1}^{\infty}$ be a complete orthonormal basis of the space $L^2(D)$ and $\dot{W}=\{\dot{W}(h(\boldsymbol{x})),\, h\in L^2(D)\}$ a zero-mean Gaussian family such that

$$\mathbb{E}\big[\dot{W}(h_1)\dot{W}(h_2)\big]=(h_1,h_2),\quad \forall h_1,h_2\in L^2(D), \tag{2.1}$$

where $D\subset\mathbb{R}^d$, $d=1,2,3$, indicates the physical domain and $(\cdot,\cdot)$ the inner product on $L^2(D)$. The (Gaussian) white noise on $L^2(D)$ is then defined as the formal series

$$\dot{W}=\sum_{k\geq 1}\dot{W}(\mathrm{u}_k)\mathrm{u}_k(\boldsymbol{x})=\sum_{k\geq 1}\xi_k\mathrm{u}_k(\boldsymbol{x}), \tag{2.2}$$

where ξ_k are independent normal random variables according to Eq. (2.1).

We then define $\mathbb{F}:=(\Omega,\mathcal{F},P)$ as a complete probability space, where $\mathcal{F}$ is the σ-algebra generated by the countably many independent identically distributed (i.i.d.) Gaussian random variables $\{\xi_k\}_{k\geq 1}$. We define a random vector $\boldsymbol{\xi}:=(\xi_1,\xi_2,\cdots)$. In practice, we often need to deal with colored noise, where the correlation between two physical points is taken into account. Such a correlation can be modeled though the *smoothed white noise*, which takes the form

$$W_\phi(\boldsymbol{x},\omega)=\sum_{k\geq 1}(\mathrm{u}_k,\phi_{\boldsymbol{x}})\xi_k, \tag{2.3}$$

where the function $\phi_{\boldsymbol{x}}(\boldsymbol{y})$ introduces correlation through the inner product with u_k. For any two physical points $\boldsymbol{x}_1,\boldsymbol{x}_2\in D$, we then have their correlation function as

$$\mathbb{E}\big[W_\phi(\boldsymbol{x}_1,\omega)W_\phi(\boldsymbol{x}_2,\omega)\big]=\sum_{k\geq 1}(\mathrm{u}_k,\phi_{\boldsymbol{x}_1})(\mathrm{u}_k,\phi_{\boldsymbol{x}_2})=(\phi_{\boldsymbol{x}_1},\phi_{\boldsymbol{x}_2}).$$

Example 2.1. Let $\mathbb{I}_B(\boldsymbol{x})$ be the indicator function, i.e., $\mathbb{I}_B(\boldsymbol{x})=1$, if $\boldsymbol{x}\in B$; $\mathbb{I}_B(\boldsymbol{x})=0$, otherwise. Let $\phi_{\boldsymbol{x}}(\boldsymbol{y})=\mathbb{I}_B(\boldsymbol{y}-\boldsymbol{x})$. We then have a nonzero correlation for two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, if $\{\boldsymbol{y}|\boldsymbol{y}-\boldsymbol{x}_1\in B\}\cap\{\boldsymbol{y}|\boldsymbol{y}-\boldsymbol{x}_2\in B\}\neq\emptyset$.

Let $\mathcal{J}$ be the collection of multi-indices $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\cdots)$ such that $\alpha_k\in\mathbb{N}_0$ and $|\boldsymbol{\alpha}|:=\sum_{k\geq 1}\alpha_k<\infty$. For $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}$, we define

$$\boldsymbol{\alpha}+\boldsymbol{\beta}=(\alpha_1+\beta_1,\alpha_2+\beta_2,\cdots),\quad |\boldsymbol{\alpha}|=\sum_{k\geq 1}\alpha_k,\quad \boldsymbol{\alpha}!=\prod_{k\geq 1}\alpha_k!.$$

By definition, $\boldsymbol{\alpha}>0$ if $|\boldsymbol{\alpha}|>0$ and $\boldsymbol{\beta}\leq\boldsymbol{\alpha}$ if

$$\beta_k\leq\alpha_k \text{ for all } k\geq 1.$$

If $\boldsymbol{\beta}\leq\boldsymbol{\alpha}$, then we define

$$\boldsymbol{\alpha}-\boldsymbol{\beta}=(\alpha_1-\beta_1,\alpha_2-\beta_2,\cdots).$$

We use $(\mathbf{0})$ to denote the multi-index with all zero entries: $(\mathbf{0})_k=0$ for all k; $\boldsymbol{\epsilon}(i)$ is the multi-index of length 1 and with the single non-zero entry at position i:

$$\boldsymbol{\epsilon}(i)_k=\begin{cases}1, & \text{if } k=i,\\ 0, & \text{if } k\neq i.\end{cases}$$

Let $\mathrm{H}_n=\mathrm{H}_n(t)$, $n=0,1,2,\cdots$, $t\in\mathbb{R}$, one-dimensional Hermite polynomial of order n:

$$\mathrm{H}_n(t)=(-1)^n e^{\frac{t^2}{2}}\frac{d^n}{dt^n}e^{-\frac{t^2}{2}}. \tag{2.4}$$

In particular,

$$\mathrm{H}_0(t)=1,\quad \mathrm{H}_1(t)=t,\quad \mathrm{H}_2(t)=t^2-1,\quad \mathrm{H}_3(t)=t^3-3t,\cdots.$$

With respect to $\boldsymbol{\xi}$, we define the collection of stochastic Hermite polynomials $\Xi=\{\mathrm{h}_{\boldsymbol{\alpha}},\boldsymbol{\alpha}\in\mathcal{J}\}$ as follows:

$$\mathrm{h}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})=\prod_{k\geq 1}\frac{1}{\sqrt{\alpha_k!}}\mathrm{H}_{\alpha_k}(\xi_k).$$

For any fixed k, the following relation holds

$$\mathbb{E}\left[\mathrm{H}_{\alpha_k}(\xi_k)\mathrm{H}_{\beta_k}(\xi_k)\right]=\delta_{\alpha_k\beta_k}\alpha_k!,\qquad \mathbb{E}\left[\mathrm{h}_{\boldsymbol{\alpha}}\mathrm{h}_{\boldsymbol{\beta}}\right]=\delta_{\boldsymbol{\alpha}\boldsymbol{\beta}}. \tag{2.5}$$

Recall the following result.

Theorem 2.1 (Cameron-Martin [5]). *The set Ξ is an orthonormal basis in $L_2(\mathbb{F})$: if $\eta\in L_2(\mathbb{F})$ and $\eta_{\boldsymbol{\alpha}}=\mathbb{E}[\eta\mathrm{h}_{\boldsymbol{\alpha}}]$, then*

$$\eta=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\eta_{\boldsymbol{\alpha}}\mathrm{h}_{\boldsymbol{\alpha}}=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\frac{\eta_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})}{\sqrt{\boldsymbol{\alpha}!}}\quad \textit{and}\quad \mathbb{E}[\eta^2]=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\eta_{\boldsymbol{\alpha}}^2.$$

Let $L^2(\mathbb{F};V)$ denote the collection of square-integrable V-valued random elements, where V is a separable Hilbert space. By Theorem 2.1, every $v\in L^2(\mathbb{F};V)$ has a unique representation

$$v=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}v_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{2.6}$$

where

$$v_{\boldsymbol{\alpha}}=\frac{\mathbb{E}\left[v\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\right]}{\boldsymbol{\alpha}!} \tag{2.7}$$

and

$$\mathbb{E}\|v\|_V^2=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\boldsymbol{\alpha}!\|v_{\boldsymbol{\alpha}}\|_V^2. \tag{2.8}$$

Then, given a collection $\mathcal{R}=\{r_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha}\in\mathcal{J}\}$ of uniformly bounded positive real numbers, we define the space $\mathcal{R}L^2(\mathbb{F};V)$ as the closure of $L^2(\mathbb{F};V)$ in the norm

$$\|v\|^2_{\mathcal{R}L^2(\mathbb{F};V)}=\sum_{\boldsymbol{\alpha}\in\mathcal{J}} r_{\boldsymbol{\alpha}}\boldsymbol{\alpha}!\|v_{\boldsymbol{\alpha}}\|^2_V. \tag{2.9}$$

The space $\mathcal{R}L^2(\mathbb{F};V)$ is called a weighted Wiener chaos space. In this work, $V=H^1_0(D)$ for the elliptic problems.

Definition 2.2. With respect to the stochastic Hermite polynomials $\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, $\boldsymbol{\alpha}\in\mathcal{J}$, the Wick product can be defined as

$$\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\diamond\mathrm{H}_{\boldsymbol{\beta}}(\boldsymbol{\xi})=\mathrm{H}_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\boldsymbol{\xi}). \tag{2.10}$$

The mathematical correspondence between the Wick product and the Malliavin divergence operator for the Skorokhod-Itô integral can be found in [9,12].

3 Stochastic elliptic models

Let D be a bounded, connected, open subset of $\mathbb{R}^d$, $d=1,2,3$, with a Lipschitz continuous boundary ∂D. We consider the following three stochastic PDEs of elliptic type:

$$\text{Model I:}\qquad -\nabla\cdot\big(a(\boldsymbol{x},\omega)\nabla u_I(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x}), \tag{3.1a}$$

$$\text{Model II:}\qquad -\nabla\cdot\big(a(\boldsymbol{x},\omega)\diamond\nabla u_{II}(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x}), \tag{3.1b}$$

$$\text{Model III:}\qquad -\nabla\cdot\big((a^{-1})^{\diamond(-1)}(\boldsymbol{x},\omega)\diamond\nabla u_{III}(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x}), \tag{3.1c}$$

in D, where homogeneous Dirichlet boundary conditions are satisfied on ∂D for all the three Eqs. (3.1a)-(3.1c), $a(\boldsymbol{x},\omega)$ is a log-normal random process, $f(\boldsymbol{x})$ is deterministic for simplicity and $(a^{-1})^{\diamond(-1)}\diamond a^{-1}=1$. All three models deal with the noise in the coefficient $a(\boldsymbol{x},\omega)$. In addition, models II and III replace the regular product with the Wick product. The difference between models II and III is clearer if the following linear systems are considered [20]:

$$\text{Model II:}\quad\begin{cases} a\diamond\nabla u_{II}=\boldsymbol{F},\\ -\nabla\cdot\boldsymbol{F}=f(\boldsymbol{x}),\end{cases}\qquad \text{Model III:}\quad\begin{cases}\nabla u_{III}=a^{-1}\diamond\boldsymbol{F},\\ -\nabla\cdot\boldsymbol{F}=f(\boldsymbol{x}).\end{cases}$$

Thinking of the Wick product as a regularization operation in the probability space, model I smooths the flux while model III smooths the gradient of $u(\boldsymbol{x},\omega)$. The following features of models I-III are observed in [20]:

- $\mathbb{E}[u_{II}]$ and $\mathbb{E}[u_{III}]$ satisfy the following deterministic PDEs

$$-\nabla\cdot\big(\mathbb{E}[a]\nabla\mathbb{E}[u_{II}]\big)=f(\boldsymbol{x}), \tag{3.2a}$$

$$-\nabla\cdot\big(\mathbb{E}[a^{-1}]^{-1}\nabla\mathbb{E}[u_{III}]\big)=f(\boldsymbol{x}), \tag{3.2b}$$

  respectively while $\mathbb{E}[u_I]$ cannot be described by a deterministic PDE due to the closure problem.

- There exists a scaling factor e^{σ^2} between u_I and u_{II}, i.e., $u_I \sim e^{\sigma^2} u_{II}$, where σ is the standard deviation of the underling stationary Gaussian random process of $a(\boldsymbol{x},\omega)$. Model III removes the scaling factor e^{σ^2}, which makes u_I and u_{III} are comparable for many cases.

In this work, we focus on the difference between u_I and u_{III}. More specifically, we discuss such a difference with respect to the two characteristic parameters of the underlying stationary Gaussian random process of $a(\boldsymbol{x},\omega)$, i.e., the standard deviation σ and the correlation length l_c.

3.1 The log-normal field $a(\boldsymbol{x},\omega)$

The log-normal random field $a(\boldsymbol{x},\omega)=e^{\diamond W_\phi(\boldsymbol{x})}$ is defined with respect to the Wick product [9]

$$e^{\diamond W_\phi(\boldsymbol{x})}=\sum_{n=0}^{\infty}\frac{1}{n!}W_\phi^{\diamond n}(\boldsymbol{x}), \tag{3.3}$$

which also corresponds to the regular log-normal random field as [9]

$$e^{\diamond W_\phi(\boldsymbol{x})}=\exp\left[W_\phi(\boldsymbol{x})-\frac{1}{2}\|\phi\|_2^2\right], \tag{3.4}$$

where $\phi\in L^2(D)$ is the function chosen for the smoothed white noise $W_\phi(\boldsymbol{x})$, see Eq. (2.3). Following are some useful properties of the log-normal random field $e^{\diamond W_\phi(x)}$:

Theorem 3.1 (see [9]). *The following statements hold for the log-normal random field $e^{\diamond W_\phi(\boldsymbol{x})}$:*

(a) $\mathbb{E}\left[e^{\diamond W_\phi}\right]=1.$

(b) $Var(e^{\diamond W_\phi})=e^{\|\phi\|_2^2}-1.$

(c) $e^{\diamond W_\phi}\diamond e^{\diamond(-W_\phi)}=1.$

Corollary 3.1. *Let $a(\boldsymbol{x},\omega)=e^{\diamond W_\phi(\boldsymbol{x})}$. Then*

$$(a^{-1})^{\diamond(-1)}=e^{-\|\phi\|_2^2}e^{\diamond W_\phi(\boldsymbol{x})}. \tag{3.5}$$

Proof. As

$$a^{-1}(\boldsymbol{x},\omega)=\left(e^{\diamond W_\phi(\boldsymbol{x})}\right)^{-1}=e^{-W_\phi(\boldsymbol{x})+\frac{1}{2}\|\phi\|_2^2}=e^{\|\phi\|_2^2}e^{\diamond(-W_\phi(\boldsymbol{x}))},$$

using property (c) in Theorem 3.1, we conclude the proof. □

Thus, for the chosen log-normal random field, the difference between $a(\boldsymbol{x},\omega)$ and $(a^{-1})^{\diamond(-1)}$ is the scaling factor $e^{-\|\phi\|_2^2}$, which is mainly related to the degree of perturbation as shown by property (b) in Theorem 3.1. More discussions about such a scaling factor can be found in [20]. Mathematically speaking, the scaling factor comes from the fact that the regular product and Wick product cannot commute.

3.2 Karhunen-Loéve expansion

We now establish a connection between the smoothed white noise and the Karhunen-Loève expansion, which is widely used in practice to approximate colored noise. Assume that the underlying Gaussian field $W_\phi(\boldsymbol{x})$ has a normalized correlation function

$$R(\boldsymbol{x}-\boldsymbol{y})=\mathbb{E}\big[W_\phi(\boldsymbol{x})W_\phi(\boldsymbol{y})\big],$$

e.g., $R(\boldsymbol{x}-\boldsymbol{y})=e^{-|\boldsymbol{x}-\boldsymbol{y}|/l_c}$, which is non-negative definite. The eigen-pairs $\{\lambda_i,\phi_i(\boldsymbol{x})\}_{i=1}^{\infty}$ of $R(\boldsymbol{x}-\boldsymbol{y})$ are defined as

$$\int_D R(\boldsymbol{x},\boldsymbol{y})\phi_i(\boldsymbol{y})d\boldsymbol{y}=\lambda_i\phi_i(\boldsymbol{x}),\qquad \int_D \phi_i(\boldsymbol{x})\phi_i(\boldsymbol{x})d\boldsymbol{x}=\delta_{ij}. \tag{3.6}$$

According to the Mercer's theorem, $\mathrm{u}_k=\phi_k(\boldsymbol{x})$ form a complete basis of $L^2(D)$.

Let $\phi_{\boldsymbol{x}}(\boldsymbol{y})=\sum_{i=1}^{\infty}\sqrt{\lambda_i}\phi_i(\boldsymbol{x})\phi_i(\boldsymbol{y})$. Using Eq. (2.3), it is easy to show that the corresponding smoothed white noise takes the form of the Karhunen-Loève (K-L) expansion

$$W_\phi(\boldsymbol{x})=\sum_{i=1}^{\infty}\sqrt{\lambda_i}\phi_i(\boldsymbol{x})\xi_i. \tag{3.7}$$

Note that

$$\|\phi_{\boldsymbol{x}}\|_2^2=\sum_{i=1}^{\infty}\lambda_i\phi_i^2(\boldsymbol{x})=R(0)=1, \tag{3.8}$$

which is the variance of the Gaussian random field $W_\phi(\boldsymbol{x})$. Using Eqs. (3.7) and (3.8) and the generating function of Hermite polynomials

$$e^{st-\frac{1}{2}s^2}=\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(t), \tag{3.9}$$

we obtain the explicit Wiener chaos expansion of a log-normal random field

$$e^{\diamond[W_\phi(\boldsymbol{x})]}=\exp\Big[W_\phi(\boldsymbol{x})-\frac{1}{2}\|\phi\|_2^2\Big]=\exp\Big[\sum_{i=1}^{\infty}(\mathrm{u}_i,\phi_{\boldsymbol{x}})\xi_i-\frac{1}{2}(\mathrm{u}_i,\phi_{\boldsymbol{x}})^2\Big]=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\frac{\Phi^{\boldsymbol{\alpha}}(\boldsymbol{x})}{\sqrt{\boldsymbol{\alpha}!}}\mathrm{h}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{3.10}$$

where

$$\Phi(\boldsymbol{x})=\big((\mathrm{u}_1,\phi_{\boldsymbol{x}}),(\mathrm{u}_2,\phi_{\boldsymbol{x}}),\cdots\big), \tag{3.11}$$

and $(\mathrm{u}_i,\phi_{\boldsymbol{x}})=\sqrt{\lambda_i}\phi_i(\boldsymbol{x})$. For a more general correlation function, e.g., $R(\boldsymbol{x}-\boldsymbol{y})=\sigma^2e^{-|\boldsymbol{x}-\boldsymbol{y}|/l_c}$, which satisfies $R(0)=\sigma^2$, we only need a small modification $(\mathrm{u}_i,\phi_{\boldsymbol{x}})=\sigma\sqrt{\lambda_i}\phi_i(\boldsymbol{x})$ to make the above formulas valid. Note that the eigen-pairs $\{\lambda_i,\phi_i(\boldsymbol{x})\}_{i=1}^{\infty}$ are subject to the correlation function $R(\boldsymbol{x}-\boldsymbol{y})$ with $\sigma=1$.

3.3 Uncertainty propagators

Based on Theorem 2.1, models I-III can be transformed into some high dimensional deterministic problems, i.e., *uncertainty propagators*, through a Galerkin procedure in the probability space. Since we will focus on the discussion of models I and III, we here only describe their uncertainty propagators. Assume that we have Wiener chaos expansions of random coefficients

$$a(\boldsymbol{x},\omega)=\sum_{\boldsymbol{\alpha}\in\mathcal{J}} a_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}},\qquad (a^{-1})^{\diamond(-1)}=\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\hat{a}_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}.$$

We then substitute the Wiener chaos expansions of solutions

$$u_I=\sum_{\boldsymbol{\beta}\in\mathcal{J}} u_{I,\boldsymbol{\beta}}\mathrm{H}_{\boldsymbol{\beta}},\qquad u_{III}=\sum_{\boldsymbol{\beta}\in\mathcal{J}} u_{III,\boldsymbol{\beta}}\mathrm{H}_{\boldsymbol{\beta}}$$

into models I and III and take the Galerkin projection in the probability space, which results in the following propagators:

$$-\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{J}}\nabla\cdot(a_{\boldsymbol{\alpha}}\nabla u_{I,\boldsymbol{\beta}})\mathbb{E}\left[\mathrm{H}_{\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\beta}}\mathrm{H}_{\gamma}\right]=f(\boldsymbol{x})\delta_{(\mathbf{0}),\gamma},\tag{3.12a}$$

$$-\sum_{|\boldsymbol{\alpha}|\le\gamma}\nabla\cdot\left(\hat{a}_{\gamma-\boldsymbol{\alpha}}\nabla u_{III,\boldsymbol{\alpha}}\right)=f(\boldsymbol{x})\delta_{(\mathbf{0}),\gamma},\tag{3.12b}$$

where $\gamma\in\mathcal{J}$. To this end, the original stochastic PDEs are transformed to a system of deterministic PDEs, where Eq. (3.12a) is a coupled PDE system of chaos coefficients $u_{I,\boldsymbol{\alpha}}$ while Eq. (3.12b) is a decoupled one which is lower triangular, i.e., $u_{III,\gamma}$ only depends on $u_{III,\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha}<\gamma$.

Define an operator $\mathrm{A}_{\boldsymbol{\alpha}}=-\nabla\cdot(\hat{a}_{\boldsymbol{\alpha}}\nabla)$. Eq. (3.12b) can be rewritten as

$$\begin{cases}\mathrm{A}_{(\mathbf{0})}u_{III,(\mathbf{0})}=f, & |\boldsymbol{\alpha}|=0,\\ \mathrm{A}_{(\mathbf{0})}u_{III,\boldsymbol{\alpha}}=-\displaystyle\sum_{\boldsymbol{\beta}\in\mathcal{J},0<\boldsymbol{\beta}\le\boldsymbol{\alpha}}\mathrm{A}_{\boldsymbol{\beta}}u_{III,\boldsymbol{\alpha}-\boldsymbol{\beta}}, & |\boldsymbol{\alpha}|>0.\end{cases}\tag{3.13}$$

Then Eq. (3.13) can be solved in an abstract form

$$\begin{cases}u_{III,(\mathbf{0})}=\mathrm{A}_{(\mathbf{0})}^{-1}f, & |\boldsymbol{\alpha}|=0,\\ u_{III,\boldsymbol{\alpha}}=-\displaystyle\sum_{\boldsymbol{\beta}\in\mathcal{J},0<\boldsymbol{\beta}\le\boldsymbol{\alpha}}\mathrm{A}_{(\mathbf{0})}^{-1}\mathrm{A}_{\boldsymbol{\beta}}u_{III,\boldsymbol{\alpha}-\boldsymbol{\beta}}, & |\boldsymbol{\alpha}|>0.\end{cases}\tag{3.14}$$

The uncertainty propagator of model I does not have such a nice property since it does not have a lower-triangular structure.

4 Compare models I and III

There exist two characteristic parameters for the underlying Gaussian random process: the standard deviation σ and the correlation length l_c. In this section we consider the difference between models I and III with respect to σ and l_c.

4.1 The difference between u_I and u_{III} with respect to σ

We first consider the difference between operations of the regular and Wick products for two random elements in $\mathcal{R}L^2(\mathbb{F};\mathbb{R})$.

Proposition 4.1. *Let $g_1,g_2,g_1(*-\diamond)g_2 \in \mathcal{R}L^2(\mathbb{F};\mathbb{R})$. Assume that $g_1=\sum_{\boldsymbol{\alpha}\in\mathcal{J}} g_{1,\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ and $g_2=\sum_{\boldsymbol{\alpha}\in\mathcal{J}} g_{2,\boldsymbol{\alpha}}\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, where $g_{1,\boldsymbol{\alpha}}=\mathcal{O}(\varepsilon^{|\boldsymbol{\alpha}|})$ and $g_{2,\boldsymbol{\alpha}}=\mathcal{O}(\varepsilon^{|\boldsymbol{\alpha}|})$ and $0<\varepsilon<1$. Then*

$$\|(g_1(*-\diamond)g_2)\|_{\mathcal{R}L^2(\mathbb{F};\mathbb{R})}=\mathcal{O}(\varepsilon^2). \tag{4.1}$$

Here $*$ denotes the regular product and $\mathcal{R}$ is used as a general operator for weighted Wiener chaos space.

Proof. We first look at the Wick product. According to the definition of Wick product, we have the Wiener chaos expansion

$$\begin{aligned}&(g_1\diamond g_2)_{(\mathbf{0})}=g_{1,(\mathbf{0})}g_{2,(\mathbf{0})},\\&(g_1\diamond g_2)_{\boldsymbol{\epsilon}(i)}=g_{1,(\mathbf{0})}g_{2,\boldsymbol{\epsilon}(i)}+g_{1,\boldsymbol{\epsilon}(i)}g_{2,(\mathbf{0})}\end{aligned}$$

for $\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ of polynomial order $|\boldsymbol{\alpha}|=0,1$. For the regular product, we have

$$\begin{aligned}&(g_1*g_2)_{(\mathbf{0})}=g_{1,(\mathbf{0})}g_{2,(\mathbf{0})}+\mathcal{O}(\varepsilon^a),\\&(g_1*g_2)_{\boldsymbol{\epsilon}(i)}=g_{1,(\mathbf{0})}g_{2,\boldsymbol{\epsilon}(i)}+g_{1,\boldsymbol{\epsilon}(i)}g_{2,(\mathbf{0})}+\mathcal{O}(\varepsilon^b),\end{aligned}$$

where the terms $\mathcal{O}(\varepsilon^a)$ and $\mathcal{O}(\varepsilon^b)$ are contributions from the higher order terms. We now identify a and b. It is not difficult to see that with respect to ε, the largest contribution to the mean is from the terms

$$g_{1,\boldsymbol{\epsilon}(i)}g_{2,\boldsymbol{\epsilon}(i)}\mathrm{H}^2_{\boldsymbol{\epsilon}(i)}=g_{1,\boldsymbol{\epsilon}(i)}g_{2,\boldsymbol{\epsilon}(i)}\mathrm{H}_{(0,\cdots,0,2,0,\cdots)}+g_{1,\boldsymbol{\epsilon}(i)}g_{2,\boldsymbol{\epsilon}(i)},$$

where $(0,\cdots,0,2,0,\cdots)$ is a multi-index such that only the i-th component is nonzero. Since $g_{1,\boldsymbol{\epsilon}(i)}g_{2,\boldsymbol{\epsilon}(i)}=\mathcal{O}(\varepsilon^2)$, we have $a=2$. Similarly, we can identify $b=3$. Then each chaos coefficient of $g_1(*-\diamond)g_2$ contains a factor ε^2, which can be taken out. Since the weighted norm forms a power series with respect to ε, the weights $\{r_{\boldsymbol{\alpha}}\}$ should depend on the coefficients of $\varepsilon^{\boldsymbol{\alpha}}$ when $\varepsilon<1$. Then we obtain the conclusion. Note here that $\mathcal{R}$ is used as a general operator for weighted Wiener chaos space, which implies that the weights for g_1, g_2 and $g_1(*-\diamond)g_2$ are possibly different. We refer to [10] for more discussions about the operator $\mathcal{R}$. □

Models I and III correspond to the following two linear systems, respectively

$$\text{I:}\quad\begin{cases}\nabla u_I=a^{-1}*\boldsymbol{F}_1,\\-\nabla\cdot\boldsymbol{F}_1=f,\end{cases}\qquad\text{III:}\quad\begin{cases}\nabla u_{III}=a^{-1}\diamond\boldsymbol{F}_3,\\-\nabla\cdot\boldsymbol{F}_3=f,\end{cases} \tag{4.2}$$

where $*$ denotes the operation of the regular product. Then the equation for u_I-u_{III} can be obtained as

$$\begin{cases} \nabla(u_I-u_{III})=a^{-1}*(\boldsymbol{F}_1-\boldsymbol{F}_3)+a^{-1}(*-\diamond)\boldsymbol{F}_3, \\ -\nabla\cdot(\boldsymbol{F}_1-\boldsymbol{F}_3)=0, \end{cases} \tag{4.3}$$

which corresponds to a second order elliptic equation for u_I-u_{III} as

$$-\nabla\cdot\big(a\nabla(u_I-u_{III})\big)=-\nabla\cdot\big(a*(a^{-1}(*-\diamond)\boldsymbol{F}_3)\big). \tag{4.4}$$

Note that we express explicitly the regular products on the right-hand side since the regular and Wick products do not commute. It is seen that Eq. (4.4) corresponds to model I while the force term is related to model III through $\boldsymbol{F}_3$.

Theorem 4.1. *Assume that $F=-\nabla\cdot(a*(a^{-1}(*-\diamond)\boldsymbol{F}_3))\in\mathcal{R}L^2(\mathbb{F};H^{-1}(D))$ subject to the weights $\{r_{\boldsymbol{\alpha}}\}$. Then there exists a set of weights $\tilde{\mathcal{R}}=\{\tilde{r}_{\boldsymbol{\alpha}},\boldsymbol{\alpha}\in\mathcal{J}\}$, such that*

$$\|u_I-u_{III}\|_{\tilde{\mathcal{R}}L^2(\mathbb{F};H_0^1(D))}=\mathcal{O}(\sigma^2). \tag{4.5}$$

Proof. It is a technical issue to study the properties of F, since this term is related to both model I and III. We will present an idea of how to study F and then just focus on the order of the difference between u_I and u_{III} with respect to σ. Both models I and III can be studied by white noise analysis, which is consistent with the weighted Wiener chaos space approach [14, 17]. Once we identify a proper space $\mathcal{R}L^2(\mathbb{F};H^{-1}(D))$ for F, we can adapt the results in [14] to find the space $\tilde{\mathcal{R}}L^2(\mathbb{F};H_0^1(D))$ in which u_I-u_{III} exists. Since in this work we are only interested in the order of $\|u_I-u_{III}\|_{\tilde{\mathcal{R}}L^2(\mathbb{F};H_0^1(D))}$ with respect to σ, the explicit definition of $\tilde{\mathcal{R}}L^2(\mathbb{F};H_0^1(D))$ is not critical for our goal.

From Eq. (4.2), we know that $\boldsymbol{F}_3=(a^{-1})^{\diamond(-1)}\diamond\nabla u_{III}$. We now discuss $u_{III,\boldsymbol{\alpha}}$ using the abstract form (3.14). Using Corollary 3.1 and Eq. (3.10), we have

$$(a^{-1})^{\diamond(-1)}=e^{-\sigma^2}\sum_{\boldsymbol{\alpha}\in\mathcal{J}}\frac{\Phi^{\boldsymbol{\alpha}}(\boldsymbol{x})}{\boldsymbol{\alpha}!}\mathrm{H}_{\boldsymbol{\alpha}}(\boldsymbol{\xi}),$$

i.e.,

$$\hat{a}_{\boldsymbol{\alpha}}(\boldsymbol{x})=e^{-\sigma^2}\frac{\Phi^{\boldsymbol{\alpha}}(\boldsymbol{x})}{\boldsymbol{\alpha}!},$$

where

$$\Phi(\boldsymbol{x})=\big(\sigma\sqrt{\lambda_1}\phi_1(\boldsymbol{x}),\sigma\sqrt{\lambda_2}\phi_2(\boldsymbol{x}),\cdots\big).$$

Thus

$$\hat{a}_{\boldsymbol{\alpha}}(\boldsymbol{x})=\mathcal{O}(\sigma^{|\boldsymbol{\alpha}|}).$$

Let $\hat{a}_{\boldsymbol{\alpha}}(\boldsymbol{x})\le C_{\boldsymbol{\alpha}}e^{-\sigma^2}\sigma^{|\boldsymbol{\alpha}|}$, $\boldsymbol{\alpha}\in\mathcal{J}$. For $\hat{v},v\in H_0^1(D)$, let $\mathrm{A}_{(\mathbf{0})}\hat{v}=\mathrm{A}_{\boldsymbol{\alpha}}v$. We have

$$e^{-\sigma^2}\|\hat{v}\|_{H_0^1(D)}^2=(\mathrm{A}_{(\mathbf{0})}\hat{v},\hat{v})=(\mathrm{A}_{\boldsymbol{\alpha}}v,\hat{v})=(\hat{a}_{\boldsymbol{\alpha}}(\boldsymbol{x})\nabla v,\nabla\hat{v})\le C_{\boldsymbol{\alpha}}\sigma^{|\boldsymbol{\alpha}|}\|v\|_{H_0^1(D)}\|\hat{v}\|_{H_0^1(D)},$$

which results in

$$\|\mathrm{A}_{(\mathbf{0})}^{-1}\mathrm{A}_{\boldsymbol{\alpha}}\| \leq C_{\boldsymbol{\alpha}}\sigma^{|\boldsymbol{\alpha}|}.$$

Thus

$$\|\mathrm{A}_{(\mathbf{0})}^{-1}\mathrm{A}_{\boldsymbol{\alpha}}\| = \mathcal{O}(\sigma^{|\boldsymbol{\alpha}|}). \tag{4.6}$$

From Eqs. (4.6) and (3.14), we have

$$\|u_{III,\boldsymbol{\alpha}}\|_{H_0^1(D)} \leq \sum_{\boldsymbol{\beta}\in\mathcal{J},0<\boldsymbol{\beta}\leq\boldsymbol{\alpha}} \|\mathrm{A}_{(\mathbf{0})}^{-1}\mathrm{A}_{\boldsymbol{\beta}}\|\|u_{III,\boldsymbol{\alpha}-\boldsymbol{\beta}}\|_{H_0^1(D)},$$

which implies that

$$\|u_{III,\boldsymbol{\alpha}}\|_{H_0^1(D)} = \mathcal{O}(\sigma^{|\boldsymbol{\alpha}|}).$$

Thus each component of $\nabla u_{III,\boldsymbol{\alpha}}$ must have a factor $\sigma^{|\boldsymbol{\alpha}|}$, i.e., $\partial_{x_i}u_{III,\boldsymbol{\alpha}} = \sigma^{|\boldsymbol{\alpha}|}C_{i,\boldsymbol{\alpha}}(\boldsymbol{x})$, $1\leq i\leq d$. Let $F_{3,i,\boldsymbol{\alpha}}$ be the chaos coefficient of the ith component of $\boldsymbol{F}_3$. Since $\boldsymbol{F}_3 = (a^{-1})^{\diamond(-1)}\diamond\nabla u_{III}$, the definition of Wick product yields that $F_{3,i,\boldsymbol{\alpha}}(\boldsymbol{x}) = \sigma^{|\boldsymbol{\alpha}|}\tilde{F}_{3,i,\boldsymbol{\alpha}}(\boldsymbol{x})$. From Proposition 4.1, we know that each Wiener chaos coefficient of $a^{-1}(*-\diamond)\boldsymbol{F}_3$ has a factor σ^2 and so does F. Based on our assumption of F, we have

$$\|u_I - u_{III}\|_{\tilde{\mathcal{R}}L^2(\mathbb{F};H_0^1(D))} \leq C\|F\|_{\mathcal{R}L^2(\mathbb{F};H^{-1}(D))},$$

where C is a general constant. It can be shown that C behaves like $e^{\theta\sigma^2}$, where θ is a constant independent of σ [14,16]. We then complete the proof from Proposition 4.1. □

From Proposition 4.1 and the proof of Theorem 4.1, we can see that the right-hand side of Eq. (4.4) takes the form

$$-\nabla\cdot\big(a*(a^{-1}(*-\diamond)\boldsymbol{F}_3)\big) = \sigma^2\tilde{f}_2(\boldsymbol{x},\boldsymbol{\xi}) + \sigma^3\tilde{f}_3(\boldsymbol{x},\boldsymbol{\xi}) + \cdots.$$

Substituting

$$a(\boldsymbol{x},\omega) = a_0 + \sigma a_1(\boldsymbol{x},\omega) + \sigma^2 a_2(\boldsymbol{x},\omega) + \cdots$$

and the following ansatz of $u_I - u_{III}$

$$u_I - u_{III} = \tilde{u}_0(\boldsymbol{x}) + \sigma\tilde{u}_1(\boldsymbol{x},\boldsymbol{\xi}) + \sigma^2\tilde{u}_2(\boldsymbol{x},\boldsymbol{\xi}) + \cdots$$

into Eq. (4.4) and comparing the coefficients of σ^i, it is not difficult to obtain that

$$\begin{aligned}
&-\nabla\cdot(a_0\nabla\tilde{u}_0) = 0,\\
&-\nabla\cdot(a_0\nabla\tilde{u}_1) = \nabla\cdot(a_1\nabla\tilde{u}_0),\\
&-\nabla\cdot(a_0\nabla\tilde{u}_2) = \nabla\cdot(a_2\nabla\tilde{u}_0) + \nabla\cdot(a_1\nabla\tilde{u}_1) + \tilde{f}_2(\boldsymbol{x},\boldsymbol{\xi}),\cdots,
\end{aligned}$$

which results in

$$\tilde{u}_0(\boldsymbol{x}) = \tilde{u}_1(\boldsymbol{x},\boldsymbol{\xi}) = 0, \quad \tilde{u}_i(\boldsymbol{x},\boldsymbol{\xi}) \neq 0, \quad i = 2,3,\cdots.$$

Thus, $u_I - u_{III}$ has the following power series expansion with respect to σ

$$u_I - u_{III} = \sigma^2 \tilde{u}_2(\boldsymbol{x},\boldsymbol{\xi}) + \sigma^3 \tilde{u}_3(\boldsymbol{x},\boldsymbol{\xi}) + \cdots. \tag{4.7}$$

Then both the mean and standard deviation of $u_I - u_{III}$ are of $\mathcal{O}(\sigma^2)$ if they exist.

We now look at the autocorrelation function of u_I and u_{III}, defined as

$$R(u_I, u_{III}) = \frac{\mathbb{E}\big[(u_I - \mathbb{E}[u_I])(u_{III} - \mathbb{E}[u_{III}])\big]}{\sigma_{u_I}\sigma_{u_{III}}}, \tag{4.8}$$

where σ_{u_I} and $\sigma_{u_{III}}$ are standard deviations of u_I and u_{III}. We assume here that σ_{u_I} and $\sigma_{u_{III}}$ is positive for any $\boldsymbol{x} \in D$. Let

$$\begin{aligned} u_I &= \hat{u}_0(\boldsymbol{x}) + \sigma \hat{u}_1(\boldsymbol{x},\boldsymbol{\xi}) + \sigma^2 \hat{u}_2^{(1)}(\boldsymbol{x},\boldsymbol{\xi}) + \sigma^3 \hat{u}_3^{(1)}(\boldsymbol{x},\boldsymbol{\xi}) + \cdots, \\ u_{III} &= \hat{u}_0(\boldsymbol{x}) + \sigma \hat{u}_1(\boldsymbol{x},\boldsymbol{\xi}) + \sigma^2 \hat{u}_2^{(3)}(\boldsymbol{x},\boldsymbol{\xi}) + \sigma^3 \hat{u}_3^{(3)}(\boldsymbol{x},\boldsymbol{\xi}) + \cdots. \end{aligned}$$

We then have the standard deviations of u_I and u_{III} as

$$\sigma_{u_I} = \sigma\big(\sigma_{\hat{u}_1}^2 + \mathcal{O}(\sigma)\big)^{\frac{1}{2}}, \qquad \sigma_{u_{III}} = \sigma\big(\sigma_{\hat{u}_1}^2 + \mathcal{O}(\sigma)\big)^{\frac{1}{2}} \tag{4.9}$$

and

$$\mathbb{E}\big[(u_I - \mathbb{E}[u_I])(u_{III} - \mathbb{E}[u_{III}])\big] = \sigma^2 \sigma_{\hat{u}_1}^2 + \mathcal{O}(\sigma^3). \tag{4.10}$$

Thus if σ is small enough, $R(u_I, u_{III}) \approx 1$, which implies that u_I and u_{III} are almost linear.

Remark 4.1. The model difference between models I and III is of $\mathcal{O}(\sigma^2)$. When σ is relatively small, model III is able to give most of the information we need from model I. However, it is much more easier to solve model III than to solve model I due to the fact that the uncertainty propagator of model III has a lower-triangular structure.

4.2 The difference between u_I and u_{III} with respect to l_c

It is not straightforward to discuss the difference between u_I and u_{III} with respect to the correlation length l_c, since we are not able to link it explicitly to the Wiener chaos expansion as we did for the discussions of σ. We will mainly focus on the one-dimensional problems which have an explicit solution, based on which we present some comments for the high-dimensional problems.

For the one-dimensional problems, we have the following exact solutions for Eqs. (3.1a)-(3.1c):

Theorem 4.2 (see [20]). *Let $D = (0,1)$, $a(x,\omega) = e^{\diamond W_\phi}$ and $f(x) \in L_1(D)$ is a deterministic function. Then the exact solutions of models I-III are:*

$$u_I = e^{\|\phi\|_2^2} \left\{ \tilde{A} \cdot \int_0^x e^{\diamond(-W_\phi(t))} dt - \int_0^x \int_0^t f(s) ds e^{\diamond(-W_\phi(t))} dt \right\}, \tag{4.11a}$$

$$u_{II} = A \diamond \int_0^x e^{\diamond(-W_\phi(t))} dt - \int_0^x \int_0^t f(s) ds e^{\diamond(-W_\phi(t))} dt, \tag{4.11b}$$

$$u_{III} = e^{\|\phi\|_2^2} u_{II}, \tag{4.11c}$$

where

$$\tilde{A}=\Big(\int_0^1 e^{\diamond(-W_\phi(t))}dt\Big)^{-1}\cdot\int_0^1\int_0^t f(s)ds e^{\diamond(-W_\phi(t))}dt,$$
$$A=\Big(\int_0^1 e^{\diamond(-W_\phi(t))}dt\Big)^{\diamond(-1)}\diamond\int_0^1\int_0^t f(s)ds e^{\diamond(-W_\phi(t))}dt.$$

4.2.1 Infinite correlation length

This case is the simplest one, since $W_\phi(x)$ does not depend on x if the correlation length is infinitely large. Then $u_i(x,\omega)$, $i=$I, II, III take the following forms [20]

$$u_I=e^{\sigma^2}u_{II}=u_{III}=e^{\sigma^2}e^{\diamond(-W_\phi)}\Delta^{-1}f(x), \tag{4.12}$$

where $W_\phi(x)$ is independent of x, Δ^{-1} is the inverse of Laplace operator satisfying the homogeneous boundary conditions. In other words, u_I and u_{III} are exactly the same when $W_\phi(x)$ is just a Gaussian random variable and u_{II} is obviously not a good approximation of u_I due to the exponential divergence.

Note that Eq. (4.12) is obtained directly from the fact that the underlying Gaussian random process is spatially-independent, which is actually valid for physical dimension $d=1,2,3$. If we combined such an observation with Theorem 4.1, we obtain the following corollary.

Corollary 4.1. *Let $\|u_I-u_{III}\|\le C(l_c)\sigma^2$ with respect to a proper weighted L^2 norm for $d=1,2,3$. Then $C(l_c)$ goes to zero as l_c goes to infinity.*

4.2.2 Small correlation length

For a small correlation length, we have the following theorem:

Theorem 4.3. *Let*

$$\phi(y)=\frac{\sigma}{\sqrt{l_c}}\mathbb{I}_{[-\frac{l_c}{2},\frac{l_c}{2}]}$$

for the definition of the smoothed white noise $W_\phi(x)$, where l_c is a constant and can be regarded as the correlation length. Then for one-dimensional stochastic elliptic problems considered in Theorem 4.2*, we have*

$$\begin{aligned}\lim_{l_c\to0}u_I(x)&=e^{\sigma^2}\lim_{l_c\to0}u_{II}(x)=\lim_{l_c\to0}u_{III}(x)\\&=e^{\sigma^2}\Big(x\int_0^1\int_0^t f(s)dsdt-\int_0^x\int_0^t f(s)dsdt\Big),\quad a.s..\end{aligned} \tag{4.13}$$

Proof. We will still use the smoothed white noise to discuss this case. However, instead of using the Karhunen-Loève expansion which is global in the physical space, we consider the step functions for the expansion of (smoothed) white noise.

Let $D=(0,1)$ and Γ_h be a uniform partition of D, i.e.,

$$\Gamma_h: 0=x_0<x_1<\cdots<x_n=1,$$

where $x_{i+1}-x_i=h=1/n$, $i=0,\cdots,n-1$. Define the following step functions:

$$\mathrm{u}_k=\frac{1}{\sqrt{h}}\mathbb{I}_{[x_k,x_{k+1})},\quad k=0,\cdots,n-1. \tag{4.14}$$

It is known that $\{\mathrm{u}_k\}$ is a complete orthonormal basis of $L^2(D)$ when $n\to\infty$, which implies that the Gaussian white noise on $L^2(D)$ can be approximated as

$$\dot{W}=\sum_{k=0}^{n-1}\frac{1}{\sqrt{h}}\mathbb{I}_{[x_k,x_{k+1})}\xi_k. \tag{4.15}$$

For the smoothed white noise, we consider a simple smoothing function

$$\phi(\boldsymbol{y})=\frac{\sigma}{\sqrt{l_c}}\mathbb{I}_{[-\frac{l_c}{2},\frac{l_c}{2}]},\quad \text{where } 0<\sigma\in\mathbb{R} \text{ and } \|\phi\|_2=\sigma.$$

Then the smoothed white noise takes the form

$$W_\phi(x)=\sum_{k=0}^{n-1}\frac{1}{\sqrt{h}}\left(\mathbb{I}_{[x_k,x_{k+1})},\phi_x\right)\xi_k, \tag{4.16}$$

where we assume that $h\le l_c$. For any x, we have

$$\begin{aligned}\mathbb{E}\left[e^{W_\phi(x)}\right]&=\mathbb{E}\left[\prod_{k=0}^{n-1}\exp\left(\frac{1}{\sqrt{h}}(\mathbb{I}_{[x_k,x_{k+1})},\phi_x)\xi_k\right)\right]=\exp\left(\sum_{k=0}^{n-1}\frac{1}{2h}(\mathbb{I}_{[x_k,x_{k+1})},\phi_x)^2\right)\\&=\exp\left(\sum_{k=0}^{n-1}\frac{1}{2}(\mathrm{u}_k,\phi_x)^2\right)=\exp\left(\frac{1}{2}\|\phi_x\|_2^2\right)=\exp\left(\frac{1}{2}\sigma^2\right),\end{aligned} \tag{4.17}$$

where we use the fact that $\phi_x\in\mathrm{span}\{\mathrm{u}_0,\cdots,\mathrm{u}_{n-1}\}$, if n is large enough. It is not difficult to verify that the properties listed in Theorem 3.1 hold for the smoothed white noise define in Eq. (4.16).

Consider the finite difference approximation of the following integral

$$\int_0^x g(t)e^{\diamond(-W_\phi(t))}dt=\lim_{h\to0}\sum_{k=0}^{n_x-1}g\left(t_{i+\frac{1}{2}}\right)e^{\diamond\left(-W_\phi\left(t_{i+\frac{1}{2}}\right)\right)}\delta x,$$

where the function $g(t)$ is bounded on $[0,1]$, n_x is an integer and $x=n_x\delta x$. We take $\delta x=l_c$, which implies that $e^{\diamond(-W_\phi(t_{i+1/2}))}$ are independent random variables. According to the strong law of large numbers, we have

$$\frac{1}{n_x}\sum_{k=0}^{n_x-1}\left(g\left(t_{i+\frac{1}{2}}\right)e^{\diamond\left(-W_\phi\left(t_{i+\frac{1}{2}}\right)\right)}-\mathbb{E}\left[g\left(t_{i+\frac{1}{2}}\right)e^{-\diamond\left(W_\phi\left(t_{i+\frac{1}{2}}\right)\right)}\right]\right)\to0,\quad \text{a.s.}$$

as $n_x \to \infty$, i.e.,

$$\int_0^x g(t)e^{\diamond(-W_\phi(t))}dt \to \lim_{\delta x\to 0}\sum_{k=0}^{n_x-1} g\left(t_{i+\frac{1}{2}}\right)\delta x = \int_0^x g(t)dt, \quad \text{a.s.},$$

where we use the fact that

$$\mathbb{E}\left[\exp\left(-\diamond\left(W_\phi\left(t_{i+\frac{1}{2}}\right)\right)\right)\right]=1.$$

Thus

$$\int_0^x e^{\diamond(-W_\phi(t))}dt \to x \quad \text{and} \quad \int_0^x\int_0^t f(s)e^{\diamond(-W_\phi(t))}dsdt \to \int_0^x\int_0^t f(s)dsdt,$$

a.s., as $\delta x = l_c \to 0$. Using the equality

$$\left(\int_0^1 e^{\diamond(-W_\phi(t))}dt\right)^{\diamond(-1)} \diamond \left(\int_0^1 e^{\diamond(-W_\phi(t))}dt\right)=1,$$

we know that

$$\left(\int_0^1 e^{\diamond(-W_\phi(t))}dt\right)^{\diamond(-1)} \to 1, \quad \text{a.s..}$$

For a more general discussion, we consider a equidistant mesh with a step size l_c

$$0=x_0<x_1<\cdots<x_{\lfloor\frac{x}{l_c}\rfloor}\le x_{\lfloor\frac{x}{l_c}\rfloor+1}=x.$$

Then

$$\begin{aligned}\int_0^x g(t)e^{\diamond(-W_\phi(t))}dt &= \sum_{i=0}^{\lfloor\frac{x}{l_c}\rfloor}\int_{x_i}^{x_{i+1}} g(t)e^{\diamond(-W_\phi(t))}dt \\ &= \Big(\sum_{i\le\lfloor\frac{x}{l_c}\rfloor \text{ and even}} + \sum_{i\le\lfloor\frac{x}{l_c}\rfloor \text{ and odd}}\Big)\int_{x_i}^{x_{i+1}} g(t)e^{\diamond(-W_\phi(t))}dt,\end{aligned}$$

where we split the summation into two parts. Since both the correlation length and the step size is l_c, $\int_{x_i}^{x_{i+1}} g(t)e^{\diamond(-W_\phi(t))}dt$ and $\int_{x_j}^{x_{j+1}} g(t)e^{\diamond(-W_\phi(t))}dt$ are independent when both i and j are odd or even. Using the strong law of large numbers, we have

$$\int_0^x g(t)e^{\diamond(-W_\phi(t))}dt \to \Big(\sum_{i\le\lfloor\frac{x}{l_c}\rfloor \text{ and even}} + \sum_{i\le\lfloor\frac{x}{l_c}\rfloor \text{ and odd}}\Big)\int_{x_i}^{x_{i+1}} g(t)dt = \int_0^x g(t)dt,$$

as $l_c \to 0$. □

Remark 4.2. When the correlation length goes to zero, both u_I and u_{III} converge almost surely to a deterministic solution, which satisfies the mean of Eq. (3.1c)

$$-\nabla\cdot\left(\frac{1}{\mathbb{E}[a^{-1}](\boldsymbol{x})}\nabla u\right)=f(\boldsymbol{x}). \tag{4.18}$$

This is actually what stochastic homogenization theory tells us for a one-dimensional problem when the random coefficient is ergodic. Although we do not have the ergodicity condition here, we see that as the correlation length decreases to zero a small length scale is introduced and the spatial integral corresponds to the strong law of large numbers, which satisfies the ergodicity condition. Similarly, the stochastic elliptic equation (3.1a) will also goes to the homogenized one as $l_c \to 0$ when the physical dimension $d > 1$. However $\mathbb{E}[u_I]$ and $\mathbb{E}[u_{III}]$ will be not the same any more, since the effective coefficient for the homogenized equation of u_I does not take the form of a harmonic mean for $d \geq 2$ while $\mathbb{E}[u_{III}]$ satisfies Eq. (4.18) for $d = 1,2,3$ [15]. Such a difference is due to the regularization of the Wick product.

Remark 4.3. In Theorem 4.3 we introduce the correlation length in a particular way through the indicator function $\phi(y)$. Such an assumption can be relaxed and it can be shown that the random solution u_I will converge to the solution of the homogenized equation (4.18) in the limit of a vanishing correlation length. In [3, 4], some central-limit-like results were given to quantify such a convergence in distribution for one-dimensional problems. In this paper, we are more interested in the limits of u_I, u_{II} and u_{III} when the correlation length goes to zero. The generalization of Theorem 4.3 is beyond the scope of this paper.

4.2.3 Moderate correlation length

We subsequently look at the cases when the correlation length is moderate. Since no scale separation can be used, we resort to numerical simulations. For the numerical study, we choose $f(x) = \sin(x)$ as the force term and $R(x-y) = \sigma^2 e^{-|x-y|/l_c}$ as the correlation function of the underlying Gaussian random field. We will examine three typical correlation lengths $l_c = 0.01, 0.1, 1$ in contrast to the computation domain $D = [0,1]$. For each correlation length, we truncate the K-L expansion such that the smallest eigenvalue is about 1% of the largest one. Let M be the number of Gaussian random variables in the truncated K-L expansion. According to our truncation criterion, $M = 300, 30, 6$ for $l_c = 0.01, 0.1, 1$, respectively. For each l_c, we will examine the cases $\sigma = 0.1, 0.5, 1$, which corresponds to the degrees 10.03%,53.29%,131.08%, respectively, of perturbation of the log-normal coefficients.

Different numerical strategies will be used for the cases $M = 300, 30, 6$, which are chosen according to several computational issues: the computation cost, the approximation errors of the Wiener chaos expansions and the difference between u_I and u_{III}. The computation cost is determined by the polynomial order and the number of random variables. The approximation errors of the Wiener chaos expansions are determined by the polynomial order. Since our main interest is the difference between u_I and u_{III}, we choose different polynomial orders for different numbers of random variables such that the computation cost is affordable and the difference between u_I and u_{III} is dominant compared to the approximation errors of the Wiener chaos expansions. If the number of random variables is too large such that the computation cost of the Galerkin method for model I

is prohibitive, we will employ the Monte Carlo approach for the numerical approximation of model I.

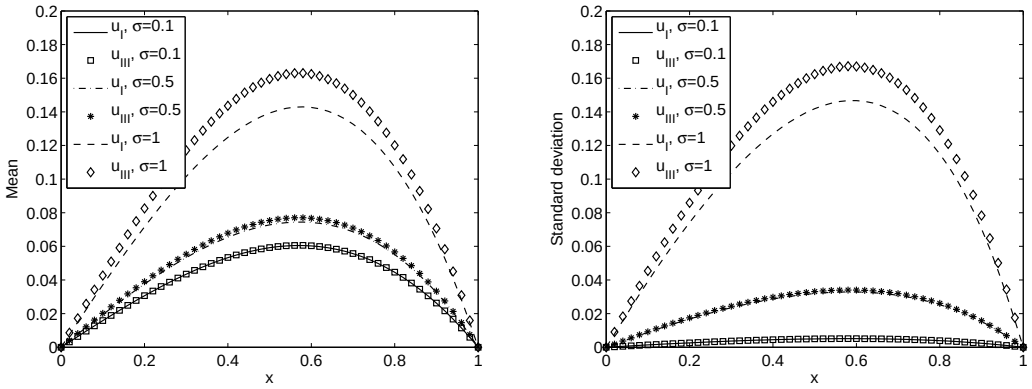

Figure 1: Statistics of u_I and u_{III}. $l_c=1$. Left: Mean; Right: Standard deviation.

Case (a): $l_c=1$.

For this case, we use fifth-order Wiener chaos expansion for both model I and III. In Fig. 1 we plot the statistics of models I and III for different σ. It is seen that for the one-dimensional problem, the first- and second-order moments of model III agree very well with those of model I when the degree of perturbation in the random coefficient is relatively low. However, the statistics of model III can be computed much more efficiently than those of model I due to the low-triangular structure of the uncertainty propagator induced by the Wick product, see Eqs. (3.12a) and (3.12b). In Fig. 2, the autocorrelation $R(u_I(x),u_{III}(x))$ is plotted, where we set $R(u_I(x),u_{III}(x))=1$ at $x=0$ and $x=1$. It is seen that a perfect correlation $R(u_I(x),u_{III}(x))\approx 1$ is obtained for different degrees of perturbations. In other words, the relation between $u_I(x)$ and $u_{III}(x)$ is almost linear. If we are only interested in the relative change of statistics with respect to x, models I and III give us almost the same information.

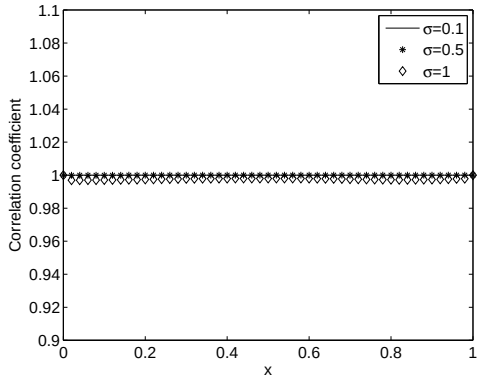

Figure 2: Correlation coefficient between u_I and u_{III}. $l_c=1$.

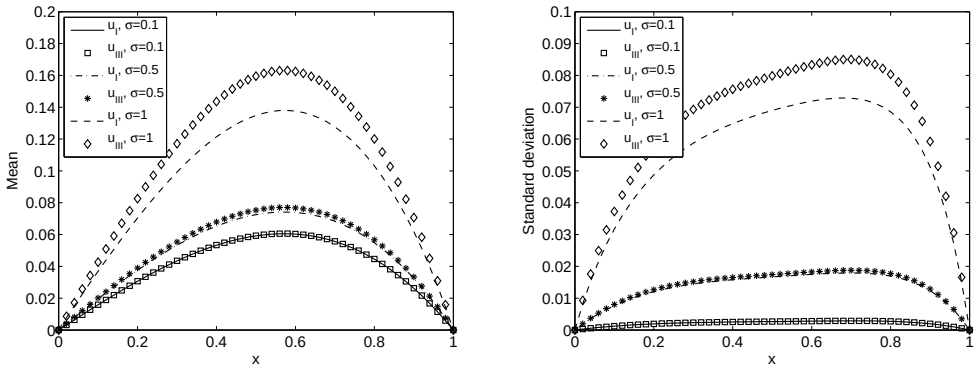

Figure 3: Statistics of u_I and u_{III}. $l_c=0.1$. Left: Mean; Right: Standard deviation.

Case (b): $l_c=0.1$**.**

For this case, we use second-order Wiener chaos expansions for both models I and III. We plot the statistics of u_I and u_{III} in Fig. 3 and the autocorrelation between u_I and u_{III} in Fig. 4. Similar phenomena are observed as in case (a). For a certain degree perturbation, the difference between statistics of u_I and u_{III} is a little bit larger than the corresponding case in case (a). By noting that for the one-dimensional problem u_I and u_{III} will converge to the same limit as l_c goes to zero or infinity, it is not surprising that the maximum difference will be reached when l_c is moderate [20].

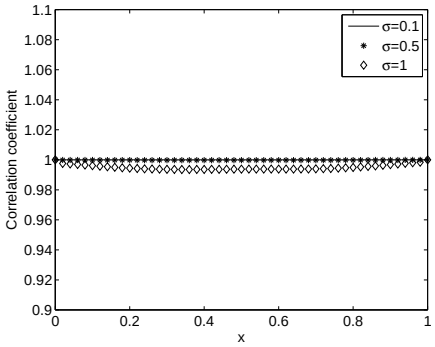

Figure 4: Correlation coefficient between u_1 and u_{III}. $l_c=0.1$.

Case (c): $l_c=0.01$**.**

For this case, we use the Monte Carlo method to approximate model I and second-order Wiener chaos to approximate model III. We plot the statistics of u_I and u_{III} in Fig. 5 and the autocorrelation between u_I and u_{III} in Fig. 6. Compared to case (b), the corresponding difference between statistics of u_I and u_{III} decreases since l_c becomes smaller.

It is observed that the largest absolute difference between u_I and u_{III} happens around $x=0.6$ for cases (a)-(c). In Fig. 7, we plot, in the log-log scale, the evolution of the difference between statistics of u_I and u_{III} at $x=0.6$ with respect to the degree of perturbation in the random coefficient. The slopes of all straight lines are 2, which confirms that the

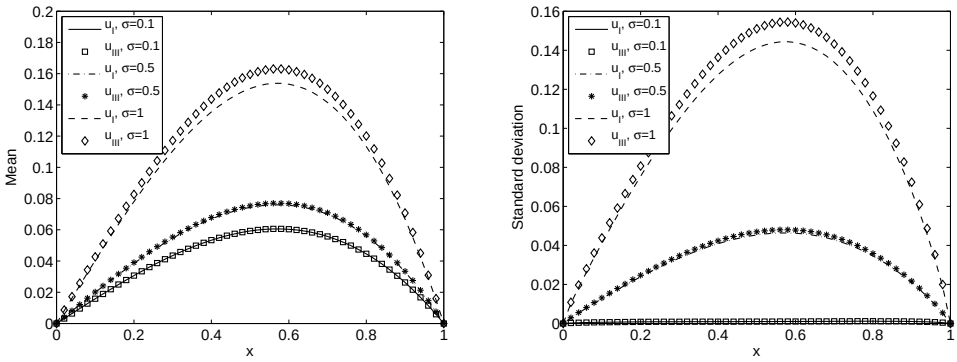

Figure 5: Statistics of u_I and u_{III}. $l_c=0.01$. Left: Mean; Right: Standard deviation.

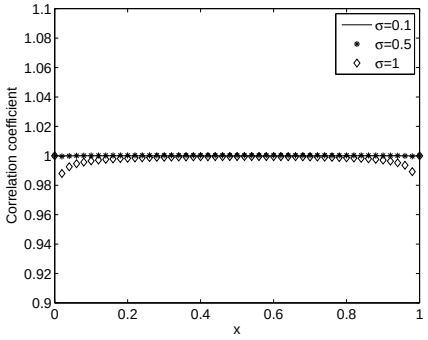

Figure 6: Correlation coefficient between u_I and u_{III}. $l_c=0.01$.

difference between the corresponding statistics of u_I and u_{III} is $C\sigma^2$ with C being a constant, as predicted by Theorem 4.1. Apparently C depends on the correlation length l_c. Numerical experiments show that C is around 0.1. The small fluctuation in the right plot of Fig. 7 for $l_c=0.01$ is because of the small statistics and the relatively low accuracy of Monte Carlo simulations.

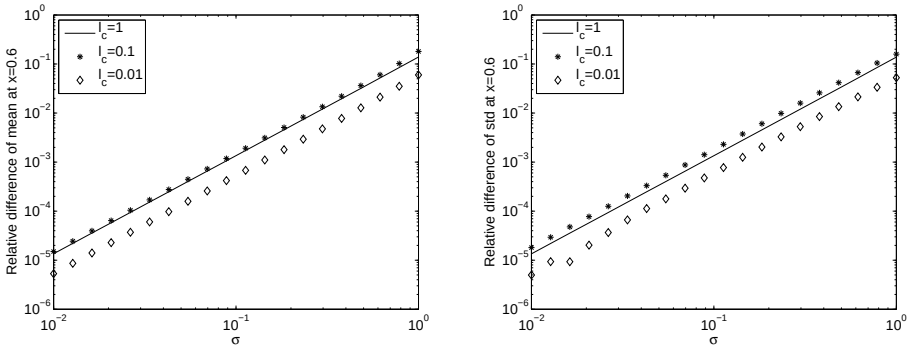

Figure 7: Relative difference between the mean and standard deviation of u_I and u_{III} at $x=0.6$. Left: Mean; Right: Standard deviation.

5 Numerical experiments for two-dimensional problems

For the two-dimensional case, we consider the computation domain $D=[0,1]^2$ and the force term $f(\boldsymbol{x})=\sin(x)\cos(y)$. We assume that the underlying Gaussian random process is subject to a Gaussian correlation function $R(\boldsymbol{x},\boldsymbol{y})=\sigma^2 e^{-|\boldsymbol{x}-\boldsymbol{y}|^2/l_c^2}$ instead of an exponential correlation function used for the one-dimensional case. The reason we choose the Gaussian correlation function is that the Gaussian correlation function is smooth and its eigenvalues and eigenfunctions can be computed accurately by a high-order numerical discretization while the exponential function has explicit formulas of eigenfunctions for one-dimensional cases [7, 20]. Three correlation lengths $l_c=5^{1/2},0.5^{1/2},0.05^{1/2}$ are examined, for which $M=5,20,100$ Gaussian random variables are used, respectively, in the Karhunen-Loève expansions. We use the Wiener chaos expansion to deal with $l_c=5^{1/2},0.5^{1/2}$ and the Monte Carlo method to deal with $l_c=0.05^{1/2}$ for model I. Almost all observations are qualitatively consistent with the one-dimensional case. For $l_c=5^{1/2}$ and $\sigma=1$, we plot in Fig. 8 the contours of the relative difference between the mean and standard deviation of u_I and u_{III}, which are normalized by the maximum absolute value of the mean or the standard deviation. The maximum relative difference occurs around the point $(0.615397,0.453080)$, which is 3.25% and 3.24% for the mean and the standard deviation, respectively. As the correlation length l_c decreases, such a relative difference increases for a certain σ. This is consistent with the fact that for two- and three-dimensional problems, models I and III have the same solution only when the correlation length is infinite. In Fig. 9 we plot the relative difference between the mean and standard deviation of u_I and u_{III} at point (0.615397,0.453080) with respect to σ. The fluctuations that occurs when $l_c=0.05^{1/2}$ and σ is relatively small are due to the low accuracy of the results given by Monte Carlo simulations. It is seen that the slope 2 is obtained, which verifies that the difference between u_I and u_{III} is of $\mathcal{O}(\sigma^2)=C(l_c)\sigma^2$. The constant $C(l_c)\approx 0.03,0.23,0.55$ for the relative difference of the mean and $C(l_c)\approx 0.03,0.21,0.53$ for

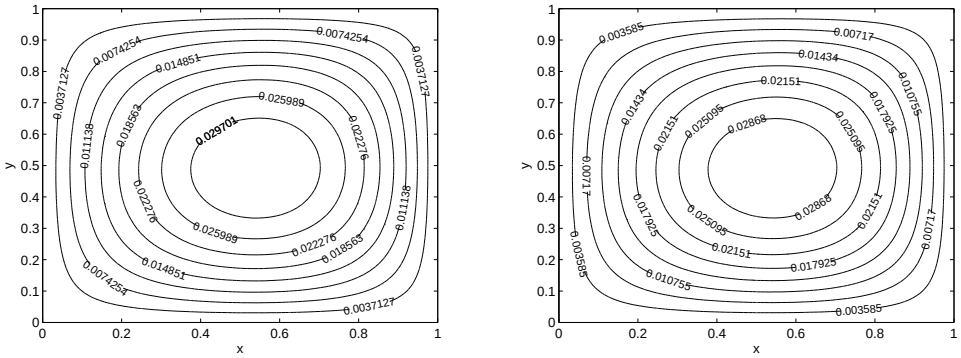

Figure 8: Contours of the relative difference between u_I and u_{III} for two-dimensional problems, which are normalized by the maximum absolute value of the mean and the standard deviation, respectively. $l_c=5^{1/2}$ and $\sigma=1$. Left: Mean; Right: Standard deviation.

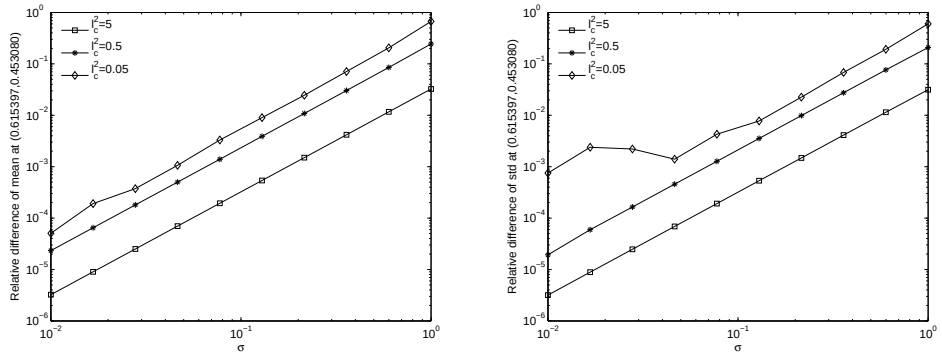

Figure 9: Relative difference between u_I and u_{III} with respect to σ at point (0.615397,0.453080). Left: Mean; Right: Standard deviation.

the relative difference of the standard deviation, corresponding to $l_c = 5^{1/2}, 0.5^{1/2}, 0.05^{1/2}$. It appears that $C(l_c)$ has a limit as l_c goes to zero. However, the discussion about such a limit of $C(l_c)$ for two- and three-dimensional problems is beyond the scope of this paper.

6 Summary

In this work, we presented a discussion on the difference between models I and III with respect to the standard deviation σ and the correlation length l_c of the underlying Gaussian random process of the log-normal random coefficient through asymptotic analysis and numerical experiments. There are several facts which do not depend on the physical dimension: (1) The difference between models I and III is of second order with respect to σ, i.e., $\|u_I - u_{III}\| \sim C(l_c)\sigma^2$, where the constant $C(l_c)$ depends on the correlation length. Note that such a difference is induced by the regularization of the Wick product. (2) When the correlation length is infinite, models I and III have the same solution. (3) When σ is relatively small, u_I and u_{III} are almost linear with respect to each other, i.e., their autocorrelation function is close to 1. We also have the following fact depending on the physical dimension: The homogenized equation of model I is the same as the mean equation of model III only when $d = 1$.

There are several open questions related to this work: (1) The constant $C(l_c)$ needs to be quantified, especially for two- and three-dimensional problems. For one-dimensional problems, $C(l_c)$ goes to zero as l_c goes to zero or infinity. For two- and three-dimensional problems, $C(l_c)$ goes to zero when l_c goes to infinity and the limit of $C(l_c)$ as l_c goes to zero needs more clarification. (2) We have seen that u_I and u_{III} can be highly correlated. Such a perfect correlation provides a sufficient condition for us to use u_{III} as a control variate for variance reduction when the Monte Carlo method is employed for u_I. (3) Since u_{III} provides a second-order approximation of u_I with a relatively small computation cost, its stiffness matrix can be a good candidate to serve as a preconditioner when the stochastic Galerkin projection method is employed for u_I.

Acknowledgments

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